Gary Chartrand; Ping Zhang

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THE FORCING CONVEXITY NUMBER OF A GRAPH

Gary Chartrand and Ping Zhang[1], Kalamazoo

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Dedicated to Frank Harary on the Occasion of His 79th Birthday

Abstract. For two vertices u and v of a connected graph G, the set $I(u,v)$ consists of all those vertices lying on a u–v geodesic in G. For a set S of vertices of G, the union of all sets $I(u,v)$ for $u, v \in S$ is denoted by $I(S)$. A set S is a convex set if $I(S) = S$. The convexity number $\mathrm{con}(G)$ of G is the maximum cardinality of a proper convex set of G. A convex set S in G with $|S| = \mathrm{con}(G)$ is called a maximum convex set. A subset T of a maximum convex set S of a connected graph G is called a forcing subset for S if S is the unique maximum convex set containing T. The forcing convexity number $f(S, \mathrm{con})$ of S is the minimum cardinality among the forcing subsets for S, and the forcing convexity number $f(G, \mathrm{con})$ of G is the minimum forcing convexity number among all maximum convex sets of G. The forcing convexity numbers of several classes of graphs are presented, including complete bipartite graphs, trees, and cycles. For every graph G, $f(G, \mathrm{con}) \leqslant \mathrm{con}(G)$. It is shown that every pair a, b of integers with $0 \leqslant a \leqslant b$ and $b \geqslant 3$ is realizable as the forcing convexity number and convexity number, respectively, of some connected graph. The forcing convexity number of the Cartesian product of $H \times K_2$ for a nontrivial connected graph H is studied.

Keywords: convex set, convexity number, forcing convexity number

MSC 2000: 05C12

1. Introduction

For two vertices u and v in a connected graph G, the *distance* $d(u,v)$ between u and v is the length of a shortest u–v path in G. A u–v path of length $d(u,v)$ is also referred to as a u–v *geodesic*. The set (interval) $I(u,v)$ consists of all those vertices lying on a u–v geodesic in G. For a set S of vertices of G, the union of all sets $I(u,v)$

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for $u, v \in S$ is denoted by $I(S)$. Hence $x \in I(S)$ if and only if x lies on some u–v geodesic, where $u, v \in S$.

A set S is *convex* if $I(S) = S$ (see [1], p. 136). Certainly, $V(G)$ is convex for every graph G. The *convex hull* $[S]$ of a set S of vertices of G is the smallest convex set containing S. So S is a convex set in G if and only if $[S] = S$. If S is a convex set in a connected graph G, then the subgraph $\langle S \rangle$ induced by S is connected.

The closed intervals and convex sets in a connected graph were studied and characterized by Nebeský [6, 7] and were also investigated extensively in the book by Mulder [5], where it was shown that these sets provide an important tool for studying metric properties of connected graphs. Convexity in graphs was also studied in [2, 3, 4]. For a connected graph G of order at least 3, the *convexity number* $\operatorname{con}(G)$ of G was defined in [2] as the maximum cardinality of a proper convex set of G, that is,

$$\operatorname{con}(G) = \max\{|S|\colon\ S \text{ is a convex set of } G \text{ and } S \neq V(G)\}.$$

Hence $2 \leqslant \operatorname{con}(G) \leqslant n - 1$ for all connected graphs G of order $n \geqslant 3$. A convex set S in G with $|S| = \operatorname{con}(G)$ is called a *maximum convex set*.

A subset T of a maximum convex set S of a connected graph G is called a *forcing subset* for S if S is the unique maximum convex set containing T. The *forcing convexity number* $f(S, \operatorname{con})$ of S is the minimum cardinality among the forcing subsets for S, and the *forcing convexity number* $f(G, \operatorname{con})$ of G is the minimum forcing convexity number among all maximum convex sets of G. Therefore, $f(G, \operatorname{con}) \leqslant \operatorname{con}(G)$ for every connected graph G. We illustrate these concepts with the graph G of Figure 1. The sets $S_1 = \{u_1, w, v_1\}$ and $S_2 = \{w, u_1, v_2\}$ are maximum convex sets of G. The remaining maximum convex sets of G are similar to S_2. Since S_1 is not the unique maximum convex set containing any of its elements, $f(S_1, \operatorname{con}) \geqslant 2$. On the other hand, S_1 is the unique maximum convex set containing u_1 and v_1. Hence $f(S_1, \operatorname{con}) = 2$. Since S_2 is the unique maximum convex set containing v_2, it follows that $f(S_2, \operatorname{con}) = 1$. Therefore, $f(G, \operatorname{con}) = 1$.

