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SECOND CENTRALIZERS OF PARTIAL TRANSFORMATIONS

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Abstract. Second centralizers of partial transformations on a finite set are determined. In particular, it is shown that the second centralizer of any partial transformation α consists of partial transformations that are locally powers of α.

Keywords: partial transformation, second centralizer

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1. Introduction

The semigroup PT_n of partial transformations on the set $X = \{1, \ldots, n\}$ consists of the functions whose domain and range are included in X, with composition as the semigroup operation. For $\alpha \in PT_n$, the sets

$$\begin{aligned} C(\alpha) &= \{\gamma \in PT_n \colon\ \alpha \circ \gamma = \gamma \circ \alpha\} \text{ and} \\ C^2(\alpha) &= \{\beta \in PT_n \colon\ \gamma \circ \beta = \beta \circ \gamma \text{ for each } \gamma \in C(\alpha)\} \end{aligned}$$

are subsemigroups of PT_n, called the (first) *centralizer* of α and the *second centralizer* of α, respectively. Note that $C^2(\alpha) \subseteq C(\alpha)$.

The purpose of this paper is to determine the second centralizers in PT_n. The second centralizers in the semigroup T_n of full transformations on the set X are described in [7].

Obviously, every power α^t $(t \geqslant 0)$ of $\alpha \in PT_n$ is an element of $C^2(\alpha)$. If α is not a nilpotent, then $\{\alpha^t \colon\ t \geqslant 0\}$ is a proper subset of $C^2(\alpha)$ since the zero (empty) transformation is in $C^2(\alpha) \setminus \{\alpha^t \colon\ t \geqslant 0\}$. Thus, in general, $C^2(\alpha)$ does not consist of just the powers of α. We show, however, that the elements of $C^2(\alpha)$ are locally powers of α.

More specifically, every $\alpha \in PT_n$ induces a partition $\{N, A_1, \ldots, A_m\}$ of the set $X = \{1, \ldots, n\}$. ($A_1, \ldots, A_m$ correspond to the weakly connected components containing a cycle in the digraph representation of α; N corresponds to the subgraph of the digraph representation obtained by removing all such components.)

Suppose that $\beta \in C^2(\alpha)$. We show that β restricted to N is equal to α^t restricted to N for some $t \geqslant 0$. Similarly, β restricted to A_i ($i = 1, \ldots, m$) is either 0 or is equal to α^{t_i} restricted to A_i for some $t_i \geqslant 0$. These necessary conditions are not sufficient for β to be in $C^2(\alpha)$. In addition, the exponents $t, t_1, \ldots, t_m$ must be related in a certain way. We prove that the "local powers" requirement together with these relations completely determine $C^2(\alpha)$.

2. First centralizers

This section introduces the terminology used throughout the paper and describes the first centralizers of partial transformations. Centralizers in PT_n have been studied in [3], [4], [5] and [6].

Let $\alpha \in PT_n$. The domain and range of α will be denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. If $\beta \in PT_n$ is such that $x\alpha = x\beta$ whenever $x \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta$, we define the *join* $\alpha\beta$ of α and β as the partial transformation with $\operatorname{dom}(\alpha\beta) = \operatorname{dom}\alpha \cup \operatorname{dom}\beta$ that coincides with α on $\operatorname{dom}\alpha$ and with β on $\operatorname{dom}\beta$. Note that the join $\alpha\beta$ (which, if defined, is simply the union of α and β) is distinct from the product (composition) $\alpha \circ \beta$.

For $k \geqslant 1$, let $i_1, i_2, \ldots, i_k$ be distinct elements of X such that $i_1\alpha = i_2$, $i_2\alpha = i_3, \ldots, i_{k-1}\alpha = i_k$. Then α restricted to the set $\{i_1, \ldots, i_{k-1}\}$ is called a *chain* in α of length k (or a k-chain in α) and denoted $(i_1 i_2 \ldots i_k]$. (Note that if $k = 1$, then $(i_1]$ is the zero transformation.) If, in addition, $i_k\alpha = i_1$ then α restricted to the set $\{i_1, i_2, \ldots, i_k\}$ is called a *circuit* in α of length k (or a k-circuit in α) and denoted $(i_1 i_2 \ldots i_k)$.

Let $\eta = (i_1 \ldots i_k]$ be a chain in α. The set $\{i_1, \ldots, i_k\}$ is called the *span* of η and denoted $\operatorname{span}\eta$. If $i_1 \notin \operatorname{ran}\alpha$ and $i_k \notin \operatorname{dom}\alpha$, we say that η is a *maximal* chain in α. Note that $(i_1]$ is a maximal chain in α if and only if $i_1 \notin \operatorname{dom}\alpha \cup \operatorname{ran}\alpha$.

If $\eta = (i_1 \ldots i_u x_r]$ is a chain in α and $\varrho = (x_0 \ldots x_{k-1})$ is a circuit in α ($u, k \geqslant 1$) such that $i_1 \notin \operatorname{ran}\alpha$ and $\{i_1, \ldots, i_u, x_r\} \cap \{x_0, \ldots, x_{k-1}\} = \{x_r\}$, we say that η is a *cilium* attached to ϱ at x_r. To distinguish cilia from maximal chains, we will use the right angle "$\rangle$" for the former and the right bracket "]" for the latter. If $\eta_1, \ldots, \eta_s$ are the cilia in α attached to ϱ, then the join $\lambda = \eta_1 \ldots \eta_s \varrho$ is called a *cell* in α. Note that an isolated circuit (with no cilia) also forms a cell.

Every partial transformation $\alpha \in PT_n$ is a join

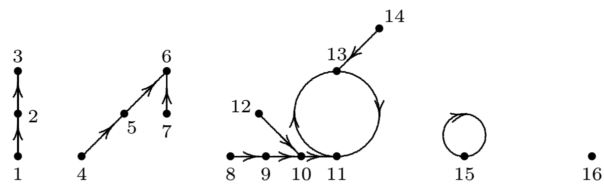
$$\eta_1 \ldots \eta_k \lambda_1 \ldots \lambda_m \tag{1}$$

of its maximal chains $\eta_1, \ldots, \eta_k$ and its cells $\lambda_1, \ldots, \lambda_m$. Join (1) is called the *chain-cell decomposition* of α.

If G is the digraph representation of α, then the maximal chains in α correspond to the simple maximal paths in G, and the cells in α correspond to the weakly connected components of G containing a cycle. For example, the transformation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 2 & 3 & - & 5 & 6 & - & 6 & 9 & 10 & 11 & 13 & 10 & 11 & 13 & 15 & - \end{pmatrix} \in PT_{16}$$

has the digraph representation



and the chain-cell decomposition

$$\alpha = \underbrace{(1\ 2\ 3]}_{\eta_1}\underbrace{(4\ 5\ 6]}_{\eta_2}\underbrace{(7\ 6]}_{\eta_3}\underbrace{(16]}_{\eta_4}\underbrace{(8\ 9\ 10\ 11\rangle(12\ 10\ 11\rangle(14\ 13\rangle(11\ 13)}_{\lambda_1}\underbrace{(15)}_{\lambda_2}.$$

If α is a full transformation on X, then there are no maximal chains in α and so $\alpha = \lambda_1 \ldots \lambda_m$ is a join of its cells. (For applications of the digraph representation of full transformations on X, see [2] and [1, 6.2].) If α is a permutation on X, then α is a join of its circuits.

Let $\alpha, \gamma \in PT_n$. Suppose that $\eta = (i_1 \ldots i_u]$ is a chain in α and $\varrho = (x_0 \ldots x_{k-1})$ is a circuit in α. If $\operatorname{dom}\gamma \cap \operatorname{span}\eta \neq \emptyset$, we say that γ *meets* η. Similarly, if $\operatorname{dom}\gamma \cap \operatorname{dom}\varrho \neq \emptyset$, we say that γ *meets* ϱ. If $\xi = (j_1 \ldots j_u]$ is a chain in α such that $i_1\gamma = j_1, \ldots, i_u\gamma = j_u$, we say that γ *maps* η *onto* ξ.

The first centralizers in PT_n are characterized in [4, Theorem 4] (also see [5, 58.8]).

Theorem 1. *Let $\alpha, \gamma \in PT_n$. Then $\gamma \in C(\alpha)$ if and only if for every maximal chain $\eta = (i_1 \ldots i_u]$ in α, every circuit $\varrho = (x_0 \ldots x_{k-1})$ in α, and every cilium $\xi = (j_1 \ldots j_v x_r\rangle$ in α attached to ϱ, the following conditions are satisfied:*

(1) *If γ meets η, then there is a maximal chain $\tau = (k_1 \ldots k_w]$ in α such that γ maps an initial segment $(i_1 \ldots i_p]$ of η ($p \leqslant u$) onto a terminal segment $(k_{w-p+1} \ldots k_w]$ of τ and γ does not meet $(i_{p+1} \ldots i_u]$;*

(2) *If γ meets ϱ, then there is a circuit $\delta = (y_0 \ldots y_{m-1})$ in α such that m divides k, γ maps the points $x_0, x_1, \ldots, x_{k-1}$ of $\operatorname{dom}\varrho$ to $y_s, y_s\alpha, \ldots, y_s\alpha^{k-1}$, and γ maps*

the points $j_1, j_2, \ldots, j_v, x_r$ *of* $\operatorname{span}\xi$ *to* $z, z\alpha, \ldots, z\alpha^{v-1}, z\alpha^v$, *where* z *is on* δ *or some cilium attached to* δ;

(3) *If* γ *does not meet* ϱ *but it meets* ξ, *then there is a maximal chain* $\tau = (k_1 \ldots k_w]$ *in* α *such that* γ *maps an initial segment* $(j_1 \ldots j_p]$ *of* ξ $(p \leqslant v)$ *onto a terminal segment* $(k_{w-p+1} \ldots k_w]$ *of* τ *and* γ *does not meet* $(j_{p+1} \ldots j_v]$.

