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A NOTE ON ONE-DIMENSIONAL STOCHASTIC EQUATIONS

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Dedicated to Ivo Vrkoč on the occasion of his 70th birthday

Abstract. We consider the stochastic equation

$$X_t = x_0 + \int_0^t b(u, X_u)\,\mathrm{d}B_u, \quad t \geqslant 0,$$

where B is a one-dimensional Brownian motion, $x_0 \in \mathbb{R}$ is the initial value, and $b\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a time-dependent diffusion coefficient. While the existence of solutions is well-studied for only measurable diffusion coefficients b, beyond the homogeneous case there is no general result on the uniqueness in law of the solution. The purpose of the present note is to give conditions on b ensuring the existence as well as the uniqueness in law of the solution.

Keywords: one-dimensional stochastic equations, time-dependent diffusion coefficients, Brownian motion, existence of solutions, uniqueness in law, continuous local martingales, representation property

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1. Introduction

We consider the one-dimensional stochastic equation

$$X_t = x_0 + \int_0^t b(u, X_u)\,\mathrm{d}B_u, \quad t \geqslant 0,$$

where B is a one-dimensional Brownian motion, $x_0 \in \mathbb{R}$ is the initial value, and $b\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a measurable diffusion coefficient.

In the homogeneous case, i.e., if $b\colon \mathbb{R} \to \mathbb{R}$ does not depend on the time parameter, existence and uniqueness in law of the solution of Eq. (1.1) are well-understood. We recall the main results (cf. [7], [9]). Let

$$E_b = \left\{x \in \mathbb{R}\colon \int_{x-\varepsilon}^{x+\varepsilon} b^{-2}(y)\,\mathrm{d}y = +\infty,\ \ \forall \varepsilon > 0\right\}, \quad N_b = \{x \in \mathbb{R}\colon\ b(x) = 0\}.$$

(Everywhere in this paper, we make the convention $0^{-1} = +\infty$ and also $0\cdot(+\infty) = 0$.) Then, for all $x_0 \in \mathbb{R}$, there exists a solution to Eq. (1.1) starting from x_0 if and only if $E_b \subseteq N_b$. If this existence condition is satisfied then, for every $x_0 \in \mathbb{R}$, the solution starting from x_0 is unique in law if and only if $E_b = N_b$.

In the general case of time- and state-dependent diffusion coefficients, T. Senf [14], [15] has shown that, for every $x_0 \in \mathbb{R}$, there exists a (possibly, exploding) solution to Eq. (1.1) starting from x_0 if b^2 as well as b^{-2} are locally integrable on $[0,+\infty) \times \mathbb{R}$. Moreover, every solution to Eq. (1.1) does not explode if only, for every $N \geqslant 1$,

$$B_N = \left\{x \in \mathbb{R}\colon \sup_{0 \leqslant t \leqslant N} b^2(t,x) < +\infty\right\} \tag{1.2}$$

has strictly positive Lebesgue measure.

However, in the nonhomogeneous case there seems to be no general result concerning the uniqueness in law of the solution. Of course, if b is (locally) Lipschitz continuous in the state variable x uniformly in the time $t \leqslant N$ ($N \geqslant 1$), then the classical result is pathwise uniqueness and hence uniqueness in law of the solution. This is also extended to coefficients b satisfying a (certain generalized) Hölder condition with exponent $\frac{1}{2}$. But what can be said about diffusion coefficients b which are only measurable at least in the state variable x?

In the present note, we will give a partial answer to this question assuming that the square b^{-2} of the reciprocal of the diffusion coefficient b satisfies a certain local Lipschitz condition in the *time variable* t where the Lipschitz constants may depend on the state variable x in such a way that they form a locally integrable function. As a result, we will obtain some existence and uniqueness statements which could be of interest in special situations. This will be illustrated by an example which gave rise to looking for a more general result.