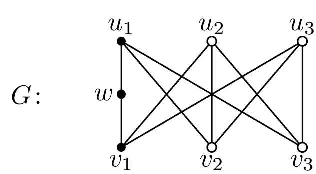

Figure 1. A graph with forcing convexity number 1

Some of the following observations were used in the previous example and all of these are fundamental to our study.

Lemma 1.1. *For a connected graph G, the forcing convexity number $f(G, \mathrm{con}) = 0$ if and only if G has a unique maximum convex set. Moreover, $f(G, \mathrm{con}) = 1$ if and only if G does not have a unique maximum convex set but some vertex of G belongs to exactly one maximum convex set.*

Corollary 1.2. *For a connected graph G, the forcing convexity number*

$$f(G, \mathrm{con}) \geqslant 2$$

if and only if every vertex of each maximum convex set belongs to at least two maximum convex sets.

Next we determine the forcing convexity number of the famous Petersen graph P shown in Figure 2.

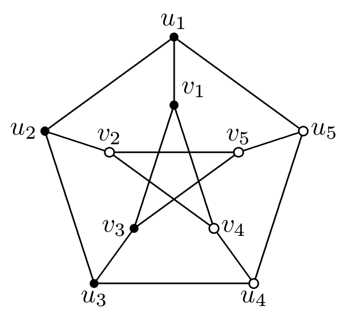

Figure 2. The Petersen graph P

It can be verified that the convexity number of P is 5 and that the maximum convex sets of P are precisely those that induce a 5-cycle. Since all such sets of cardinality 5 are similar in P, we consider the set $S = \{u_1, u_2, u_3, v_1, v_3\}$. For every $w \in S$, there exists a maximum convex set $S' \neq S$ such that $w \in S'$. For example, $S' = \{u_1, u_4, u_5, v_1, v_4\}$ is another maximum convex set containing u_1. Therefore, every vertex of each maximum convex set of P belongs to at least two maximum convex sets. Hence $f(P, \mathrm{con}) \geqslant 2$ by Corollary 1.2. For every $u, v \in S$, there exists a maximum convex set $S^* \neq S$ such that $u, v \in S^*$. For example, $S_1^* = \{u_1, u_2, u_3, u_4, u_5\}$ is another maximum convex set containing u_i, u_j in S for $1 \leqslant i \neq j \leqslant 3$, and $S_2^* = \{u_1, u_5, v_1, v_3, v_5\}$ is another maximum convex set containing u_1, v_k in S for $k = 1, 3$. Hence $f(S, \mathrm{con}) \geqslant 3$. Moreover, for $S_0 = \{u_1, v_3, u_3\}$, it follows that $[S_0] = S$. This implies that S is the unique maximum convex set containing S_0 and so $f(S, \mathrm{con}) = 3$. Therefore, $f(P, \mathrm{con}) = 3$.

The following theorem gives the forcing convexity numbers of some well known graphs, all of whose convexity numbers were determined in [2]. Since the proof is straightforward, we omit it.

Theorem 1.3. (a) *For* $n \geqslant 3$, $f(K_n, \mathrm{con}) = \mathrm{con}(K_n) = n - 1$.

(b) *For* $n \geqslant 4$, $f(C_n, \mathrm{con}) = 2$ *and* $\mathrm{con}(C_n) = \lceil n/2 \rceil$.

(c) *For integers* $k, n_1, n_2, \ldots, n_k \geqslant 2$, $f(K_{n_1,n_2,\ldots,n_k}, \mathrm{con}) = \mathrm{con}(K_{n_1,n_2,\ldots,n_k}) = k$.

(d) *For a tree* T *of order* $n \geqslant 2$ *with* k *end-vertices,* $f(T, \mathrm{con}) = k - 1$ *and* $\mathrm{con}(T) = n - 1$.

2. Graphs with prescribed forcing convexity number and convexity number

We have already noted that if G is a connected graph with $f(G, \mathrm{con}) = a$ and $\mathrm{con}(G) = b$, then $0 \leqslant a \leqslant b$, where $b \geqslant 2$. We now establish a converse result. First we need an additional definition.

A vertex v in a graph G is called a *complete vertex* if the subgraph induced by its neighborhood $N(v)$ is complete. Connected graphs of order $n \geqslant 3$ containing a complete vertex are precisely those having convexity number $n-1$, as was established in [2].

Theorem A. *Let* G *be a noncomplete connected graph of order* $n \geqslant 3$. *Then* $\mathrm{con}(G) = n - 1$ *if and only if* G *contains a complete vertex.*

We first determine the forcing convex numbers of all nontrivial connected graphs with forcing convexity number 2.