3. Second centralizers

Let $\alpha \in PT_n$ and let λ be a cell in α. We define the *radius* of λ, written $r(\lambda)$, as the largest integer u such that $(i_1 \ldots i_u x\rangle$ is a cilium in λ. If λ has no cilia, we define $r(\lambda)$ to be 0. Let $\eta_1 \ldots \eta_k$ be the join of all maximal chains in α and let $N = \operatorname{span}\eta_1 \cup \ldots \cup \operatorname{span}\eta_k$. We define the *diameter* of N, written $d(N)$, as the largest integer u such that $(i_1 \ldots i_u]$ is a maximal chain in α. If $N = \emptyset$ (that is, if α has no maximal chains), we define $d(N)$ to be 0.

For example, for $\alpha = \underbrace{(1\ 2]}_{\eta_1}\underbrace{(3]}_{\eta_2}\underbrace{(6\ 7\ 4\rangle(8\ 4\rangle(4\ 5)}_{\lambda_1}\underbrace{(10\ 9\rangle(9)}_{\lambda_2}\underbrace{(11\ 12)}_{\lambda_3}$, we have $N = \{1, 2, 3\}$, $d(N) = 2$, $r(\lambda_1) = 2$, $r(\lambda_2) = 1$, and $r(\lambda_3) = 0$.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of nonnegative integers. We introduce an element $-\infty \notin \mathbb{N}$ and agree that for every $a \in \mathbb{N}$, $-\infty < a$, and that for every $\beta \in PT_n$, $\beta^{-\infty} = 0$, where 0 is the zero (empty) transformation. For $\beta \in PT_n$ and a subset A of X, $\beta\big|A$ will denote the restriction of β to A. Finally, the length of a circuit ϱ will be denoted by $\ell(\varrho)$.

The following theorem determines the second centralizers of partial transformations.

Theorem 2. *Let* $\alpha, \beta \in PT_n$, *let* $\alpha = \eta_1 \ldots \eta_k \lambda_1 \ldots \lambda_m$ *be the chain-cell decomposition of* α, *let* $N = \operatorname{span}\eta_1 \cup \ldots \cup \operatorname{span}\eta_k$, *and let* ϱ_i *be the circuit in the cell* λ_i. *Then* $\beta \in C^2(\alpha)$ *if and only if there are* $t \in \mathbb{N}$ *and* $t_1, \ldots, t_m \in \mathbb{N} \cup \{-\infty\}$ *such that for all* $i, j \in \{1, \ldots, m\}$:

(1) $\beta\big|N = \alpha^t\big|N$;

(2) $\beta\big|\operatorname{dom}\lambda_i = \alpha^{ti}\big|\operatorname{dom}\lambda_i$;

(3) *If either* $t < \min\{d(N), r(\lambda_i)\}$ *or* $0 \leqslant t_i < \min\{d(N), r(\lambda_i)\}$, *then* $t_i = t$;

(4) *If* $\ell(\varrho_i)$ *divides* $\ell(\varrho_j)$, *then:*

  a) *If* $t_i \geqslant 0$ *and* $t_j \geqslant 0$, *then* $t_i \equiv t_j \pmod{\ell(\varrho_i)}$;

  b) *If either* t_i *or* t_j *is less than* $\min\{r(\lambda_i), r(\lambda_j)\}$, *then* $t_i = t_j$.

Note that (4b) and the convention $-\infty < a$ for every $a \in \mathbb{N}$ imply that if $\ell(\varrho_i)$ divides $\ell(\varrho_j)$, then $t_i = -\infty \iff t_j = -\infty$. To illustrate Theorem 2, we consider the following transformations in PT_{12}:

$$\begin{aligned}
\alpha &= \eta_1\lambda_1\lambda_2 = (1\ \ 2\ \ 3](6\ \ 7\ \ 4\rangle(4\ \ 5)(12\ \ 8\rangle(8\ \ 9\ \ 10\ \ 11),\\
\beta_1 &= (6\ \ 2](7\ \ 3],\\
\beta_2 &= (1\ \ 2\ \ 3],\\
\beta_3 &= (6\ \ 4\rangle(4)(7\ \ 5\rangle(5)(12\ \ 8\rangle(8\ \ 9\ \ 10\ \ 11),\\
\beta_4 &= (1\ \ 2\ \ 3](6\ \ 7\ \ 4\rangle(4\ \ 5),\\
\beta_5 &= (6\ \ 4\rangle(4)(7\ \ 5\rangle(5)(8)(9)(10)(11)(12),\\
\beta_6 &= (1\ \ 3](6\ \ 7\ \ 4\rangle(4\ \ 5)(12\ \ 10\rangle(8\ \ 11\ \ 10\ \ 9),\\
\beta_7 &= (1\ \ 3](6\ \ 5\rangle(7\ \ 4\rangle(4\ \ 5)(12\ \ 8\rangle(8\ \ 9\ \ 10\ \ 11).
\end{aligned}$$

By Theorem 1, each β_i is in $C(\alpha)$. Note that $N = \{1,2,3\}$, $d(N) = 3$, $\ell(\varrho_1) = 2$, $\ell(\varrho_2) = 4$, $r(\lambda_1) = 2$, and $r(\lambda_2) = 1$. We apply Theorem 2 to each β_i.

(1) $\beta_1 \notin C^2(\alpha)$ since β_1 restricted to $\operatorname{dom}\lambda_1$ is not equal to any power of α restricted to $\operatorname{dom}\lambda_1$.

(2) $\beta_2\big|N = \alpha^1\big|N$, $\beta_2\big|\operatorname{dom}\lambda_1 = \alpha^{-\infty}\big|\operatorname{dom}\lambda_1$, and $\beta_2\big|\operatorname{dom}\lambda_2 = \alpha^{-\infty}\big|\operatorname{dom}\lambda_2$, but $\beta_2 \notin C^2(\alpha)$ since $1 < \min\{d(N), r(\lambda_1)\}$ and $1 \neq -\infty$.

(3) $\beta_3\big|N = \alpha^3\big|N$, $\beta_3\big|\operatorname{dom}\lambda_1 = \alpha^2\big|\operatorname{dom}\lambda_1$, and $\beta_3\big|\operatorname{dom}\lambda_2 = \alpha^1\big|\operatorname{dom}\lambda_2$, but $\beta_3 \notin C^2(\alpha)$ since $2 \not\equiv 1 \pmod{\ell(\varrho_1)}$.

(4) $\beta_4\big|N = \alpha^1\big|N$, $\beta_4\big|\operatorname{dom}\lambda_1 = \alpha^1\big|\operatorname{dom}\lambda_1$, and $\beta_4\big|\operatorname{dom}\lambda_2 = \alpha^{-\infty}\big|\operatorname{dom}\lambda_2$, but $\beta_4 \notin C^2(\alpha)$ since $-\infty < \min\{r(\lambda_1), r(\lambda_2)\}$ and $1 \neq -\infty$.

(5) $\beta_5\big|N = \alpha^3\big|N$, $\beta_5\big|\operatorname{dom}\lambda_1 = \alpha^2\big|\operatorname{dom}\lambda_1$, $\beta_5\big|\operatorname{dom}\lambda_2 = \alpha^0\big|\operatorname{dom}\lambda_2$, and $2 \equiv 0 \pmod{\ell(\varrho_1)}$, but $\beta_5 \notin C^2(\alpha)$ since $0 < \min\{r(\lambda_1), r(\lambda_2)\}$ and $2 \neq 0$.

(6) $\beta_6\big|N = \alpha^2\big|N$, $\beta_6\big|\operatorname{dom}\lambda_1 = \alpha^1\big|\operatorname{dom}\lambda_1$, $\beta_6\big|\operatorname{dom}\lambda_2 = \alpha^3\big|\operatorname{dom}\lambda_2$, and $3 \equiv 1 \pmod{\ell(\varrho_1)}$, but $\beta_6 \notin C^2(\alpha)$ since $1 < \min\{d(N), r(\lambda_1)\}$ and $1 \neq 3$.

(7) $\beta_7\big|N = \alpha^2\big|N$, $\beta_7\big|\operatorname{dom}\lambda_1 = \alpha^3\big|\operatorname{dom}\lambda_1$, $\beta_7\big|\operatorname{dom}\lambda_2 = \alpha^1\big|\operatorname{dom}\lambda_2$, and (3) and (4) of Theorem 2 are satisfied, so $\beta_7 \in C^2(\alpha)$.

The remainder of the paper will be devoted to proving Theorem 2. It is convenient to lay out the proof of the "only if" part of the theorem as a series of lemmas.

The following two lemmas show that for $\alpha \in PT_n$ and $\beta \in C^2(\alpha)$, β restricted to N is equal to some power of α restricted to N. (In other words, such a β satisfies (1) of Theorem 2.)