Unless otherwise noted, it will always be assumed that the diffusion coefficient b satisfies the following two conditions:

(C.1) For every $N \geqslant 1$, there exists a locally integrable function $L_N\colon\ \mathbb{R} \to [0, +\infty]$ such that

$$|b^{-2}(s,x) - b^{-2}(t,x)| \leqslant L_N(x)|t-s|, \quad s,t \in [0,N].$$

(C.2) For every $N \geqslant 1$, there exists a measurable function $h_N\colon\ \mathbb{R} \to [0,+\infty)$ such that h_N^{-1} is locally integrable and

$$h_N(x) \leqslant b^2(t,x) \text{ for all } (t,x) \in [0,N] \times \mathbb{R}.$$

Note that in condition (C.1), the function L_N may have the value $+\infty$ on an exceptional set of Lebesgue measure zero. Conditon (C.1) means that the function b^{-2} is locally Lipschitz continuous in t for Lebesgue almost all $x \in \mathbb{R}$, with a local Lipschitz constant $L_N(x)$ depending on $x \in \mathbb{R}$ and having a moderate growth.

Condition (C.2) is formulated in accordance with condition (E_2) of [4], as part of the existence condition $(E(x_0))$ used there. However, in the light of (C.1) it takes a quite simple form: Indeed, as can easily be verified, conditions (C.1) and (C.2) are equivalent to conditions (C.1) and (C.2′) where

(C.2′) The function $b^{-2}(0,\cdot)\colon\ \mathbb{R} \longrightarrow \mathbb{R}$ is locally integrable.

In the homogeneous case, this is just a necessary and sufficient condition for the existence of a nontrivial solution $(X,\mathbb{F})$ to Eq. (1.1) for every starting point $x_0 \in \mathbb{R}$ (cf. [8]). (We recall that a solution $(X,\mathbb{F})$ to Eq. (1.1) is called trivial if $\mathbf{P}(X_t = x_0,\ \forall t \geqslant 0) = 1$.) Thus condition (C.2′) can hardly be missed in the general case.

By $\langle X\rangle$ we denote the square variation process of a continuous local martingale $(X,\mathbb{F})$. If $(X,\mathbb{F})$ is a (nonexploding) solution of Eq. (1.1) starting from $x_0 \in \mathbb{R}$ then, obviously,

$$A_t^* := \langle X\rangle_t = \int_0^t b^2(s,X_s)\,\mathrm{d}s, \quad t \geqslant 0. \tag{2.1}$$

We define the right inverse T^* of the increasing process A^* by

$$T_t^* = \inf\{s \geqslant 0\colon\ A_s^* > t\}, \quad t \geqslant 0. \tag{2.2}$$

We also set

$$U_\infty = \inf\{s \geqslant 0\colon\ A_s^* = A_\infty^*\} \tag{2.3}$$

where, of course, $A^*_\infty = \sup_{t\geqslant 0} A^*_t$. We consider the time changed process $(W^*, \mathbb{G}^*)$ with

$$W^*_t = X_{T^*_t} - x_0, \quad \mathscr{G}^*_t = \mathscr{F}_{T^*_t}, \quad t \geqslant 0. \tag{2.4}$$

It is well-known that $(W^*, \mathbb{G}^*)$ is a Brownian motion stopped at A^*_∞. Enlarging the probability space, without loss of generality we can, and always will, assume that $(W^*, \mathbb{G}^*)$ is extended to a full Brownian motion, again denoted by $(W^*, \mathbb{G}^*)$.

Let us introduce the following notions (cf. [4], Definition 5.1; [5], Definition 4.4).

Definition 2.1. Let $(X, \mathbb{F})$ be a solution to Eq. (1.1).

(i) $(X, \mathbb{F})$ is called *basic* if

$$\int_0^{U_\infty} \mathbf{1}_{\{b=0\}}(s, X_s)\, \mathrm{d}s = 0 \quad \mathbf{P}\text{-a.s.}$$

(ii) $(X, \mathbb{F})$ is said to be *nonabsorbing* if $U_\infty = +\infty$ $\mathbf{P}$-a.s.

The main purpose of the present note is to give a proof of the following theorem. While the result on the existence is borrowed from [4], the emphasis lies on the uniqueness in law.