Theorem 2.1. *For a connected graph* G *with* $\mathrm{con}(G) = 2$,

$$f(G, \mathrm{con}) = \begin{cases} 1 & \text{if } G = P_3, \\ 2 & \text{otherwise.} \end{cases}$$

P r o o f. Since $\mathrm{con}(G) = 2$, every pair of adjacent vertices forms a maximum convex set of G. Hence G does not contain a unique maximum convex set and so $f(G, \mathrm{con}) \geqslant 1$. If $G = P_3$, then $f(G, \mathrm{con}) = 1$ by Theorem 1.3. Otherwise, G contains no end-vertices by Theorem A and so every vertex of G belongs to at least two maximum convex sets. Therefore, $f(G, \mathrm{con}) = 2$ by Corollary 1.2. □

By Theorem 2.1, there is no connected graph with convexity number 2 and forcing convexity number 0. Next we show that every pair a, b of integers with $0 \leqslant a \leqslant b$ and $b \geqslant 3$ is realizable as the forcing convexity number and convexity number, respectively, of some connected graph.

Theorem 2.2. *For every pair* a, b *of integers with* $0 \leqslant a \leqslant b$ *and* $b \geqslant 3$, *there exists a connected graph* G *with* $f(G, \mathrm{con}) = a$ *and* $\mathrm{con}(G) = b$.

P r o o f. We have already seen that $f(K_{b+1}, \mathrm{con}) = \mathrm{con}(K_{b+1}) = b$. Thus, we assume that $0 \leqslant a < b$. If $a \geqslant 1$, then any tree of order $b+1$ having $a+1$ end-vertices has the desired property by Theorems A and 1.3(b). Thus we may assume that $a = 0$.

We construct a connected graph G with $f(G, \mathrm{con}) = 0$ and $\mathrm{con}(G) = b$. In order to do this, we first construct three graphs F_1, F_2 and F. First let $F_1 = \overline{K}_2 + H$, where H is any graph of order $b-2 \geqslant 1$ and $V(\overline{K}_2) = \{u, v\}$. Next let F_2 be a graph with vertex set $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ such that $x_i x_{i+1}, y_i y_{i+1}, x_i y_{i+1}, y_i x_{i+1} \in E(F_2)$ for $i = 1, 2$. Then the graph F is obtained from F_1 and F_2 by adding edges ux_1, uy_1, vx_3, vy_3. The graphs F_1, F_2, and F are shown in Figure 3.

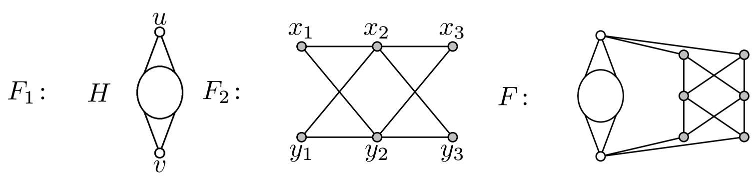

Figure 3. The graphs F_1, F_2, and F

The graph G is then obtained from F by adding two new vertices x and y and the edges (1) xx_1, xy_1, xy_3, yx_3, yy_1, yy_3 and (2) xw, yw for every $w \in V(H)$, where H is the subgraph of F. In particular, if $b = 3$, then $H = K_1$ and the graph G is shown in Figure 4. We claim that G has the desired properties, that is, $f(G, \mathrm{con}) = 0$ and $\mathrm{con}(G) = b$. We show only that the graph G in Figure 4 (where $b = 3$) has forcing number 0 and convexity number 3 since the proofs for the cases when $b \geqslant 4$ are similar.

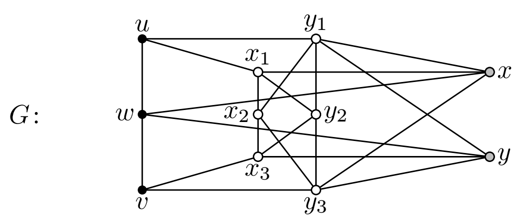

Figure 4. The graph G for $b = 3$

First we make an observation. Let

$$W = V(F_2) \cup \{x, y\} = V(G) - \{u, v, w\}.$$

For any two nonadjacent vertices z', z'' of W, we have $[\{z', z''\}] = V(G)$. Hence if S_0 is any set of vertices containing two nonadjacent vertices of W, then $[S_0] = V(G)$.