Lemma 3. *Let $\alpha, \beta \in PT_n$ be such that $\beta \in C^2(\alpha)$, and let $\eta = (i_1 \ldots i_u]$ be a maximal chain in α. Then there is $t \in \{0, \ldots, u\}$ such that $\beta\big|\operatorname{span}\eta = \alpha^t\big|\operatorname{span}\eta$.*

Proof. If $\operatorname{dom}\beta \cap \operatorname{span}\eta = \emptyset$, then $\beta\big|\operatorname{span}\eta = \alpha^u\big|\operatorname{span}\eta$. Otherwise, by Theorem 1, $i_1 \in \operatorname{dom}\beta$ and one of the following two cases holds.

Case 1. $i_1\beta = i_p$ for some $p \in \{1, \ldots, u\}$.

Then $\beta\big|\operatorname{span}\eta = \alpha^{p-1}\big|\operatorname{span}\eta$ by Theorem 1.

Case 2. There is a maximal chain $\xi = (j_1 \dots j_u]$ in α such that for some $p \in \{1, \dots, v\}$, $j_p \notin \{i_1, \dots, i_u\}$ and $i_1\beta = j_p$.

We will construct $\gamma \in C(\alpha)$ such that $i_1 \notin \operatorname{dom}\gamma$ and $j_p \in \operatorname{dom}\gamma$. Set $\operatorname{dom}\gamma = \{x \in \operatorname{dom}\alpha\colon x\alpha^q = j_p \text{ for some } q \geqslant 0\}$. Define the values of γ so that for every maximal chain $\mu = (m_1 \dots m_d j_p \dots j_v]$ $(d \geqslant 0)$ in α whose span contains j_p, γ maps the initial segment $(m_1 \dots m_d j_p]$ of μ onto a terminal segment of μ. By Theorem 1 and the construction of γ, we have $\gamma \in C(\alpha)$, $i_1 \notin \operatorname{dom}\gamma$ (since $j_p \notin \{i_1, \dots, i_u\}$), and $j_p \in \operatorname{dom}\gamma$. Thus $i_1(\beta \circ \gamma) = j_p\gamma$ is defined and $i_1(\gamma \circ \beta)$ is undefined. It follows that $\gamma \notin C(\beta)$, which is a contradiction. □

Recall that for a circuit ϱ, $\ell(\varrho)$ denotes the length of ϱ. Similarly, for a chain η, $\ell(\eta)$ will denote the length of η.

Lemma 4. *Let $\alpha, \beta \in PT_n$ be such that $\beta \in C^2(\alpha)$, and let $\eta = (i_1 \dots i_u]$ and $\xi = (j_1 \dots j_v]$ be maximal chains in α. Suppose that $t \in \{0, \dots, u\}$ and $w \in \{0, \dots, v\}$ are integers such that $\beta\big|\operatorname{span}\eta = \alpha^t\big|\operatorname{span}\eta$ and $\beta\big|\operatorname{span}\xi = \alpha^w\big|\operatorname{span}\xi$. If $t > w$, then $\beta\big|\operatorname{span}\xi = \alpha^t\big|\operatorname{span}\xi$.*

Proof. Suppose $t > w$. Proceeding by induction on $\ell(\eta) + \ell(\xi)$, we assume that the lemma is true for all maximal chains η' and ξ' in α with $\ell(\eta') + \ell(\xi') > \ell(\eta) + \ell(\xi)$. We consider three cases.

Case 1. $w = 0$.

Then $i_u\beta = i_u\alpha^t$ is undefined (since $t > w = 0$) and $j_1\beta = j_1\alpha^0 = j_1$. Define $\gamma \in PT_n$ by: $\operatorname{dom}\gamma = \{j_1\}$ and $j_1\gamma = i_u$. By Theorem 1, $\gamma \in C(\alpha)$. Since $j_1(\beta \circ \gamma) = j_1\gamma = i_u$ and $j_1(\gamma \circ \beta) = i_u\beta$ is undefined, $\gamma \notin C(\beta)$, which is a contradiction.

Case 2. $w = v$.

Then $\beta\big|\operatorname{span}\xi = 0\big|\operatorname{span}\xi = \alpha^t\big|\operatorname{span}\xi$ (since $t > w$).

Case 3. $1 \leqslant w < v$.

Then $j_1\beta = j_{w+1}$. Let $m = \min\{u, v\}$. Since $w < v$ and $w < t \leqslant u$, $w + 1 \leqslant m$. Let $\tau = (k_1 \dots k_b j_m \dots j_v]$ $(b \geqslant 0)$ be a longest maximal chain in α whose span contains j_m. We consider two cases.

Case 3.1. $b \leqslant u - 1$.

Then we can construct $\gamma \in C(\alpha)$ that maps the initial segment $(j_1 \dots j_m]$ of ξ onto a terminal segment of η. Set $\operatorname{dom}\gamma = \{x \in \operatorname{dom}\alpha\colon x\alpha^q = j_m \text{ for some } q \geqslant 0\}$. Let $\mu = (m_1 \dots m_d j_m \dots j_v]$ $(d \geqslant 0)$ be any maximal chain in α whose span contains j_m. Since $\ell(\tau) \geqslant \ell(\mu)$ and $b \leqslant u - 1$, we have $u - d \geqslant u - b \geqslant 1$ and so $u \geqslant d + 1$. Thus we can define γ so that it maps $(m_1 \dots m_d j_m]$ onto a terminal segment of η. In particular, γ maps the initial segment $(j_1 \dots j_m]$ of ξ onto a terminal segment of η, say $(i_r \dots i_u]$.

By Theorem 1 and the construction of γ, we have $\gamma \in C(\alpha)$, $j_1\gamma = i_r$, and $j_{w+1}\gamma = i_{r+w}$. Since $i_r\beta = i_r\alpha^t$, either $i_r\beta$ is undefined or $i_r\beta = i_{r+t}$. Thus $j_1(\beta\circ\gamma) = j_{w+1}\gamma = i_{r+w}$ and either $j_1(\gamma\circ\beta) = i_r\beta$ is undefined or $j_1(\gamma\circ\beta) = i_r\beta = i_{r+t}$. In either case, since $t > w$, it follows that $\gamma \notin C(\beta)$, which is a contradiction.

Case 3.2. $b \geqslant u$.

Then $\ell(\tau) = b + v - m + 1 \geqslant u + v - m + 1 \geqslant m + v - m + 1 = v + 1 > \ell(\xi)$ and $\ell(\tau) = b + v - m + 1 \geqslant u + v - m + 1 \geqslant u + m - m + 1 = u + 1 > \ell(\eta)$. By Lemma 3, $\beta\big|\operatorname{span}\tau = \alpha^p\big|\operatorname{span}\tau$ for some $p \in \{0,\ldots,\ell(\tau)\}$. Suppose $p > w$. Then, by the inductive hypothesis applied to τ and ξ, $\beta\big|\operatorname{span}\xi = \alpha^p\big|\operatorname{span}\xi$. It follows that $j_1\beta = j_1\alpha^p \neq j_{w+1} = j_1\beta$, which is a contradiction. Thus $p \leqslant w$. Then $t > p$ and so, by the inductive hypothesis applied to η and τ, $\beta\big|\operatorname{span}\tau = \alpha^t\big|\operatorname{span}\tau$. Note that, since $\ell(\tau) > \ell(\eta)$ and $t \in \{0,\ldots,\ell(\eta)\}$, we also have $t \in \{0,\ldots,\ell(\tau)\}$. Now repeat the argument used in the case $p > w$ above (with p replaced by t) to obtain a contradiction. This concludes the proof. □

The next three lemmas show that for $\alpha \in PT_n$ and $\beta \in C^2(\alpha)$, β restricted to the domain of a cell λ_i is equal to some power (possibly $-\infty$) of α restricted to that domain. (In other words, such a β satisfies (2) of Theorem 2.)

Lemma 5. *Let $\alpha,\beta \in PT_n$ be such that $\beta \in C^2(\alpha)$, and let $\varrho = (x_0\ldots x_{k-1})$ be a circuit in α. If $\operatorname{dom}\beta\cap\operatorname{dom}\varrho \neq \emptyset$, then there is $t \in \{0,\ldots,k-1\}$ such that $\beta\big|\operatorname{dom}\varrho = \alpha^t\big|\operatorname{dom}\varrho$.*

Proof. By Theorem 1, $x_0 \in \operatorname{dom}\beta$ and one of the following two cases holds.

Case 1. $x_0\beta = x_t$ for some $t \in \{0,\ldots,k-1\}$.

Then $\beta\big|\operatorname{dom}\varrho = \alpha^t\big|\operatorname{dom}\varrho$ by Theorem 1.

Case 2. There is a circuit $\delta = (y_0\ldots y_{m-1})$ in α such that $\delta \neq \varrho$, m divides k, and $x_0\beta = y_p$ for some $p \in \{0,\ldots,m-1\}$.