Theorem 2.2. *Let conditions* (C.1) *and* (C.2) *be satisfied. Then, for every initial state* $x_0 \in \mathbb{R}$, *there exists a (nonexploding) nonabsorbing and basic solution* $(X, \mathbb{F})$ *of Eq.* (1.1). *Moreover, the nonabsorbing and basic solution* $(X, \mathbb{F})$ *of Eq.* (1.1) *is unique in law.*

Next we give the following slight modification of Theorem 2.2. For this we state

(C.3) For every $(t, x) \in [0, +\infty) \times \mathbb{R}$, $b(t, x) \neq 0$.

Obviously, under (C.3) every solution $(X, \mathbb{F})$ to Eq. (1.1) is nonabsorbing and basic. From Theorem 2.2 we therefore obtain

Theorem 2.3. *Suppose that conditons* (C.1)–(C.3) *are satisfied. Then, for every starting point* $x_0 \in \mathbb{R}$, *there exists a solution* $(X, \mathbb{F})$ *of Eq.* (1.1). *This solution is unique in law.*

As an illustration we give the following example.

Example 2.4. For arbitrary $\alpha \in \mathbb{R}$, let

$$b(t, x) = f(x) + \exp(-\alpha t)\, g(x), \quad (t, x) \in [0, +\infty) \times \mathbb{R},$$

where f and g are Borel functions on $\mathbb{R}$. We assume that the following conditions are satisfied:

a) g^{-2} is locally integrable.

b) If $g(x) \neq 0$, then $\operatorname{sgn}(f(x)) = \operatorname{sgn}(g(x))$, $x \in \mathbb{R}$, where we put

$$\operatorname{sgn}(z) = \frac{z}{|z|}.$$

By N_f and N_g we denote the set of zeros of f and g, respectively. Obviously, N_g has Lebesgue measure zero. For any $x \in N_g^c$ we have

$$b^{-2}(t,x) = (|f(x)| + \exp(-\alpha t)\,|g(x)|)^{-2} \tag{2.5}$$

and hence

$$\frac{\partial b^{-2}(t,x)}{\partial t} = 2\alpha|g(x)|\exp(-\alpha t)(|f(x)| + \exp(-\alpha t)|g(x)|)^{-3}.$$

This gives

$$\sup_{0\leqslant t\leqslant N}\left|\frac{\partial b^{-2}(t,x)}{\partial t}\right| \leqslant 2|\alpha|\exp(2|\alpha|N)g^{-2}(x)$$

and, setting $L_N(x)$ equal to the right hand side for $x \in N_g^c$ and equal to $+\infty$ otherwise, we observe that (C.1) is satisfied. From (2.5) it follows immediately that (C.2′) (and hence (C.2)) hold true. If we additionally assume that

c) $N_f \cap N_g = \emptyset$

holds then (C.3) is also satisfied. Now Theorem 2.3 immediately implies that, for every starting point $x_0 \in \mathbb{R}$, there exists a solution to Eq. (1.1) which is, moreover, unique in law.

However, if $N_f \cap N_g \neq \emptyset$ then the uniqueness in law fails. Indeed, in this case we can only assert that there exists a unique nonabsorbing and basic solution X starting from x_0. But if $x_0 \in N_f \cap N_g$ then there also is the trivial solution staying forever at x_0, the law of which is, obviously, different from that of X. More generally, if $x_0 \in \mathbb{R}$ is arbitrary and if the nonabsorbing and basic solution X starting from x_0 reaches $N_f \cap N_g$ in finite time with strictly positive probability then the process obtained by stopping X at the first time it reaches $N_f \cap N_g$ is again a solution to Eq. (1.1) which has a law different from that of X.

As a particular example, we consider functions f and g defined by

$$f(x) = |x|^{\beta}\operatorname{sgn}(x), \quad g(x) = \operatorname{sgn}(x), \quad x \in \mathbb{R},$$

where $\beta \in \mathbb{R}$. Then we have $N_f \cap N_g = \{0\}$. Let $(X, \mathbb{F})$ be an arbitrary solution to Eq. (1.1) starting from $x_0 \neq 0$. Below it will be proved that the following property is satisfied:

(R) The point 0 will be reached by X with probability 1 (resp., 0) if and only if $\beta < 1$ (resp., $1 \leqslant \beta$).