Next we show that $\operatorname{con}(G) = 3$. Since $S = \{u, v, w\}$ is a convex set in G, it follows that $\operatorname{con}(G) \geqslant 3$. Assume, to the contrary, that there exists a convex set S' with $|S'| \geqslant 4$ and $S' \neq V(G)$. Since S' cannot contain two nonadjacent vertices of W, it follows that S' contains at most two vertices of W. On the other hand, $|S'| \geqslant 4$ and so S' contains at least two vertices of $\{u, v, w\}$. Since $u, v \in S'$ implies that $w \in S'$, it follows that either $\{u, v, w, z\} \subseteq S'$, where $z \in W$, or, without loss of generality, that $\{u, w, z_1, z_2\} \subseteq S'$, where $z_1, z_2 \in W$ and $z_1 z_2 \in E(G)$. In each case, it is routine to verify that $[S'] = V(G)$. Since S' is a convex set, $S' = [S'] = V(G)$, which is a contradiction. Therefore, $\operatorname{con}(G) = 3$.

Finally, we show that S is the unique maximum convex set in G, implying that $f(G, \operatorname{con}) = 0$. Therefore, assume, to the contrary, that there exists a convex set S^* of G such that $S^* \neq S$ and $|S^*| = 3$. Necessarily, $\langle S^* \rangle = P_3$. Again, since S^* cannot contain two nonadjacent vertices of W, it follows that S^* contains at most two vertices of W. Hence S^* contains at least one and at most two vertices of $\{u, v, w\}$. We consider two cases.

Case 1 *S^* contains exactly one vertex in $\{u, v, w\}$.* First assume that $w \in S^*$. Then S^* contains exactly one of x and y as well as a neighbor z of this vertex. However, either u or v lies on a $w-z$ geodesic and so S^* is not convex, a contradiction. Hence either $u \in S^*$ or $v \in S^*$, say $v \in S^*$. Thus $S^* = \{v, z_1, z_2\}$, where $z_1, z_2 \in W$ and $z_1 z_2 \in E(G)$. Since $z_1 z_2 \in E(G)$, one of z_1 and z_2 is at distance 2 from v, say $d(v, z_1) = 2$. Thus either (1) $x_2 \in S^*$ or $y_2 \in S^*$, or (2) $x \in S^*$ or $y \in S^*$. In (1), we must have $x_3, y_3 \in S^*$, while in (2), we must have $w \in S^*$. In either cases, we have a contradiction.

Case 2 *S^* contains exactly two vertices in $\{u, v, w\}$, namely $v, w \in S^*$ or $u, w \in S^*$, say the former.* Then $S^* = \{v, w, z\}$, where $z \in W$. Similarly to that described above, either (1) $z = x_3$ or $z = y_3$, or (2) $z = x$ or $z = y$. In (1), we must have $y \in S^*$, while in (2), $y_3 \in S^*$. Hence S^* is not convex, producing a contradiction.

Therefore, S is the unique maximum convex set of G. □

3. The forcing convexity number of $H \times K_2$

In this section, we consider the relationship between $f(H, \operatorname{con})$ and $f(H \times K_2, \operatorname{con})$ for a connected graph H. Let $H \times K_2$ be formed from two copies H_1 and H_2 of H, where corresponding vertices of H_1 and H_2 are adjacent. Let $S_i \subseteq V(H_i)$ for $i = 1, 2$. Then S_2 is called the *projection* of S_1 onto H_2 if S_2 is the set of vertices in H_2 corresponding to the vertices of H_1 that are in S_1. The following two results appeared in [3].

Lemma B. *For a nontrivial connected graph H, let $H \times K_2$ be formed from two copies H_1 and H_2 of H, where corresponding vertices of H_1 and H_2 are adjacent. Then every convex set of $H \times K_2$ is either*

(1) *a convex set in H_1,*

(2) *a convex set in H_2, or*

(3) *$S_1 \cup S_2$, where S_1 is convex in H_1 and S_2 is the projection of S_1 onto H_2.*

Theorem C. *If H is a nontrivial connected graph of order n, then*

$$\operatorname{con}(H \times K_2) = \max\{2\operatorname{con}(H), n\}.$$

In order to establish a relationship between $f(H \times K_2, \operatorname{con})$ and $f(H, \operatorname{con})$ for a nontrivial connected graph H, we first verify the following lemma. For a graph G and a set S of vertices of G, we write $f_G(S, \operatorname{con})$ to indicate the forcing convexity number of S in the graph G.

