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A NOTE ON MAXIMAL INEQUALITY FOR STOCHASTIC CONVOLUTIONS

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Dedicated to Ivo Vrkoč on the occasion of his 70th birthday

Abstract. Using unitary dilations we give a very simple proof of the maximal inequality for a stochastic convolution

$$\int_0^t S(t-s)\psi(s)\,\mathrm{d}W(s)$$

driven by a Wiener process W in a Hilbert space in the case when the semigroup $S(t)$ is of contraction type.

Keywords: infinite-dimensional Wiener process, stochastic convolution, maximal inequality

MSC 2000: 60H15

Stochastic convolution integrals and estimates thereof play an important rôle in the theory of stochastic partial differential equations. We have found it interesting that one of the basic maximal inequalities, due to L. Tubaro, may be given a very easy proof, as we will show in this paper.

Let H and Υ_0 be real separable Hilbert spaces and (e^{At}) a C_0-semigroup on H. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathsf{P})$ be a stochastic basis carrying a Q-Wiener process W in Υ_0, with $Q \in \mathscr{L}(\Upsilon_0)$ a self-adjoint and non-negative operator (not necessarily nuclear). Set $\Upsilon = \operatorname{Rng} Q^{1/2}$ and endow Υ with the norm $\|x\|_\Upsilon = \|Q^{-1/2}x\|_{\Upsilon_0}$, $Q^{-1/2}$ being the pseudoinverse. Let us denote, for any Hilbert spaces V and Z, by $\mathscr{L}(V,Z)$ and

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$\mathscr{J}_2(V,Z)$ the spaces of all bounded and all Hilbert-Schmidt operators from V to Z, respectively, equipped with their standard norms. We shall use $\boldsymbol{M}^p$ to denote the space of all progressively measurable processes $\psi\colon\ [0,T]\times\Omega\longrightarrow\mathscr{J}_2(\Upsilon,H)$ such that

$$\mathsf{E}\int_0^T\|\psi(s)\|^p_{\mathscr{J}_2(\Upsilon,H)}\,\mathrm{d}s<\infty$$

(with an obvious modification if $p=\infty$). For any $\psi\in\boldsymbol{M}^2$ we may define the stochastic convolution integral

$$W_A(t)=\int_0^t \mathrm{e}^{A(t-s)}\psi(s)\,\mathrm{d}W(s),\quad t\in[0,T].$$

The properties of the process W_A are crucial when regularity of mild solutions to stochastic evolution equations is studied, see the treatise [6] for a systematic account of the theory of mild solutions to infinite-dimensional stochastic equations. Unfortunately, the process W_A is not a martingale, and standard tools of the martingale theory, yielding e.g. continuity of trajectories or L^p-estimates like the Burkholder-Davis-Gundy inequality, are not available. The first maximal inequality for the process W_A (and, consequently, a proof of continuity of paths) is due to P. Kotelenez ([10], [11]) who proved that

$$\mathsf{E}\sup_{0\leqslant t\leqslant T}\left\|\int_0^t \mathrm{e}^{A(t-s)}\psi(s)\,\mathrm{d}W(s)\right\|^2\leqslant L_2\mathsf{E}\int_0^T\|\psi(s)\|^2_{\mathscr{J}_2(\Upsilon,H)}\,\mathrm{d}s$$

holds for a constant $L_2<\infty$ and every $\psi\in\boldsymbol{M}^2$, provided the semigroup (e^{At}) is contractive, that is, $\|\mathrm{e}^{At}\|_{\mathscr{L}(H)}\leqslant 1$ for all $t\geqslant 0$. The proofs in [10], [11] are rather complicated. Much simpler approach to maximal inequalities, based on the factorization method proposed by G. Da Prato, S. Kwapień and J. Zabczyk in [4], appeared in [5] (cf. also [6], § 7.1). It was shown there that for each $p>2$ a constant $L_p<\infty$ may be found (which depends on p, T and on $\sup\limits_{t\in[0,T]}\|\mathrm{e}^{At}\|_{\mathscr{L}(H)}$) such that

$$\mathsf{E}\sup_{0\leqslant t\leqslant T}\left\|\int_0^t \mathrm{e}^{A(t-s)}\psi(s)\,\mathrm{d}W(s)\right\|^p\leqslant L_p\mathsf{E}\int_0^T\|\psi(s)\|^p_{\mathscr{J}_2(\Upsilon,H)}\,\mathrm{d}s \tag{1}$$

for every $\psi\in\boldsymbol{M}^p$. The factorization method requires no contractivity assumptions on the semigroup and may be modified to yield, under suitable hypotheses, also estimates of W_A in norms of interpolation spaces between H and $\mathrm{Dom}(A)$ (see e.g. [8],