Let $\beta < 1$ and consider a nonabsorbing and basic solution starting from $x_0 \neq 0$. Then the process obtained by stopping X at the first time it reaches 0 is again a solution, but with a different law. Clearly, both solutions are basic and hence nontrivial. The first solution is nonabsorbing, but the second absorbing.

On the other hand, if $1 \leqslant \beta$ then every solution X starting from $x_0 \neq 0$ does not reach 0 **P**-a.s. and consequently, is nonabsorbing and basic. Hence, if $1 \leqslant \beta$ then the solution starting from $x_0 \neq 0$ is unique in law.

Remark 2.5. Using the theorem of Girsanov, the results can be extended to stochastic equations of type

$$X_t = x_0 + \int_0^t a(u, X_u)\,\mathrm{d}u + \int_0^t b(u, X_u)\,\mathrm{d}B_u, \quad t \geqslant 0,$$

with drift and diffusion coefficients a and b. The simplest condition is to require that, additionally to the conditions used above, the ratio a/b be bounded.

Remark 2.6. The results also remain true if the driving Brownian motion B is replaced by a symmetric α-stable process S. In this case, the function b^{-2} in condition (C.1) must be replaced by $|b|^{-\alpha}$. Moreover, condition (C.2) has to be substituted by condition (E_2) which is part of the existence condition $(\mathrm{E}(x_0))$ stated in Theorem 5.3 of [4], for every $x_0 \in \mathbb{R}$.

3. Proofs of the results

P r o o f of Existence. The existence of a (possibly, exploding) nonabsorbing and basic solution to Eq. (1.1) immediately follows from [4], Theorem 5.3. Moreover, Theorem 5.4 in [4] shows that every solution $(X, \mathbb{F})$ to Eq. (1.1) does not explode if $\lambda(B_N) > 0$ for all $N \geqslant 1$, where λ is the Lebesgue measure on $\mathbb{R}$ and B_N is defined by (1.2), which is guaranteed by (C.1). We notice that [4] deals with stochastic equations driven by symmetric α-stable processes where the parameter α is from $(0, 2]$. Of course, this includes the case of a Brownian motion (with variance function $2t$) for $\alpha = 2$. We also notice that in [4] for this existence and nonexplosion result, instead of condition (C.1), only an, obviously, weaker condition is used, namely, that $b^2(\cdot, x)$ is continuous for Lebesgue almost all $x \in \mathbb{R}$. Under the additional assumption that b^2 is locally integrable in $[0,+\infty)\times\mathbb{R}$, existence of a solution to Eq. (1.1) is also established in [14] and [15].

We now come to some preparations for the proof of the uniqueness in law. For the formulation of the following lemma, from now on we extend the function b to $[0,+\infty]\times\mathbb{R}$ by setting $b(+\infty,x)=+\infty$ (and hence $b^{-2}(+\infty,x)=0$).

Lemma 3.1. *For every (nonexploding) nonabsorbing and basic solution* $(X,\mathbb{F})$ *of Eq.* (1.1) *starting from* $x_0\in\mathbb{R}$ *we have* $\mathbf{P}$-a.s.

$$\text{(3.1)}\qquad \begin{cases} T_t^* = \displaystyle\int_0^t b^{-2}(T_s^*, x_0+W_s^*)\,\mathrm{d}s, & t\geqslant 0,\\ A_t^* < A_\infty^*, & t\geqslant 0,\end{cases}$$

where T^*, W^* *and* A^* *are given by* (2.2), (2.4) *and* (2.1), *respectively.*

Proof. Because $(X,\mathbb{F})$ is basic and nonabsorbing, we get

$$\int_0^\infty \mathbf{1}_{\{b=0\}}(s,X_s)\,\mathrm{d}s = 0 \quad \mathbf{P}\text{-a.s.}$$