Lemma 3.1. *Let H be a connected graph of order $n \geqslant 2$ such that $\operatorname{con}(H \times K_2) = 2\operatorname{con}(H) > n$. Moreover, let $H \times K_2$ be formed from two copies H_1 and H_2 of H, where corresponding vertices of H_1 and H_2 are adjacent. If $S = S_1 \cup S_2$ is a maximum convex set in $H \times K_2$, where $S_i \subseteq V(H_i)$ for $i = 1, 2$, then*

$$f_{H \times K_2}(S, \operatorname{con}) = f_{H_i}(S_i, \operatorname{con}).$$

Proof. Since $S = S_1 \cup S_2$ is a maximum convex set in $H \times K_2$, it follows by Lemma B that S_i is a maximum convex set of H_i for $i = 1, 2$ and that S_2 is the projection of S_1 onto H_2. Certainly, $f_{H_1}(S_1, \operatorname{con}) = f_{H_2}(S_2, \operatorname{con})$. We first show that $f_{H \times K_2}(S, \operatorname{con}) \leqslant f_{H_1}(S_1, \operatorname{con})$. Let T_1 be a minimum forcing subset for S_1. Thus $|T_1| = f(S_1, \operatorname{con})$ and S_1 is the unique maximum convex set in H_1 containing T_1. Let T_2 be the projection of T_1 onto S_2 in H_2. We claim that S is the unique maximum convex set in $H \times K_2$ containing T_1. Assume, to the contrary, that there exists a maximum convex set S' in $H \times K_2$ containing T_1 such that $S' \neq S$. Hence $S' = S'_1 \cup S'_2$, where $S'_i \subseteq V(H_i)$, $i = 1, 2$. Again, by Lemma B, S'_i is a maximum convex set in H_i containing T_i, $i = 1, 2$, and S'_2 is the projection of S'_1 onto H_2. Since $S' \neq S$, it follows that $S'_1 \neq S_1$. This implies that S_1 is not the unique maximum convex set in H_1 containing T_1 since S'_1 contains T_1 as well, contrary to our assumption. Hence S is the unique maximum convex set in G containing T_1, as claimed. Therefore, $f_{H \times K_2}(S, \operatorname{con}) \leqslant |T_1| = f_{H_1}(S_1, \operatorname{con})$.

It remains to verify the reverse inequality $f_{H \times K_2}(S, \operatorname{con}) \geqslant f_{H_1}(S_1, \operatorname{con})$. Assume, to the contrary, that $f_{H \times K_2}(S, \operatorname{con}) < f_{H_1}(S_1, \operatorname{con})$. Let T be a minimum forcing subset for S. Then $|T| = f_{H\times K_2}(S, \mathrm{con})$ and S is the unique maximum convex set in $H \times K_2$ containing T. We consider two cases.

Case 1. *$T \subseteq S_1$ or $T \subseteq S_2$, say the former.* Since $|T| < f_{H_1}(S_1, \mathrm{con})$, it follows that S_1 is not the unique maximum convex set containing T in H_1. So there exists a maximum convex set W_1 containing T in H_1 such that $W_1 \neq S_1$. Let W_2 be the projection of W_1 onto H_2 and let $W = W_1 \cup W_2$. Then $W \neq S$ and W is a maximum convex set containing T in $H \times K_2$, a contradiction.

Case 2. $T \cap S_i \neq \emptyset$, $i = 1, 2$. Then $T = T_1 \cup T_2$, where $\emptyset \neq T_i \subseteq S_i$ for $i = 1, 2$. Let $\pi(T_1)$ be the projection of T_1 onto H_2 and $\pi^{-1}(T_2)$ be the (inverse) projection T_2 onto H_1. Then the set $T' = T_1 \cup \pi^{-1}(T_2)$ is a subset of S_1. Since $|T'| \leqslant |T_1| + |\pi^{-1}(T_2)| = |T_1| + |T_2| = |T| < f_{H_1}(S_1, \mathrm{con})$, it follows that S_1 is not the unique maximum convex set containing T' in H_1. So there exists a maximum convex set U_1 containing T' in H_1 such that $U_1 \neq S_1$. Let U_2 be the projection of U_1 onto H_2 and let $U = U_1 \cup U_2$. Then $U \neq S$ and U is also a maximum convex set containing T in $H \times K_2$, a contradiction. □

We now determine $f(H \times K_2)$ for almost all graphs H.

Theorem 3.2. *Let H be a connected graph of order $n \geqslant 2$ for which $\mathrm{con}(H) \neq n/2$, and let $H \times K_2$ be formed from two copies H_1 and H_2 of H whose corresponding vertices are adjacent. Then*

$$f(H \times K_2, \mathrm{con}) = \begin{cases} 1 & \text{if } \mathrm{con}(H \times K_2) = n \\ f(H, \mathrm{con}) & \text{if } \mathrm{con}(H \times K_2) = 2\,\mathrm{con}(H). \end{cases}$$

Proof. Let $G = H \times K_2$. Assume first that $\mathrm{con}(G) = n$. By Theorem C, $n > 2\,\mathrm{con}(H)$. It follows by Lemma B that G contains exactly two maximum convex sets, namely $S_1 = V(H_1)$ and $S_2 = V(H_2)$. Hence $f(H \times K_2, \mathrm{con}) = 1$.