[12] or [6], § 5.4), to cover the case when $\psi \notin \boldsymbol{M}^2$ but the semigroup (e^{At}) is Hilbert-Schmidt ([6]) or the case of Banach space valued stochastic integrals ([1]). On the other hand, the factorization method does not work for $p = 2$ and the estimate (1) is not necessarily sharp. In fact, it was proved by L. Tubaro in [17] that if (e^{At}) is contractive and $p \in [2, \infty[$ then

$$(2) \qquad \mathsf{E} \sup_{0 \leqslant t \leqslant T} \left\| \int_0^t \mathrm{e}^{A(t-s)} \psi(s) \, \mathrm{d}W(s) \right\|^p \leqslant C_p \mathsf{E} \left(\int_0^T \|\psi(s)\|^2_{\mathscr{J}_2(\Upsilon, H)} \, \mathrm{d}s \right)^{\frac{p}{2}}$$

for all $\psi \in \boldsymbol{M}^2$, the constant C_p depending only on p. Later, A. Ichikawa [9] extended the estimate (2) also to $p \in]0, 2[$. Let us recall in this connection the relevant part of the Burkholder-Davis-Gundy inequality: for any $p \in]0, \infty[$ there exists a constant $C_p < \infty$ such that

$$(3) \qquad \mathsf{E} \sup_{0 \leqslant t \leqslant T} \left\| \int_0^t \varphi(s) \, \mathrm{d}W(s) \right\|^p \leqslant C_p \mathsf{E} \left(\int_0^T \|\varphi(s)\|^2_{\mathscr{J}_2(\Upsilon, H)} \, \mathrm{d}s \right)^{\frac{p}{2}}$$

for all $\varphi \in \boldsymbol{M}^2$. Note that the right hand sides in (2) and (3) are of the same type, moreover, the constants C_p may be taken the same. The proof in [17] is based on a lucid basic idea of applying the Itô formula to the function $\|\cdot\|^p$ and to a suitable smooth approximation of the process ψ, but it cannot be called completely elementary. (A closely related procedure was used recently in [2] to get an analogue to (2) for Banach space valued stochastic convolutions.)

On the other hand, it has been noted already in [3] that the properties of W_A are rather obvious if (e^{At}) is a group, since then

$$W_A(t) = \mathrm{e}^{At} \int_0^t \mathrm{e}^{-As} \psi(s) \, \mathrm{d}W(s) \equiv \mathrm{e}^{At} M_t,$$

where M is a martingale. It is the purpose of this paper to show that an almost trivial proof of (2) follows from this observation if one takes into account that every semigroup of contractions on H is a projection of a unitary strongly continuous *group* on a superspace of H by Sz.-Nagy's theorem on unitary dilations ([14], [15], see e.g. [7], § 7.2, or [16], Theorem I.8.1, for more recent expositions). More precisely: suppose that (e^{At}) is a semigroup of contractions on H, then there exist a Hilbert space $\mathfrak{H}$ and a unitary C_0-group $(U_t)_{t \in \mathbb{R}}$ on $\mathfrak{H}$ such that H embeds isometrically into $\mathfrak{H}$ and $PU_t = \mathrm{e}^{At}$ on H for all $t \geqslant 0$, P being the orthogonal projection from $\mathfrak{H}$

onto H. Therefore, taking $p \in]0,\infty[$ and $\psi \in \boldsymbol{M}^2$ we get

$$\begin{aligned}
&\mathsf{E} \sup_{0\leqslant t\leqslant T} \left\| \int_0^t \mathrm{e}^{A(t-s)} \psi(s)\,\mathrm{d}W(s) \right\|_H^p \\
&\quad = \mathsf{E} \sup_{0\leqslant t\leqslant T} \left\| \int_0^t PU_{t-s}\psi(s)\,\mathrm{d}W(s) \right\|_{\mathfrak{H}}^p \\
&\quad = \mathsf{E} \sup_{0\leqslant t\leqslant T} \left\| PU_t \int_0^t U_{-s}\psi(s)\,\mathrm{d}W(s) \right\|_{\mathfrak{H}}^p \\
&\quad \leqslant \|P\|_{\mathscr{L}(\mathfrak{H})} \mathsf{E} \sup_{0\leqslant t\leqslant T} \|U_t\|_{\mathscr{L}(\mathfrak{H})} \left\| \int_0^t U_{-s}\psi(s)\,\mathrm{d}W(s) \right\|_{\mathfrak{H}}^p \\
&\quad \leqslant \mathsf{E} \sup_{0\leqslant t\leqslant T} \left\| \int_0^t U_{-s}\psi(s)\,\mathrm{d}W(s) \right\|_{\mathfrak{H}}^p \\
&\quad \leqslant C_p \mathsf{E} \left(\int_0^T \|U_{-s}\psi(s)\|^2_{\mathscr{J}_2(\Upsilon,\mathfrak{H})}\,\mathrm{d}s \right)^{\frac{p}{2}} \\
&\quad \leqslant C_p \mathsf{E} \left(\int_0^T \|U_{-s}\|^2_{\mathscr{L}(\mathfrak{H})} \|\psi(s)\|^2_{\mathscr{J}_2(\Upsilon,\mathfrak{H})}\,\mathrm{d}s \right)^{\frac{p}{2}} \\
&\quad \leqslant C_p \mathsf{E} \left(\int_0^T \|\psi(s)\|^2_{\mathscr{J}_2(\Upsilon,\mathfrak{H})}\,\mathrm{d}s \right)^{\frac{p}{2}} \\
&\quad = C_p \mathsf{E} \left(\int_0^T \|\psi(s)\|^2_{\mathscr{J}_2(\Upsilon,H)}\,\mathrm{d}s \right)^{\frac{p}{2}},
\end{aligned}$$