This yields

$$T_t^* = \int_0^{T_t^*} b^{-2}(s,X_s)b^2(s,X_s)\,\mathrm{d}s = \int_0^{T_t^*} b^{-2}(s,X_s)\,\mathrm{d}A_s^* \quad \mathbf{P}\text{-a.s.}$$

and, changing the time in the integral (cf. [8], Lemma 1.6),

$$T_t^* = \int_0^{A^*_{T_t^*}} b^{-2}(T_s^*, x_0+W_s^*)\,\mathrm{d}s = \int_0^{t\wedge A_\infty^*} b^{-2}(T_s^*, x_0+W_s^*)\,\mathrm{d}s \quad \mathbf{P}\text{-a.s.},$$

the latter equality being valid since $A^*_{T_t^*}=t\wedge A_\infty^*$ in view of the continuity of A^*. Hence the first equation of (3.1) is true on the set $\{t<A_\infty^*\}$ and, moreover,

$$T_t^* \leqslant \int_0^t b^{-2}(T_s^*, x_0+W_s^*)\,\mathrm{d}s \qquad \mathbf{P}\text{-a.s.}$$

But on $\{A_\infty^*\leqslant t\}$, we have $T_t^*=+\infty$, which proves the first equation of (3.1) on this set, too. Since $(X,\mathbb{F})$ is nonexploding we have $A_t^*<+\infty$ $\mathbf{P}$-a.s. and hence the inequality in (3.1) on $\{A_\infty^*=+\infty\}$ holds true. Finally, $A_t^*<A_\infty^*$ on the set $\{A_\infty^*<+\infty\}$ is satisfied, because $(X,\mathbb{F})$ is nonabsorbing. □

In a second step, we investigate the stochastic equation (3.1). A solution $(T, \mathbb{G})$ to Eq. (3.1) is a right continuous and increasing process T taking values in $[0, +\infty]$, defined on a (complete) probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and adapted to the filtration $\mathbb{G}$ (satisfying the usual conditions), such that there exists a Brownian motion $(W, \mathbb{G})$ with the property that Eq. (3.1) is satisfied (with T, W, A instead of T^*, W^*, A^*). Here the process A is defined as the right inverse of T:

$$A_t = \inf\{s \geqslant 0\colon\ T_s > t\}, \quad t \geqslant 0. \tag{3.2}$$

Lemma 3.2. *The solution* $(T, \mathbb{G})$ *to Eq.* (3.1) *is pathwise unique.*

Proof. The main idea of the proof is borrowed from [10], Theorem 1.2. Let $(T^1, \mathbb{G})$ and $(T^2, \mathbb{G})$ be two solutions to Eq. (3.1) on the same probability space $(\Omega, \mathscr{F}, \mathbf{P})$, with the same filtration $\mathbb{G}$ and with the same Brownian motion $(W, \mathbb{G})$. We have to show $T^1 = T^2$ $\mathbf{P}$-a.s. For this we set $\tau_N = A_N^1 \wedge A_N^2$ for every $N \geqslant 1$. In view of

$$\lim_{N\to\infty} T^i_{A^i_N} = +\infty, \quad i = 1, 2, \quad \mathbf{P}\text{-a.s.},$$

as a consequence of Eq. (3.1), it is sufficient to show that

$$T^1_{t\wedge\tau_N} = T^2_{t\wedge\tau_N}, \quad t \geqslant 0, \quad \mathbf{P}\text{-a.s.}$$

for every $N \geqslant 1$. We fix $N \geqslant 1$ and introduce the set

$$C_N = \left\{\omega \in \Omega\colon\ \int_0^t L_N(x_0 + W_u(\omega))\,\mathrm{d}u < +\infty,\ \ \forall t \geqslant 0\right\}$$

where L_N is the (state-dependent) Lipschitz constant from condition (C.1). The function L_N being locally integrable, Theorem 1 from [6] yields that $\mathbf{P}(C_N) = 1$. Obviously, we have $T^i_{t\wedge\tau_N} \leqslant N$, $i = 1, 2$, and setting $S_t := T^1_{t\wedge\tau_N} - T^2_{t\wedge\tau_N}$, $t \geqslant 0$, on the set C_N we can estimate