Let λ be the cell in α that has δ as the circuit. Define $\gamma \in PT_n$ by $\operatorname{dom}\gamma = \operatorname{dom}\lambda$ and $y\gamma = y$ for every $y \in \operatorname{dom}\lambda$. By Theorem 1 and the construction of γ, we have $\gamma \in C(\alpha)$, $x_0 \notin \operatorname{dom}\gamma$, and $y_p\gamma = y_p$. Since $x_0(\beta\circ\gamma) = y_p\gamma = y_p$ and $x_0(\gamma\circ\beta)$ is undefined, $\gamma \notin C(\beta)$, which is a contradiction. □

Lemma 6. *Let $\alpha,\beta \in PT_n$ be such that $\beta \in C^2(\alpha)$, and let $\eta = (i_1\ldots i_u x_0\rangle$ be a cilium in α attached to a circuit $\varrho = (x_0\ldots x_{k-1})$. If $\operatorname{dom}\beta\cap\operatorname{span}\eta \neq \emptyset$, then there is $t \in \{0,\ldots,u+k-1\}$ such that $\beta\big|\operatorname{span}\eta = \alpha^t\big|\operatorname{span}\eta$.*

Proof. Let λ be the cell in α that has ϱ as the circuit. By Theorem 1 and Lemma 5, $i_1 \in \operatorname{dom}\beta$ and one of the following four cases holds.

Case 1. $i_1\beta = i_p$ for some $p \in \{1,\ldots,u\}$.

Then $\beta\big|\operatorname{span}\eta = \alpha^{p-1}\big|\operatorname{span}\eta$ by Theorem 1.

Case 2. $i_1\beta = x_p$ for some $p \in \{0, \ldots, k-1\}$.

Then $\beta\big| \operatorname{span}\eta = \alpha^{u+p}\big| \operatorname{span}\eta$ by Theorem 1.

Case 3. There is a cilium $\xi = (j_1 \ldots j_v x_s\rangle$ in λ such that for some $p \in \{1, \ldots, v\}$, $i_1\beta = j_p$ and $j_p \notin \{i_1, \ldots, i_u\}$.

We consider two cases.

Case 3.1. $s \neq 0$, i.e., η and ξ meet ϱ at different points.

We will construct $\gamma \in C(\alpha)$ such that $i_1\gamma = i_1$ and $j_p\gamma \neq j_p$. Set $\operatorname{dom}\gamma = \operatorname{dom}\lambda$. Let $\mu = (m_1 \ldots m_d x_s\rangle$ be any cilium in λ attached to ϱ at x_s and let $x_h \in \operatorname{dom}\varrho$ be such that $x_h\alpha^d = x_s$. Define γ so that it maps the points $m_1, m_2, \ldots, m_d, x_s$ of $\operatorname{span}\mu$ to $x_h, x_h\alpha, \ldots, x_h\alpha^{d-1}, x_h\alpha^d = x_s$. If $y \in \operatorname{dom}\lambda$ is not in the span of any cilium attached to ϱ at x_s, define $y\gamma = y$. By Theorem 1 and the construction of γ, we have $\gamma \in C(\alpha)$, $i_1\gamma = i_1$ (since η is not attached to ϱ at x_s), and $j_p\gamma \neq j_p$ (since ξ is attached to ϱ at x_s and so $j_p\gamma \in \operatorname{dom}\varrho$). Since $i_1(\beta \circ \gamma) = j_p\gamma \neq j_p$ and $i_1(\gamma \circ \beta) = i_1\beta = j_p$, $\gamma \notin C(\beta)$, which is a contradiction.

Case 3.2. $s = 0$, i.e., η and ξ are attached to ϱ at the same point.

Since both η and ξ meet ϱ at x_0 and $j_p \notin \{1, \ldots, u\}$, we have $\eta = (i_1 \ldots i_q z \ldots\rangle$, $\xi = (j_1 \ldots j_p \ldots j_r z \ldots\rangle$ ($q \geqslant 1, r \geqslant p$), and $\{i_1, \ldots, i_q\} \cap \{j_1, \ldots, j_r\} = \emptyset$. (Note that z may be equal to x_0.) Let $\tau = (k_1 \ldots k_b j_r z \ldots\rangle$ ($b \geqslant 0$) be a longest cilium in λ whose span contains j_r. If $j_p \in \operatorname{span}\tau$, we may assume that $\xi = \tau$. We consider three cases.

Case 3.2.1. $\tau \neq \xi$ (which implies $j_p \notin \operatorname{span}\tau$).

We will construct $\gamma \in C(\alpha)$ such that $i_1\gamma = i_1$ and $j_p\gamma \neq j_p$. Set $\operatorname{dom}\gamma = \operatorname{dom}\lambda$. Let $\mu = (m_1 \ldots m_d x_0\rangle$ be any cilium in λ whose span contains j_r. Since $\ell(\tau) \geqslant \ell(\mu)$, we can define γ so that it maps μ onto a terminal segment of τ. If $y \in \operatorname{dom}\lambda$ is not in the span of any cilium in λ whose span contains j_r, define $y\gamma = y$. By Theorem 1 and the construction of γ, we have $\gamma \in C(\alpha)$, $i_1\gamma = i_1$ (since $j_r \notin \operatorname{span}\eta$), and $j_p\gamma \neq j_p$ (since $j_p\gamma \in \operatorname{span}\tau$ and $j_p \notin \operatorname{span}\tau$), which leads to a contradiction as in Case 3.1.

Case 3.2.2. $\tau = \xi$ and $\ell(\eta) \geqslant \ell(\xi)$.

Again, we will construct $\gamma \in C(\alpha)$ such that $i_1\gamma = i_1$ and $j_p\gamma \neq j_p$. Let $\mu = (m_1 \ldots m_d x_0\rangle$ be any cilium in λ whose span contains j_r. Since $\ell(\eta) \geqslant \ell(\xi) \geqslant \ell(\mu)$, we can define γ so that it maps μ onto a terminal segment of η. If $y \in \operatorname{dom}\lambda$ is not in the span of any cilium in λ whose span contains j_r, define $y\gamma = y$. By Theorem 1 and the construction of γ, we have $\gamma \in C(\alpha)$, $i_1\gamma = i_1$ (since $j_r \notin \operatorname{span}\eta$), and $j_p\gamma \neq j_p$ (since $j_p\gamma \in \operatorname{span}\eta$ and $j_p \notin \operatorname{span}\eta$), which leads to a contradiction as in Case 3.1.

Case 3.2.3. $\tau = \xi$ and $\ell(\eta) < \ell(\xi)$.

We will construct $\gamma \in C(\alpha)$ such that $j_1\gamma = i_1$ and $j_p \notin \operatorname{ran}\gamma$. Set $\operatorname{dom}\gamma = \operatorname{dom}\lambda$. Let $a \in \{0, \ldots, k-1\}$ be such that $a \equiv v - u \pmod{k}$. Let $\mu = (m_1 \ldots m_d x_0\rangle$ be any cilium in λ whose span contains j_r and let $c = v - d + 1$. Since $\ell(\xi) \geqslant \ell(\mu)$, $c \geqslant 1$. If $c \leqslant u$, define γ so that it maps the points $m_1, m_2, \ldots, m_d, x_0$ of $\operatorname{dom}\mu$ to

$i_c, i_c\alpha, \ldots, i_c\alpha^{d-1}, i_c\alpha^d$. Note that $i_c\alpha^d = x_a$. If $c > u$, select $x_h \in \operatorname{dom}\varrho$ so that $x_h\alpha^d = x_a$ and define γ so that it maps the points $m_1, m_2, \ldots, m_d, x_0$ of $\operatorname{dom}\mu$ to $x_h, x_h\alpha, \ldots, x_h\alpha^{d-1}, x_h\alpha^d = x_a$. Note that if $\mu = \xi$, then $c = v-d+1 = v-v+1 = 1$ and $j_1\gamma = m_1\gamma = i_c = i_1$. If $y \in \operatorname{dom}\lambda$ is not in the span of any cilium whose span contains j_r, we define $y\gamma = y\alpha^a$.

By Theorem 1 and the construction of γ, we have $\gamma \in C(\alpha)$, $j_1\gamma = i_1$, and $j_p \notin \operatorname{ran}\gamma$. (Indeed, let $y \in \operatorname{dom}\lambda$. If y is in the span of a cilium whose span contains j_r, then $y\gamma$ is in the set $\{i_1, \ldots, i_u\} \cup \operatorname{dom}\varrho$. Thus $y\gamma \neq j_p$ since j_p is not in that set. If y is not in the span of any such cilium, then $y\gamma = y\alpha^a \neq j_p$ since otherwise we would have $y\alpha^{a+r-p} = j_r$, which cannot happen if y is not in the span of a cilium whose span contains j_r. Hence $j_p \notin \operatorname{ran}\gamma$.) Since $\operatorname{ran}(\beta \circ \gamma) \subseteq \operatorname{ran}\gamma$, $j_p \notin \operatorname{ran}(\beta \circ \gamma)$. Since $j_1(\gamma \circ \beta) = i_1\beta = j_p$, $j_p \in \operatorname{ran}(\gamma \circ \beta)$. It follows that $\gamma \notin C(\beta)$, which is a contradiction.

Case 4. There is a maximal chain $\xi = (j_1 \ldots j_v]$ in α such that $i_1\beta = j_p$ for some $p \in \{1, \ldots, v\}$.