We now assume that $\mathrm{con}(H \times K_2) = 2\,\mathrm{con}(H)$. By Theorem C, $2\,\mathrm{con}(H) > n$. Again, by Lemma B, the maximum convex sets of G are of the form $S_1 \cup S_2$, where S_1 is a maximum convex set in H_1 and S_2 is the projection of S_1 onto H_2. Moreover, by Lemma 3.1, $f_G(S, \mathrm{con}) = f_{H_1}(S_1, \mathrm{con})$ for every maximum convex set S in G. Hence

$$\begin{aligned} f(G, \mathrm{con}) &= \min\{f(S, \mathrm{con})\colon S \text{ is a maximum convex set in } G\} \\ &= \min\{f_{H_1}(S_1, \mathrm{con})\colon S_1 \text{ is a maximum convex set in } H_1\} \\ &= f(H_1, \mathrm{con}) = f(H, \mathrm{con}). \end{aligned}$$

This completes the proof. □

If H is a connected graph of order $n \geqslant 2$ containing a complete vertex, then $\operatorname{con}(H) = n - 1$ by Theorem A. Certainly, if $n \geqslant 3$, then $2\operatorname{con}(H) > n$. This observation yields the following corollary.

Corollary 3.3. *If H is a connected graph of order $n \geqslant 3$ containing complete vertices, then $f(H \times K_2, \operatorname{con}) = 1$.*

What remains to consider then are connected graphs H of order $n \geqslant 4$ with $\operatorname{con}(H) = n/2$. Certainly, n is even then. For such a graph H, a maximum convex set in $H \times K_2$ is either $V(H_i)$, $i = 1, 2$, or is of the form $S_1 \cup S_2$, where S_i is a maximum convex set of cardinality $n/2$ in H_i, $i = 1, 2$, and S_2 is the projection of S_1 onto H_2. Since $H \times K_2$ contains more than one maximum convex set, $f(H \times K_2, \operatorname{con}) \geqslant 1$. If H contains a vertex v that belongs to no maximum convex set of H, then $V(H_1)$ is the unique maximum convex set in $H \times K_2$ containing v_1, where v_1 is the corresponding vertex of v in H_1 of $H \times K_2$. Therefore, in this case, $f(H \times K_2, \operatorname{con}) = 1$. This observation yields the following result.

Proposition 3.4. *If H is a connected graph of order $n \geqslant 4$ with $\operatorname{con}(H) = n/2$ such that H contains a vertex that belongs to no maximum convex set of H, then $f(H \times K_2, \operatorname{con}) = 1$.*

Assume now that H is a connected graph of order $n \geqslant 4$ with $\operatorname{con}(H) = n/2$ such that every vertex of H belongs to some maximum convex set of H. Consequently, every vertex in $H \times K_2$ belongs to at least two maximum convex sets in $H \times K_2$. For example, let v_1 be a vertex in $H \times K_2$ such that $v_1 \in V(H_1)$ and let S_1 be a maximum convex set in H_1 containing v_1. Then $V(H_1)$ and $S = S_1 \cup S_2$, where S_2 is the projection of S_1 onto H_2, are both maximum convex sets in $H \times K_2$ containing v_1. By Corollary 1.2,

$$f(H \times K_2, \operatorname{con}) \geqslant 2. \tag{1}$$

Moreover, equality holds in (1) when $f(H, \operatorname{con}) = 1$ as we show next.

Proposition 3.5. *If H is a connected graph of order $n \geqslant 4$ with $\operatorname{con}(H) = n/2$ and $f(H, \operatorname{con}) = 1$ such that every vertex of H belongs to some maximum convex set, then $f(H \times K_2, \operatorname{con}) = 2$.*

Proof. By the discussion above, we see that $f(H \times K_2, \operatorname{con}) \geqslant 2$. Since $f(H, \operatorname{con}) = 1$, there exists a maximum convex set S_1 in $V(H_1)$ and a vertex $v_1 \in S_1$ such that S_1 is the unique maximum convex set in H_1 containing v_1. Let $S = S_1 \cup S_2$, where S_2 is the projection of S_1 onto H_2. Then S contains both v_1 and its corresponding vertex v_2 in H_2. We claim that S is the unique maximum convex set in $H \times K_2$ containing v_1 and v_2. Assume, to the contrary, that there exists a maximum convex set S' in $H \times K_2$ containing v_1 and v_2 such that $S' \neq S$. Then $S' \neq V(H_i)$, $i = 1, 2$, and so $S' = S'_1 \cup S'_2$, where S'_1 is the maximum convex set in H_1 containing v_1 and S'_2 is the projection of S'_1 onto H_2. Since $S \neq S'$, it follows that $S'_1 \neq S_1$, implying that S_1 is not the unique maximum convex set in H_1 containing v_1, which is a contradiction. Hence $f(S, \mathrm{con}) = 2$. Therefore, $f(H \times K_2, \mathrm{con}) = 2$. $\square$