$$\begin{aligned}
S_t^2 &= 2\int_0^t S_u\,\mathrm{d}S_u = 2\int_0^{t\wedge\tau_N} S_u\,\mathrm{d}S_u \\
&= 2\int_0^{t\wedge\tau_N} S_u\big[b^{-2}(T^1_u, x_0 + W_u) - b^{-2}(T^2_u, x_0 + W_u)\big]\,\mathrm{d}u \\
&\leqslant 2\int_0^{t\wedge\tau_N} |S_u|\,|b^{-2}(T^1_u, x_0 + W_u) - b^{-2}(T^2_u, x_0 + W_u)|\,\mathrm{d}u \\
&\leqslant 2\int_0^{t\wedge\tau_N} |S_u| L_N(x_0 + W_u)|T^1_u - T^2_u|\,\mathrm{d}u \\
&\leqslant 2\int_0^t S_u^2 L_N(x_0 + W_u)\,\mathrm{d}u.
\end{aligned}$$

Setting

$$H_t = 2\int_0^t L_N(x_0 + W_u)\,\mathrm{d}u, \quad t \geqslant 0,$$

on the set C_N, from the above inequality we obtain

$$\begin{aligned} S_t^2 \exp(-H_t) &= \int_0^t S_u^2 \mathrm{d}\exp(-H_u) + \int_0^t \exp(-H_u)\,\mathrm{d}S_u^2 \\ &\leqslant \int_0^t S_u^2 \exp(-H_u)\big(-2L_N(x_0 + W_u)\big)\,\mathrm{d}u \\ &\quad + 2\int_0^t \exp(-H_u) S_u^2 L_N(x_0 + W_u)\,\mathrm{d}u = 0. \end{aligned}$$

This implies $S_t^2 = 0$ on C_N for all $t \geqslant 0$ and hence the assertion. □

First proof of uniqueness. Now the proof of the uniqueness is easily accomplished. If $(X, \mathbb{F})$ is a nonabsorbing and basic solution to Eq. (1.1) then $(T^*, \mathbb{G}^*)$, defined by (2.2) and (2.4), is a solution to Eq. (3.1) by Lemma 3.1. This solution is pathwise unique by Lemma 3.2. This implies that the joint distribution of (T^*, W^*) is unique, see [2], Proposition 2 or Theorem 3, for this fact. (This can also be seen using the existence of an $\mathbb{F}^{W^*}$-adapted solution T of Eq. (3.1) which is ensured by (C.1) and (C.2) (cf. [4], Theorem 3.1). Together with Lemma 3.2 it is now easy to understand that the joint distribution of (T^*, W^*) is unique.) Now, because of

$$X_t = W^*_{A^*_t}, \quad A^*_t = \inf\{s \geqslant 0\colon\ T^*_s > t\}, \quad t \geqslant 0,$$

X is a measurable functional of (T^*, W^*) and, the distribution of (T^*, W^*) being unique, the nonabsorbing and basic solution X of Eq. (1.1) is unique in law. □

Remark 3.3. The uniqueness proof (outside of the parentheses) only uses (C.1) but not (C.2). A somewhat weaker version of this result was given in [14] (Theorem 4.3.6) under stronger conditions on b, exploiting the representation property of continuous local martingales. The following lemma prepares this alternative reasoning.

Lemma 3.4. *Let condition* (C.1) *be satisfied. If* $(X, \mathbb{F})$ *is a nonabsorbing and basic solution to Eq.* (1.1) *then the continuous local martingale* $(X, \mathbb{F}^X)$, *where* $\mathbb{F}^X$ *is the filtration generated by* X, *possesses the representation property.*

Proof. First we recall that a continuous local martingale $(X, \mathbb{F}^X)$ is said to satisfy the representation property if every (local) martingale $(M, \mathbb{F}^X)$ can be represented as

$$M_t = M_0 + \int_0^t H_s\,\mathrm{d}X_s, \quad t \geqslant 0,$$

for some $\mathbb{F}^X$-previsible integrand H (cf. [11] or [13]). We know that $(T^*, \mathbb{G}^*)$ defined by (2.2) and (2.4) satisfies Eq. (3.1). On the other hand, the solution of Eq. (3.1) is pathwise unique by Lemma 3.2. By a version of the theorem of T. Yamada and S. Watanabe [16] (also see [2], Theorem 3; [12], Corollaries 14 and 15, where the equation for A^* is considered), $(T^*, \mathbb{G}^*)$ is a strong solution to Eq. (3.1), i.e., T^* is $\mathbb{F}^{W^*}$-adapted. Consequently, the process A^* defined by (2.1), just being the right inverse of T^* defined by (3.2) (replacing T by T^*), is a (strictly increasing) $\mathbb{F}^{W^*}$-time change and the assertion follows from [3], Theorem 5.