Define $\gamma \in PT_n$ by: $\operatorname{dom}\gamma$ is the union of spans of all maximal chains in α, and $y\gamma = y$ for all $y \in \operatorname{dom}\gamma$. By Theorem 1 and the construction of γ, we have $\gamma \in C(\alpha)$, $j_p \in \operatorname{dom}\gamma$, and $i_1 \notin \operatorname{dom}\gamma$. Since $i_1(\beta \circ \gamma) = j_p\gamma$ is defined and $i_1(\gamma \circ \beta) = (i_1\gamma)\beta$ is undefined, $\gamma \notin C(\beta)$, which is a contradiction. □

Lemma 7. *Let $\alpha, \beta \in PT_n$ be such that $\beta \in C^2(\alpha)$, and let $\eta = (i_1 \ldots i_u x_0\rangle$ and $\xi = (j_1 \ldots j_v x_s\rangle$ be cilia in α attached to a circuit $\varrho = (x_0 \ldots x_{k-1})$ such that $\beta\big|\operatorname{dom}\varrho = \alpha^e\big|\operatorname{dom}\varrho$ for some $e \in \{0, \ldots, k-1\}$. Suppose that $t \in \{0, \ldots, u+k-1\}$ and $w \in \{0, \ldots, v+k-1\}$ are integers such that $\beta\big|\operatorname{span}\eta = \alpha^t\big|\operatorname{span}\eta$ and $\beta\big|\operatorname{span}\xi = \alpha^w\big|\operatorname{span}\xi$. If $t > w$, then $\beta\big|\operatorname{span}\xi = \alpha^t\big|\operatorname{span}\xi$.*

Proof. Suppose $t > w$ and let λ be the cell in α that has ϱ as the circuit. Proceeding by induction on $\ell(\eta) + \ell(\xi)$, we assume that the lemma is true for all cilia η' and ξ' in λ with $\ell(\eta') + \ell(\xi') > \ell(\eta) + \ell(\xi)$.

Since $x_0\alpha^t = x_0\alpha^e$ and $x_s\alpha^w = x_s\alpha^e$, we have $t \equiv e \pmod k$ and $w \equiv e \pmod k$. Thus $t \equiv w \pmod k$ and so, since $t > w$, $t = w + lk$ for some $l \geqslant 1$. We consider three cases.

Case 1. $w = 0$.

Then $i_u\beta = i_u\alpha^{lk} = x_{k-1}$ and $j_1\beta = j_1\alpha^0 = j_1$. We will construct $\gamma \in C(\alpha)$ such that $j_1\gamma = i_u$. Set $\operatorname{dom}\gamma = \operatorname{dom}\lambda$. Select $q \in \{0, \ldots, k-1\}$ so that $q \equiv v - 1 \pmod k$ and define γ so that it maps the points $j_1, j_2, \ldots, j_v, x_s$ of $\operatorname{span}\xi$ to $i_u, i_u\alpha = x_0, \ldots, i_u\alpha^{v-1}, i_u\alpha^v = x_q$. Let $a \in \{0, \ldots, k-1\}$ be such that $a \equiv q - s \pmod k$. For any cilium $\mu = (m_1 \ldots m_d x_c\rangle$ in λ with $\mu \neq \xi$, select $x_h \in \operatorname{dom}\varrho$ so that $x_h\alpha^d = x_{a+c}$ and define γ so that it maps the points $m_1, m_2, \ldots, m_d, x_c$ of $\operatorname{span}\mu$ to $x_h, x_h\alpha, \ldots, x_h\alpha^{d-1}, x_h\alpha^d = x_{a+c}$. By Theorem 1 and the construction of γ, we have $\gamma \in C(\alpha)$ and $j_1\gamma = i_u$. Since $j_1(\beta \circ \gamma) = j_1\gamma = i_u$ and $j_1(\gamma \circ \beta) = i_u\beta = x_{k-1}$, $\gamma \notin C(\beta)$, which is a contradiction.

Case 2. $w \geqslant v$.

Then for each $p \in \{1, \ldots, v\}$, $j_p\beta = j_p\alpha^w = x_q$ for some $q \in \{0, \ldots, k-1\}$. Thus $j_p\alpha^t = j_p\alpha^{w+lk} = (j_p\alpha^w)\alpha^{lk} = x_q\alpha^{lk} = x_q = j_p\beta$. Similarly, $x_s\alpha^t = x_s\beta$ and so $\beta\big|\operatorname{span}\xi = \alpha^t\big|\operatorname{span}\xi$.

Case 3. $1 \leqslant w < v$.

Then $j_1\beta = j_1\alpha^w = j_{w+1}$. Let $m = \min\{u, v\}$. Since $w < v$ and $w = t - lk \leqslant u + k - 1 - lk = u - (l-1)k - 1 < u$, $w + 1 \leqslant m$. Let $\tau = (k_1 \ldots k_b j_m \ldots j_v x_s\rangle$ $(b \geqslant 0)$ be a longest cilium in λ whose span contains j_m. We consider two cases.

Case 3.1. $b \leqslant u - 1$.

Then we can construct $\gamma \in C(\alpha)$ that maps the initial segment $(j_1 \ldots j_m]$ of ξ onto a terminal segment of $(i_1 \ldots i_u]$. Set $\operatorname{dom}\gamma = \operatorname{dom}\lambda$. Select $q \in \{0, \ldots, k-1\}$ such that $q \equiv v - m \pmod{k}$. Let $\mu = (m_1 \ldots m_d j_m \ldots j_v x_s\rangle$ $(d \geqslant 0)$ be any cilium in λ whose span contains j_m. Since $\ell(\tau) \geqslant \ell(\mu)$ and $b \leqslant u - 1$, we have $u \geqslant b + 1 \geqslant d + 1$. Thus we can define γ so that it maps the initial segment $(m_1 \ldots m_d j_m]$ of μ onto a terminal segment of $(i_1 \ldots i_u]$ and the remaining points of $\operatorname{span}\mu$, $j_{m+1}, j_{m+2}, \ldots, j_v$, x_s, to $x_0, x_0\alpha, \ldots, x_0\alpha^{v-m-1}$, $x_0\alpha^{v-m} = x_q$. In particular, γ maps the initial segment $(j_1 \ldots j_m]$ of ξ onto a terminal segment of $(i_1 \ldots i_u]$, say $(i_r \ldots i_u]$. Let $\mu = (m_1 \ldots m_d x_c\rangle$ be any cilium in λ whose span does not contain j_m. Let $a \in \{0, \ldots, k-1\}$ be such that $a \equiv q - s \pmod{k}$. Select $x_h \in \operatorname{dom}\varrho$ so that $x_h\alpha^d = x_{c+a}$ and define γ so that it maps the points $m_1, m_2, \ldots, m_d, x_c$ of $\operatorname{span}\mu$ to $x_h, x_h\alpha, \ldots, x_h\alpha^{d-1}, x_h\alpha^d = x_{c+a}$.

By Theorem 1 and the construction of γ, we have $\gamma \in C(\alpha)$, $j_1\gamma = i_r$, and $j_{w+1}\gamma = i_{r+w}$. Since $j_1(\beta \circ \gamma) = j_{w+1}\gamma = i_{r+w}$ and $j_1(\gamma \circ \beta) = i_r\beta = i_r\alpha^t \neq i_{r+w}$ (since $t > w$), $\gamma \notin C(\beta)$, which is a contradiction.

Case 3.2. $b \geqslant u$.

Then $\ell(\tau) = b + v - m + 2 \geqslant u + v - m + 2 \geqslant m + v - m + 2 = v + 2 > \ell(\xi)$ and $\ell(\tau) = b + v - m + 2 \geqslant u + v - m + 2 \geqslant u + m - m + 2 = u + 2 > \ell(\eta)$. By Lemma 6, $\beta\big|\operatorname{span}\tau = \alpha^p\big|\operatorname{span}\tau$ for some $p \in \{0, \ldots, \ell(\tau) + k - 2\}$. Suppose $p > w$. Then, by the inductive hypothesis applied to τ and ξ, $\beta\big|\operatorname{span}\xi = \alpha^p\big|\operatorname{span}\xi$. It follows that $j_1\beta = j_1\alpha^p \neq j_{w+1} = j_1\beta$, which is a contradiction. Thus $p \leqslant w$. Then $t > p$ and so, by the inductive hypothesis applied to η and τ, $\beta\big|\operatorname{span}\tau = \alpha^t\big|\operatorname{span}\tau$. Note that, since $\ell(\tau) > \ell(\eta)$ and $t \in \{0, \ldots, \ell(\eta) + k - 2\}$, we also have $t \in \{0, \ldots, \ell(\tau) + k - 2\}$. Now repeat the argument used in the case $p > w$ above (with p replaced by t) to obtain a contradiction. This concludes the proof. □

Lemmas 3–7 imply that if $\alpha \in PT_n$ and $\beta \in C^2(\alpha)$, then β satisfies (1) and (2) of Theorem 2, that is, $\beta\big|N = \alpha^t\big|N$ and $\beta\big|\operatorname{dom}\lambda_i = \alpha^{t_i}\big|\operatorname{dom}\lambda_i$ for some $t \in \mathbb{N}$ and

$t_i \in \mathbb{N} \cup \{-\infty\}$ $(i = 1, \ldots, m)$. The next two lemmas show that the exponents t and t_i satisfy (3) of Theorem 2.