We are now only concerned with determining $f(H \times K_2, \mathrm{con})$, where H is a connected graph of order $n \geqslant 4$ having the three properties (1) $\mathrm{con}(H) = n/2$, (2) $f(H, \mathrm{con}) \geqslant 2$, and (3) every vertex of H belongs to at least one maximum convex set of H. We now introduce a new term. For a connected graph H of order $n \geqslant 3$, the *anti-convexity number* $\mathrm{acon}(H)$ of H is the minimum number of vertices of H that belongs to no maximum convex set of H. For graphs H satisfying the three properties listed above, $\mathrm{acon}(H) \geqslant 2$. Each graph H_i, $i = 1, 2$, of Figure 5 satisfies the properties (1)–(3). In particular, $\mathrm{con}(H_1) = 3$ and $\mathrm{con}(H_2) = 4$. Observe that $\mathrm{acon}(H_i) = 2$ for $i = 1, 2$, where a 2-element set $\{u, v\}$ of vertices belonging to no maximum convex set is indicated in each graph. We note also that $f(H_1, \mathrm{con}) = 3$ and $f(H_2, \mathrm{con}) = 2$. We have already seen that the Petersen graph P, which has order 10 and is shown in Figure 2, has $\mathrm{con}(P) = 5$ and $f(P, \mathrm{con}) = 3$. It is also the case that $\mathrm{acon}(P) = 3$. We now determine $f(H \times K_2, \mathrm{con})$ for graphs H satisfying properties (1)–(3) in terms of $f(H, \mathrm{con})$ and $\mathrm{acon}(H)$.

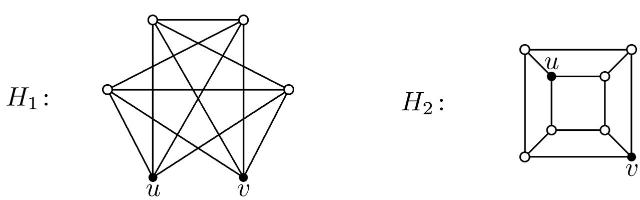

Figure 5. Two graphs whose convexity numbers are half their order

Theorem 3.6. *Let H be a connected graph of order $n \geqslant 4$ satisfying* (1) $\mathrm{con}(H) = n/2$, (2) $f(H, \mathrm{con}) \geqslant 2$, *and* (3) *every vertex of H belongs to at least one maximum convex set of H. Then*

$$f(H \times K_2, \mathrm{con}) = \min\{\mathrm{acon}(H), f(H, \mathrm{con})\}.$$

P r o o f. Let S be a maximum convex set of $H \times K_2$. There are two possibilities for S.

Case 1. $S = V(H_1)$ *or* $S = V(H_2)$, *say the former.* Let T be a minimum forcing subset for S in $H \times K_2$ such that $|T| = f_{H \times K_2}(S, \mathrm{con})$. Hence S is the unique maximum convex set in $H \times K_2$ containing T. Since $S = V(H_1)$, it follows that T belongs to no maximum convex set in H_1. So $\operatorname{acon}(H) \leqslant |T|$. We claim, in fact, that $|T| = \operatorname{acon}(H)$, that is, we claim that T is a minimum set of vertices of H_1 that belongs to no maximum convex set in H_1. Assume, to the contrary, that there is a set T' of vertices of H_1 such that $|T'| < |T|$ and T' belongs to no maximum convex set in H_1. Then S is the unique maximum convex set in $H \times K_2$ containing T' and so $f_{H\times K_2}(S, \operatorname{con}) \leqslant |T'| < |T|$, contrary to the fact that $|T| = f_{H\times K_2}(S, \operatorname{con})$. So $|T| = \operatorname{acon}(H)$, as claimed. Therefore, in this case $f_{H\times K_2}(S, \operatorname{con}) = \operatorname{acon}(H)$.