Remark 3.5. If we assume that, additionally to (C.1), condition (C.2) is satisfied then Theorems 3.1 and 3.2 of [4] ensure the existence of an $\mathbb{F}^W$-adapted solution T to Eq. (3.1) for any given Brownian motion W. Together with the pathwise uniqueness stated in Lemma 3.2, this again yields that the solution $(T^*, \mathbb{G}^*)$ in the proof of Lemma 3.4 is $\mathbb{F}^{W^*}$-adapted, giving a direct proof of Lemma 3.4 without referring to the theorem of T. Yamada and S. Watanabe.

Second proof of uniqueness. For the proof of uniqueness based on the representation property and Lemma 3.4 we assume that X^1 and X^2 are two nonabsorbing and basic solutions to Eq. (1.1). By Lemma 3.4, X^1 and X^2 possess the representation property. We consider their distributions Q^1 and Q^2 on the space of continuous functions $C([0, +\infty))$ and set $Q = \frac{1}{2}(Q^1 + Q^2)$. It is easy to verify that the canonical process on $C([0, +\infty))$ with respect to Q is again a nonabsorbing and basic solution of Eq. (1.1) and hence possesses the representation property. It is well-known (cf. [11] or [13]) that then Q must be an extremal point in the set of continuous local martingale measures. But this is only possible if $Q^1 = Q^2$, which proves the claim. □

Proof of (R). Let $(X, \mathbb{F})$ be an arbitrary solution to Eq. (1.1) starting from $x_0 \neq 0$ and introduce A^*, T^* and W^* by (2.1), (2.2) and (2.4), respectively. We then have the representation

$$X_t = x_0 + W^*_{A^*_t}, \quad t \geqslant 0.$$

By τ we denote the first time W^* reaches $-x_0$. Obviously, (R) is equivalent to the assertion

$$\mathbf{P}(\tau < A^*_\infty) = 0 \text{ or } 1$$

in dependence of $1 \leqslant \beta$ or $\beta < 1$. Since

$$\{\tau < A^*_\infty\} = \{T^*_\tau < +\infty\}$$

we have to explore conditions under which T^*_τ converges **P**-a.s. (diverges **P**-a.s.). However, T^*_τ can be represented as the integral

$$T^*_\tau = \int_0^\tau b^{-2}(T^*_s, W^*_s)\,\mathrm{d}s \quad \mathbf{P}\text{-a.s.}$$

This can be verified in the same way as Lemma 3.1 using $b(s,x) \neq 0$ for all $x \neq 0$. The integrand

$$b^{-2}(T^*_s, x_0 + W^*_s) = (|x_0 + W^*_s|^\beta + \exp(-\alpha T^*_s))^{-2}$$

is continuous in $s < \tau$ and behaves like $|x_0 + W^*_s|^{-2\beta}$ for $s \uparrow \tau$. Therefore, T^*_τ is finite (infinite) if and only if

$$\int_0^\tau |x_0 + W^*_s|^{-2\beta}\,\mathrm{d}s$$

is finite (infinite). But this integral is finite **P**-a.s. if and only if

$$\int_0^{-x_0} |x_0 + y|^{-2\beta}(-x_0 - y)\,\mathrm{d}y < +\infty \tag{3.3}$$

holds (to be definite, we have assumed $x_0 < 0$ here). Otherwise the above integral is infinite **P**-a.s. (cf. [1], Lemma 2). But, obviously, (3.3) is satisfied if and only if $\beta < 1$. This completes the proof of (R). □

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