Lemma 8. *Let $\alpha, \beta \in PT_n$ be such that $\beta \in C^2(\alpha)$, let $\eta = (i_1 \ldots i_u]$ be a maximal chain in α, let $\varrho = (x_0 \ldots x_{k-1})$ be a circuit in α, and let $\xi = (j_1 \ldots j_v x_0\rangle$ be a longest cilium attached to ϱ. Suppose that t is a nonnegative integer such that $\beta\big|\operatorname{span}\eta = \alpha^t\big|\operatorname{span}\eta$. If $t < \min\{u, v\}$, then $\operatorname{dom}\beta \cap \operatorname{span}\xi \neq \emptyset$.*

Proof. Let $t < \min\{u, v\}$. Suppose, by way of contradiction, that $\operatorname{dom}\beta \cap \operatorname{span}\xi = \emptyset$. Let $m = \min\{u, v\}$. We will construct $\gamma \in C(\alpha)$ that maps the initial segment $(j_1 \ldots j_m]$ of ξ onto a terminal segment of η. Let λ be the cell in α that has ϱ as the circuit. Set $\operatorname{dom}\gamma = \{x \in \operatorname{dom}\lambda\colon x\alpha^q = j_m$ for some $q \geqslant 0\}$. Let $\mu = (m_1 \ldots m_d j_m \ldots j_v x_0\rangle$ $(d \geqslant 0)$ be a cilium in λ whose span contains j_m. Since $\ell(\mu) \leqslant \ell(\xi)$, $d+1 \leqslant m \leqslant u$. Thus we can define γ so that it maps the initial segment $(m_1 \ldots m_d j_m]$ of μ onto a terminal segment of η. In particular, γ maps the initial segment $(j_1 \ldots j_m]$ of ξ onto a terminal segment of η, say $(i_r \ldots i_u]$.

By Theorem 1 and the construction of γ, we have $\gamma \in C(\alpha)$ and $j_1\gamma = i_r$. Since $j_1(\beta \circ \gamma)$ is undefined (since $j_1 \notin \operatorname{dom}\beta$) and $j_1(\gamma \circ \beta) = i_r\beta = i_r\alpha^t = i_{r+t}$ (since $t < m$), $\gamma \notin C(\beta)$, which is a contradiction. Thus $\operatorname{dom}\beta \cap \operatorname{span}\xi \neq \emptyset$. □

Lemma 9. *Let $\alpha, \beta \in PT_n$ be such that $\beta \in C^2(\alpha)$, let $\eta = (i_1 \ldots i_u]$ be a maximal chain in α, let $\varrho = (x_0 \ldots x_{k-1})$ be a circuit in α, and let $\xi = (j_1 \ldots j_v x_0\rangle$ be a longest cilium attached to ϱ. Suppose that t and w are nonnegative integers such that $\beta\big|\operatorname{span}\eta = \alpha^t\big|\operatorname{span}\eta$ and $\beta\big|\operatorname{span}\xi = \alpha^w\big|\operatorname{span}\xi$. If either t or w is less than $\min\{u, v\}$, then $t = w$.*

Proof. Let $m = \min\{u, v\}$. Let λ be the cell in α that has ϱ as the circuit. As in the proof of Lemma 8, we can construct $\gamma \in C(\alpha)$ such that $\operatorname{dom}\gamma = \{x \in \operatorname{dom}\lambda\colon x\alpha^q = j_m$ for some $q \geqslant 0\}$ and γ maps the initial segment $(j_1 \ldots j_m]$ of ξ onto a terminal segment of η, say $(i_r \ldots i_u]$. Then $j_1(\beta \circ \gamma) = (j_1\alpha^w)\gamma$ and $j_1(\gamma \circ \beta) = i_r\beta = i_r\alpha^t$. Since $\gamma \in C(\beta)$, $(j_1\alpha^w)\gamma = i_r\alpha^t$.

Suppose $t < m$. Then $i_r\alpha^t$ is defined and $i_r\alpha^t = i_{r+t}$. Thus w must be less than m (otherwise $j_1\alpha^w$ would not be in $\operatorname{dom}\gamma$) and so $(j_1\alpha^w)\gamma = j_{w+1}\gamma = i_{r+w}$. Hence $i_{r+t} = i_{r+w}$ and so $t = w$.

Suppose $w < m$. Then $(j_1\alpha^w)\gamma$ is defined and $(j_1\alpha^w)\gamma = j_{w+1}\gamma = i_{r+w}$. Thus t must be less than m (otherwise $i_r\alpha^t$ would be undefined) and so $i_r\alpha^t = i_{r+t}$. Hence $i_{r+t} = i_{r+w}$ and so $t = w$. □

We already proved (Lemmas 5–7) that if $\alpha \in PT_n$ and $\beta \in C^2(\alpha)$, then β satisfies (2) of Theorem 2, that is, $\beta\big|\operatorname{dom}\lambda_i = \alpha^{t_i}\big|\operatorname{dom}\lambda_i$ for some $t_i \in \mathbb{N} \cup \{-\infty\}$ $(i = 1, \ldots, m)$. The next three lemmas show that the exponents t_i satisfy (4) of Theorem 2.

Lemma 10. *Let $\alpha, \beta \in PT_n$ be such that $\beta \in C^2(\alpha)$, and let $\varrho = (x_0 \dots x_{k-1})$ and $\delta = (y_0 \dots y_{m-1})$ be circuits in α such that k divides m. Then* $\operatorname{dom}\beta \cap \operatorname{dom}\varrho = \emptyset$ *if and only if* $\operatorname{dom}\beta \cap \operatorname{dom}\delta = \emptyset$.

Proof. We will construct $\gamma \in C(\alpha)$ such that $y_0\gamma = x_0$. Let λ be the cell in α that has δ as the circuit. Set $\operatorname{dom}\gamma = \operatorname{dom}\lambda$. Define γ so that it maps the points $y_0, y_1, \dots, y_{m-1}$ of $\operatorname{dom}\delta$ to $x_0, x_0\alpha, \dots, x_0\alpha^{m-1}$. Let $\xi = (j_1 \dots j_v y_p\rangle$ be any cilium in α attached to δ. Select $x_h \in \operatorname{dom}\varrho$ so that $x_h\alpha^v = x_0\alpha^p$ and define γ so that it maps the points $j_1, j_2, \dots, j_v, y_p$ of $\operatorname{span}\xi$ to $x_h, x_h\alpha, \dots, x_h\alpha^{v-1}, x_h\alpha^v$.

By Theorem 1 and the construction of γ, we have $\gamma \in C(\alpha)$ and $y_0\gamma = x_0$. Suppose $\operatorname{dom}\beta \cap \operatorname{dom}\varrho = \emptyset$ and $\operatorname{dom}\beta \cap \operatorname{dom}\delta \neq \emptyset$. Then $y_0 \in \operatorname{dom}\beta$ and $y_0\beta \in \operatorname{dom}\delta$ (by Lemma 5). Thus $y_0(\beta \circ \gamma)$ is defined and $y_0(\gamma \circ \beta) = x_0\beta$ is undefined. Suppose $\operatorname{dom}\beta \cap \operatorname{dom}\delta = \emptyset$ and $\operatorname{dom}\beta \cap \operatorname{dom}\varrho \neq \emptyset$. Then $x_0 \in \operatorname{dom}\beta$ (by Theorem 1). Thus $y_0(\beta \circ \gamma)$ is undefined and $y_0(\gamma \circ \beta) = x_0\beta$ is defined. In either case, $\gamma \notin C(\beta)$, which is a contradiction. The result follows. □

Lemma 11. *Let $\alpha, \beta \in PT_n$ be such that $\beta \in C^2(\alpha)$, and let $\varrho = (x_0 \dots x_{k-1})$ and $\delta = (y_0 \dots y_{m-1})$ be circuits in α such that k divides m. Suppose that t and w are nonnegative integers such that $\beta\big|\operatorname{dom}\varrho = \alpha^t\big|\operatorname{dom}\varrho$ and $\beta\big|\operatorname{dom}\delta = \alpha^w\big|\operatorname{dom}\delta$. Then $w \equiv t \pmod{k}$.*

Proof. Let $t' \in \{0, 1, \dots, k-1\}$ and $w' \in \{0, 1, \dots, m-1\}$ be such that $t' \equiv t \pmod{k}$ and $w' \equiv w \pmod{m}$. Note that $w' \equiv w \pmod{k}$ (since k divides m), $x_0\beta = x_{t'}$, and $y_0\beta = y_{w'}$. As in the proof of Lemma 10, we can construct $\gamma \in C(\alpha)$ that maps $y_0, y_1, \dots, y_{m-1}$ to $x_0, x_0\alpha, \dots, x_0\alpha^{m-1}$. Note that $y_{w'}\gamma = x_{w''}$, where $w'' \in \{0, 1, \dots, k-1\}$ and $w'' \equiv w' \pmod{k}$. On the other hand, $y_{w'}\gamma = y_0(\beta \circ \gamma) = y_0(\gamma \circ \beta) = x_0\beta = x_{t'}$. Hence $t' = w''$ and so $t \equiv t' = w'' \equiv w' \equiv w \pmod{k}$. □

Lemma 12. *Let $\alpha, \beta \in PT_n$ be such that $\beta \in C^2(\alpha)$, let $\varrho = (x_0 \dots x_{k-1})$ and $\delta = (y_0 \dots y_{l-1})$ be circuits in α such that k divides l, let $\eta = (i_1 \dots i_u x_0\rangle$ be a cilium attached to ϱ, and let $\xi = (j_1 \dots j_v y_0\rangle$ be a longest cilium attached to δ. Suppose that t and w are nonnegative integers such that $\beta\big|\operatorname{span}\eta = \alpha^t\big|\operatorname{span}\eta$ and $\beta\big|\operatorname{span}\xi = \alpha^w\big|\operatorname{span}\xi$. If either $t < \min\{u, v\}$ or $w < \min\{u, v\}$, then $w = t$.*

Proof. Let $m = \min\{u, v\}$. We will construct $\gamma \in C(\alpha)$ that maps the initial segment $(j_1 \dots j_m]$ of ξ onto a terminal segment of $(i_1 \dots i_u]$. Set $\operatorname{dom}\gamma = \operatorname{dom}\lambda$, where λ is the cell in α that has δ as the circuit. Let $q \in \{0, \dots, k-1\}$ be such that $q \equiv v - m \pmod{k}$.