Case 2. *$S = S_1 \cup S_2$, where S_1 is a maximum convex set in H_1 and S_2 is the projection of S_1 onto H_2.* An argument similar to the one employed in Lemma 3.1 shows that $f_{H_1}(S_1, \operatorname{con}) \leqslant f_{H\times K_2}(S, \operatorname{con})$. Thus, it remains to show that $f_{H\times K_2}(S, \operatorname{con}) \leqslant f_{H_1}(S_1, \operatorname{con})$. Let $T_1 = \{t_1, t_2, \dots, t_k\}$ be a minimum forcing set for S_1 in H_1. Thus $|T_1| = f_{H_1}(S_1, \operatorname{con})$. Since $f(H, \operatorname{con}) \geqslant 2$, it follows that $|T_1| = k \geqslant 2$. Let t'_k be the corresponding vertex of t_k in H_2 and let $T^* = \{t_1, t_2, \dots, t_{k-1}, t'_k\}$. Next we show that S is the unique maximum convex set in $H \times K_2$ containing T^*. Assume, to the contrary, that there exists a maximum convex set S' in $H \times K_2$ such that S' contains T^* and $S' \neq S$. Since S' contains the vertex t_1 of H_1 and the vertex t'_k of H_2, it follows $S' \neq V(H_i)$ for $i = 1, 2$. Hence $S' = S'_1 \cup S'_2$, where S'_1 is a maximum convex set in H_1 and S'_2 is the projection of S'_1 onto H_2. Then S'_1 is a maximum convex set in H_1 containing T_1. Since $S' \neq S$, it follows that $S'_1 \neq S_1$ and so S_1 is not the unique maximum convex set containing T_1 in H_1, a contradiction. Therefore, in this case $f_{H\times K_2}(S, \operatorname{con}) = f_{H_1}(S_1, \operatorname{con})$.

Combining Cases 1 and 2, we have

$$\begin{aligned} f(H \times K_2) &= \min\{f_{H\times K_2}(S, \operatorname{con})\colon\ S \text{ is a maximum convex set of } H \times K_2\} \\ &= \min\{\operatorname{acon}(H),\ \min\{f_H(S, \operatorname{con})\colon\ S \text{ is a maximum convex set in } H\}\} \\ &= \min\{\operatorname{acon}(H), f(H, \operatorname{con})\}. \end{aligned}$$

This completes the proof. □

We have seen examples of graphs H satisfying the properties (1)-(3) in Theorem 3.6 such that $\operatorname{acon}(H) = f(H, \operatorname{con})$ and $\operatorname{acon}(H) < f(H, \operatorname{con})$. Thus, in both cases, $f(H \times K_2, \operatorname{con}) = \operatorname{acon}(H)$. Of course, if $\operatorname{acon}(H) \leqslant f(H, \operatorname{con})$ for all graphs H satisfying (1)-(3), then $f(H \times K_2, \operatorname{con}) = \operatorname{acon}(H)$. However, we know of no example of a graph H satisfying (1)-(3) for which $f(H \times K_2, \operatorname{con}) \neq \operatorname{acon}(H)$. If such an example does exist, then $f(H \times K_2, \operatorname{con}) = \operatorname{acon}(H) - 1$ as we now show.

Theorem 3.7. *For every nontrivial connected graph H,*

$$\operatorname{acon}(H) \leqslant f(H, \operatorname{con}) + 1.$$

P r o o f. Let S be a maximum convex set in H such that $f(S,\mathrm{con}) = f(H,\mathrm{con})$, and let T be a minimum forcing subset for S. For $v \in T$, the set $T - \{v\}$ is not a forcing set for S. Hence there exists a maximum convex set S' distinct from S containing $T - \{v\}$. For $w \in S' - S$, let $T' = T \cup \{w\}$. Then T' belongs to no maximum convex set in H. Therefore,

$$\mathrm{acon}(H) \leqslant |T'| = |T| + 1 = f(H,\mathrm{con}) + 1,$$

completing the proof. □

As a consequence of the results presented in this section, we are able to state the forcing convexity numbers of $f(H \times K_2,\mathrm{con})$ of some well known graphs H.

Corollary 3.8. (a) *For $n \geqslant 3$, $f(K_n \times K_2,\mathrm{con}) = 1$.*
(b) *If T is a tree of order least 3, then $f(T \times K_2,\mathrm{con}) = 1$.*
(c) *For $n \geqslant 4$, $f(C_n \times K_2,\mathrm{con}) = 2$.*
(d) *For integers $k, n_1, n_2, \ldots, n_k \geqslant 2$ with $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$,*

$$f(K_{n_1,n_2,\ldots,n_k} \times K_2,\mathrm{con}) = \begin{cases} 1 & \text{if } n_k \geqslant 3 \\ 2 & \text{otherwise.} \end{cases}$$

(e) *For $n \geqslant 3$, $f(Q_n,\mathrm{con}) = 2$.*

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Author's address: Dept. of Mathematics and Statistics, Western Michigan University, Kalamazoo, MI 49008, U.S.A., e-mails: `chartrand@wmich.edu; ping.zhang@wmich.edu`.