where we have used (3) and the fact that $\mathscr{J}_2(\Upsilon,\mathfrak{H})$- and $\mathscr{J}_2(\Upsilon,H)$-norms of ψ coincide, as ψ is $\mathscr{L}(\Upsilon,H)$-valued.

The case of a generalized contraction semigroup satisfying $\|\mathrm{e}^{At}\|_{\mathscr{L}(H)} \leqslant \mathrm{e}^{\mu t}$ for some $\mu \geqslant 0$ and all $t \geqslant 0$ may be reduced easily to the case considered above by passing to the contraction semigroup $\left(\mathrm{e}^{-\mu t}\mathrm{e}^{At}\right)_{t\geqslant 0}$, as was done also in the quoted papers.

The very simple argument we have just presented yields the Tubaro theorem:

Theorem. *Suppose that* (e^{At}) *is a* C_0*-semigroup on* H *satisfying* $\|\mathrm{e}^{At}\|_{\mathscr{L}(H)} \leqslant \mathrm{e}^{\mu t}$ *for some* $\mu \geqslant 0$ *and all* $t \geqslant 0$*. Then for every* $p \in]0,\infty[$ *there exists a constant* $C_p < \infty$ *such that*

$$\mathsf{E} \sup_{0\leqslant t\leqslant T} \left\| \int_0^t e^{A(t-s)}\psi(s)\,\mathrm{d}W(s) \right\|^p \leqslant C_p \mathrm{e}^{\mu p T} \mathsf{E} \left(\int_0^T \|\psi(s)\|^2_{\mathscr{J}_2(\Upsilon,H)}\,\mathrm{d}s \right)^{\frac{p}{2}} \tag{4}$$

for all $\psi \in \boldsymbol{M}^2$.

Remark 1. Obviously, the proof yields also that W_A has a modification with continuous trajectories in H, for the martingale

$$\int_0^t U_{-s}\psi(s)\,\mathrm{d}W(s), \quad 0 \leqslant t \leqslant T,$$

has a modification with paths continuous in $\mathfrak{H}$ and the mapping $\mathbb{R} \longrightarrow \mathscr{L}(\mathfrak{H}, H)$, $t \longmapsto PU_t$ is strongly continuous. Further, by a standard localization procedure Theorem may be extended to progressively measurable processes ψ such that $\|\psi\|_{\mathscr{J}_2(\Upsilon,H)} \in L^2([0,T])$ P-almost surely.

Remark 2. The constant C_p in (4) is the same as in the Burkholder-Davis-Gundy inequality (3), in particular, C_p may be defined by

$$C_p = \left(\frac{4p}{p-1}\right)^p \left(p + \frac{1}{2}\right)^{\frac{p}{2}}$$

for $p \geqslant 2$. Hence $C_p^{1/p} = O(p^{1/2})$, $p \to +\infty$, and the Zygmund extrapolation theorem implies that

$$\mathsf{E}\exp\Big(\lambda \sup_{0\leqslant t\leqslant T} \|W_A(t)\|^2\Big) \leqslant K$$

for some constants $K < \infty$, $\lambda > 0$ and all $\psi \in \boldsymbol{M}^\infty$ with $\operatorname{ess\,sup} \|\psi\|_{\mathscr{J}_2(\Upsilon,H)} \leqslant 1$, see [13] for details.

Remark 3. In [10], [11] and [9], stochastic convolution integrals driven by general martingales were studied; the proof based on unitary dilations may be extended to such integrals. This topic will be treated in a separate paper.

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