Let $\mu = (m_1 \dots m_d j_m \dots j_v y_0\rangle$ $(d \geqslant 0)$ be any cilium in λ whose span contains j_m. Since $\ell(\xi) \geqslant \ell(\mu)$, $m - 1 \geqslant d$ and so $u \geqslant m \geqslant d + 1$. Thus we can define γ so that it maps the initial segment $(m_1 \dots m_d j_m]$ of μ onto a terminal segment of $(i_1 \dots i_u]$

and the remaining points of $\operatorname{span}\mu$, $j_{m+1}, j_{m+2}, \ldots, j_v, y_0$, to $x_0, x_0\alpha, \ldots, x_0\alpha^{v-m-1}$, $x_0\alpha^{v-m} = x_q$. In particular, γ maps the initial segment $(j_1 \ldots j_m]$ of ξ onto a terminal segment of $(i_1 \ldots i_u]$, say $(i_r \ldots i_u]$. Let $\mu = (m_1 \ldots m_d y_s\rangle$ be any cilium in λ whose span does not contain j_m. Select $a \in \{0, \ldots, k-1\}$ such that $a \equiv q + s \pmod{k}$ and $x_h \in \operatorname{dom}\varrho$ such that $x_h\alpha^d = x_a$. Define γ so that it maps the points $m_1, m_2, \ldots, m_d$, y_s of $\operatorname{span}\mu$ to the points $x_h, x_h\alpha, \ldots, x_h\alpha^{d-1}$, $x_h\alpha^d = x_a$.

By Theorem 1 and the construction of γ, $\gamma \in C(\alpha)$ and it maps $(j_1 \ldots j_m]$ onto $(i_r \ldots i_u]$. (Note that this implies $u - r = m - 1$ and so $r + m - 1 = u$.) Then $j_1(\beta \circ \gamma) = (j_1\alpha^w)\gamma$ and $j_1(\gamma \circ \beta) = i_r\beta = i_r\alpha^t$. Since $\gamma \in C(\beta)$, $(j_1\alpha^w)\gamma = i_r\alpha^t$.

Suppose $t < m$. Then $r + t \leqslant r + m - 1 = u$ and so $i_r\alpha^t = i_{r+t}$. Thus w must be less than m (otherwise, by the construction of γ, $(j_1\alpha^w)\gamma$ would be in $\operatorname{dom}\varrho$ and so it could not be equal to i_{r+t}) and so $(j_1\alpha^w)\gamma = j_{w+1}\gamma = i_{r+w}$. Hence $i_{r+t} = i_{r+w}$ and so $t = w$.

Suppose $w < m$. Then $(j_1\alpha^w)\gamma = j_{w+1}\gamma = i_{r+w}$. Thus t must be less than m (otherwise $i_r\alpha^t$ would be in $\operatorname{dom}\varrho$ and so it could not be equal to i_{r+w}) and so $i_r\alpha^t = i_{r+t}$. Hence $i_{r+t} = i_{r+w}$ and so $t = w$. □

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. Suppose that $\beta \in C^2(\alpha)$. Suppose that $k \geqslant 1$, that is, α has at least one maximal chain. Let $i \in \{1, \ldots, k\}$. By Lemma 3, $\beta\big|\operatorname{span}\eta_i = \alpha^{w_i}\big|\operatorname{span}\eta_i$ for some $w_i \in \{0, \ldots, \ell(\eta_i)\}$. Let $t = \max\{w_1, \ldots, w_k\}$. By Lemma 4, $\beta\big|\operatorname{span}\eta_i = \alpha^t\big|\operatorname{span}\eta_i$ for each $i \in \{1, \ldots, k\}$. Since $N = \operatorname{span}\eta_1 \cup \ldots \cup \operatorname{span}\eta_k$, $\beta\big|N = \alpha^t\big|N$. If $k = 0$, that is, $N = \emptyset$, then $\beta\big|N = \alpha^0\big|N$. Thus, in any case, there is an integer t that satisfies condition (1).

Let $i \in \{1, \ldots, m\}$. By Lemma 5, $\beta\big|\operatorname{dom}\varrho_i = \alpha^w\big|\operatorname{dom}\varrho_i$ for some $w \in \{0, \ldots, \ell(\varrho_i) - 1\} \cup \{-\infty\}$. If $\lambda_i = \varrho_i$, that is, if ϱ_i is an isolated circuit, take $t_i = w$. Suppose that $\lambda_i \neq \varrho_i$, that is, λ_i has at least one cilium. Let $\eta_1, \ldots, \eta_b$ be the cilia in λ_i. Suppose $w = -\infty$, that is, $\operatorname{dom}\beta \cap \operatorname{dom}\varrho_i = \emptyset$. Then $\operatorname{dom}\beta \cap \operatorname{dom}\eta_p = \emptyset$ for each $p \in \{1, \ldots, b\}$ (by Lemma 6), and so $\beta\big|\operatorname{dom}\lambda_i = \alpha^w$. Thus if $w = -\infty$, take $t_i = w$. Suppose $w \neq -\infty$. Then, by Lemma 6, for each $p \in \{1, \ldots, b\}$, there is $w_p \in \{0, \ldots, \ell(\eta_p) + \ell(\varrho_i) - 2\}$ such that $\beta\big|\operatorname{span}\eta_p = \alpha^{w_p}\big|\operatorname{span}\eta_p$. Let $t_i = \max\{w_1, \ldots, w_b\}$. By Lemma 7, $\beta\big|\operatorname{span}\eta_p = \alpha^{t_i}\big|\operatorname{span}\eta_p$ for each $p \in \{1, \ldots, b\}$. Let y be the point at which η_1 meets ϱ_i. Then $y\alpha^w = y\beta = y\alpha^{t_i}$, which implies $t_i \equiv w \pmod{\ell(\varrho_i)}$. Let $x \in \operatorname{dom}\lambda_i$. If $x \in \operatorname{span}\eta_p$ for some $p \in \{1, \ldots, b\}$, then $x\beta = x\alpha^{t_i}$. If $x \in \operatorname{dom}\varrho_i$, then $x\beta = x\alpha^w = x\alpha^{t_i}$ (since $t_i \equiv w \pmod{\ell(\varrho_i)}$). Since $\operatorname{dom}\lambda_i = \operatorname{span}\eta_1 \cup \ldots \cup \operatorname{span}\eta_b \cup \operatorname{dom}\varrho_i$, $\beta\big|\operatorname{dom}\lambda_i = \alpha^{t_i}\big|\operatorname{dom}\lambda_i$. Thus for each $i \in \{1, \ldots, m\}$, there is $t_i \in \mathbb{N} \cup \{-\infty\}$ that satisfies condition (2). Moreover, it follows from Lemma 8 and Lemma 9 that for each $i \in \{1, \ldots, m\}$, t and t_i satisfy condition (3), and it follows from Lemma 10, Lemma 11, and Lemma 12 that for all $i, j \in \{1, \ldots, m\}$, t_i and t_j satisfy condition (4).

Conversely, suppose that there are $t \in \mathbb{N}$ and $t_1, \dots, t_m \in \mathbb{N} \cup \{-\infty\}$ such that conditions (1)–(4) are satisfied for all $i, j \in \{1, \dots, m\}$. Let $\gamma \in C(\alpha)$. We need to prove that $\beta \circ \gamma = \gamma \circ \beta$. Let $x \in X$ and consider four cases.

Case 1. $x \in N$ and $x \in \operatorname{dom}(\gamma \circ \beta)$.

Then $x \in \operatorname{dom} \gamma$ and $x\gamma \in \operatorname{dom} \beta$. Since $x\gamma \in N$ (by Theorem 1), $x\gamma \in \operatorname{dom} \alpha^t$ (by (1)). Thus $x \in \operatorname{dom}(\gamma \circ \alpha^t)$ and so, since γ commutes with α^t, $x \in \operatorname{dom}(\alpha^t \circ \gamma)$. Thus, since $\beta\big|N = \alpha^t\big|N$, $x \in \operatorname{dom}(\beta \circ \gamma)$ and $x(\beta \circ \gamma) = (x\beta)\gamma = (x\alpha^t)\gamma = x(\alpha^t \circ \gamma) = x(\gamma \circ \alpha^t) = (x\gamma)\alpha^t = (x\gamma)\beta = x(\gamma \circ \beta)$.

Case 2. $x \in N$ and $x \in \operatorname{dom}(\beta \circ \gamma)$.

By an argument similar to that used in Case 1, $x \in \operatorname{dom}(\gamma \circ \beta)$ and $x(\beta \circ \gamma) = x(\gamma \circ \beta)$.

Case 3. $x \in \operatorname{dom} \lambda_j$ for some $j \in \{1, \dots, m\}$ and $x \in \operatorname{dom}(\gamma \circ \beta)$.

Then $x \in \operatorname{dom} \gamma$ and $x\gamma \in \operatorname{dom} \beta$. By Theorem 1, one of the following two cases holds.

Case 3.1. $x\gamma \in \operatorname{dom} \lambda_i$ for some $i \in \{1, \dots, m\}$.

Then, by Theorem 1, $\ell(\varrho_i)$ divides $\ell(\varrho_j)$ and $\operatorname{dom} \lambda_j \subseteq \operatorname{dom} \gamma$. Since $\beta\big|\operatorname{dom} \lambda_i = \alpha^{t_i}\big|\operatorname{dom} \lambda_i$ (by (2)) and $x\gamma \in \operatorname{dom} \beta$, t_i cannot be $-\infty$. Thus $t_j \neq -\infty$ by (4b). It follows that $x \in \operatorname{dom}(\alpha^{t_j} \circ \gamma)$ and so, since $\beta\big|\operatorname{dom} \lambda_j = \alpha^{t_j}\big|\operatorname{dom} \lambda_j$, $x \in \operatorname{dom}(\beta \circ \gamma)$.

Since $x\gamma \in \operatorname{dom} \lambda_i$, $x(\gamma \circ \beta) = (x\gamma)\beta = (x\gamma)\alpha^{t_i} \in \operatorname{dom} \lambda_i$. Since $x \in \operatorname{dom} \lambda_j$, $x(\beta \circ \gamma) = (x\beta)\gamma = (x\alpha^{t_j})\gamma \in \operatorname{dom} \lambda_i$. Thus $x\gamma$, $x(\gamma \circ \beta)$, and $x(\beta \circ \gamma)$ are all in $\operatorname{dom} \lambda_i$. Let $\varrho_i = (x_0 \dots x_{a-1})$, let $\varrho_j = (y_0 \dots y_{b-1})$, and consider two cases.

Case 3.1.1. $x(\gamma \circ \beta) \in \operatorname{dom} \varrho_i$.

We claim that $x(\beta \circ \gamma)$ is also in $\operatorname{dom} \varrho_i$. Suppose, by way of contradiction, that $x(\beta \circ \gamma) \notin \operatorname{dom} \varrho_i$. Then $x \notin \operatorname{dom} \varrho_j$ since otherwise $x\beta = x\alpha^{t_j}$ would be in $\operatorname{dom} \varrho_j$ and so $x(\beta \circ \gamma) = (x\beta)\gamma$ would be in $\operatorname{dom} \varrho_i$ (by Theorem 1). Thus there is a cilium $\xi = (m_1 \dots m_v y_r\rangle$ in λ_j such that $x = m_p$ for some $p \in \{1, \dots, v\}$. We observed in the foregoing argument that $m_p\beta = x\beta$ cannot be in $\operatorname{dom} \varrho_j$. It follows that $p + t_j \leqslant v$ and $m_p\beta = m_p\alpha^{t_j} = m_{p+t_j}$. Since $p + t_j \leqslant v$, $t_j \leqslant v - p < v$. Since $m_{p+t_j}\gamma = (m_p\beta)\gamma = m_p(\beta \circ \gamma) = x(\beta \circ \gamma) \notin \operatorname{dom} \varrho_i$, it follows by Theorem 1 that there is a cilium $\eta = (k_1 \dots k_u x_s\rangle$ in λ_i such that for some $q \in \{1, \dots, u\}$, $m_p\gamma = k_q$, $q + t_j \leqslant u$, and $m_{p+t_j}\gamma = k_{q+t_j}$. Since $q + t_j \leqslant u$, $t_j \leqslant u - q < u$. Hence $t_j < \min\{u, v\} \leqslant \min\{r(\lambda_i), r(\lambda_j)\}$ and so $t_i = t_j$ by (4b). But then $x(\gamma \circ \beta) = (m_p\gamma)\beta = k_q\beta = k_q\alpha^{t_i} = k_q\alpha^{t_j} = k_{q+t_j} \notin \operatorname{dom} \varrho_i$, which is a contradiction.

Thus both $x(\gamma \circ \beta)$ and $x(\beta \circ \gamma)$ are in $\operatorname{dom} \varrho_i$ and so $x(\gamma \circ \beta) = x_p$ and $x(\beta \circ \gamma) = x_q$ for some $p, q \in \{0, \dots, a-1\}$. By (4a), $t_i \equiv t_j \pmod{a}$ and so there is an integer $l \geqslant 0$ such that either $t_i = t_j + lk$ or $t_j = t_i + lk$. In the former case, we have:

$$\begin{aligned} x_p &= x(\gamma \circ \beta) = (x\gamma)\beta = (x\gamma)\alpha^{t_i} = x(\gamma \circ \alpha^{t_j} \circ \alpha^{lk}) = x(\alpha^{t_j} \circ \gamma \circ \alpha^{lk}) \\ &= (x\alpha^{t_j})(\gamma \circ \alpha^{lk}) = (x\beta)(\gamma \circ \alpha^{lk}) = (x(\beta \circ \gamma))\alpha^{lk} = x_q\alpha^{lk} = x_q. \end{aligned}$$

And in the latter case, we have:

$$x_q = x(\beta \circ \gamma) = (x\beta)\gamma = (x\alpha^{t_j})\gamma = x(\alpha^{t_j} \circ \gamma) = x(\gamma \circ \alpha^{t_j}) = x(\gamma \circ \alpha^{t_i} \circ \alpha^{lk})$$
$$= ((x\gamma)\alpha^{t_i})\alpha^{lk} = ((x\gamma)\beta)\alpha^{lk} = (x(\gamma \circ \beta))\alpha^{lk} = x_p\alpha^{lk} = x_p.$$

Thus $x(\gamma \circ \beta) = x_p = x_q = x(\beta \circ \gamma)$.

Case 3.1.2. $x(\gamma \circ \beta) \notin \operatorname{dom} \varrho_i$.

Then $x \notin \operatorname{dom} \varrho_j$ since otherwise $x\gamma$ would be in $\operatorname{dom} \varrho_i$ (by Theorem 1) and so $x(\gamma \circ \beta) = (x\gamma)\beta = (x\gamma)\alpha^{t_i}$ would also be in $\operatorname{dom} \varrho_i$. Thus there is a cilium $\xi = (m_1 \dots m_v y_r\rangle$ in λ_j such that $x = m_p$ for some $p \in \{1, \dots, v\}$. We observed in the foregoing argument that $x\gamma$ cannot be in $\operatorname{dom} \varrho_i$. It follows that there is a cilium $\eta = (k_1 \dots k_u x_s\rangle$ in λ_i such that $x\gamma = m_p\gamma = k_q$ for some $q \in \{1, \dots, u\}$. Since $k_q\alpha^{t_i} = k_q\beta = (m_p\gamma)\beta = m_p(\gamma \circ \beta) \notin \operatorname{dom} \varrho_i$, we must have $q + t_i \leqslant u$ and $m_p(\gamma \circ \beta) = k_q\alpha^{t_i} = k_{q+t_i}$. Since $q + t_i \leqslant u$, $t_i \leqslant u - q < u$. Since (by Theorem 1) either $m_v\gamma = k_u$ or $m_v\gamma \in \operatorname{dom} \varrho_i$, the fact that $m_p\gamma = k_q$ coupled with Theorem 1 implies that $u - q \leqslant v - p$. Thus $t_i \leqslant u - q \leqslant v - p < v$. Hence $t_i < \min\{u, v\}$ and so $t_i = t_j$ by (4b). Thus

$$x(\beta \circ \gamma) = (x\beta)\gamma = (x\alpha^{t_j})\gamma = x(\alpha^{t_j} \circ \gamma) = x(\gamma \circ \alpha^{t_j}) = (x\gamma)\alpha^{t_j} = (x\gamma)\alpha^{t_i}$$
$$= (x\gamma)\beta = x(\gamma \circ \beta).$$

Case 3.2. $x\gamma \in N$.

Then, by Theorem 1, there is a cilium $\xi = (m_1 \dots m_v y_r\rangle$ in λ_j and a maximal chain $\eta_i = (k_1 \dots k_u]$ in α such that for some $p \in \{1, \dots, v\}$, $x = m_p$ and γ maps an initial segment $(m_1 \dots m_p \dots]$ of $(m_1 \dots m_v]$ onto a terminal segment of η_i. Let $m_p\gamma = k_q$ ($q \in \{1, \dots, u\}$). Since $k_q = m_p\gamma = x\gamma \in \operatorname{dom} \beta$ and $\beta|N = \alpha^t|N$, $k_q \in \operatorname{dom} \alpha^t$, which implies $q + t \leqslant u$ and $k_q\beta = k_q\alpha^t = k_{q+t}$. Since γ maps an initial segment of $(m_1 \dots m_v]$ onto a terminal segment of $(k_1 \dots k_u]$, $m_p\gamma = k_q$ and $q + t \leqslant u$ imply that $p + t \leqslant v$ and $m_{p+t} \in \operatorname{dom} \gamma$. Thus $t < \min\{u, v\} \leqslant \min\{d(N), r(\lambda_j)\}$ and so $t = t_j$ (by (3)). Hence $x \in \operatorname{dom} \beta$ (since $t_j = t \geqslant 0$ and $\beta|\operatorname{dom} \lambda_j = \alpha^{t_j}|\operatorname{dom} \lambda_j$) and $x\beta = m_p\beta = m_p\alpha^{t_j} = m_p\alpha^t = m_{p+t} \in \operatorname{dom} \gamma$. Thus $x \in \operatorname{dom}(\beta \circ \gamma)$ and, since γ commutes with α^t, $x(\beta \circ \gamma) = (x\beta)\gamma = (x\alpha^{t_j})\gamma = (x\alpha^t)\gamma = x(\alpha^t \circ \gamma) = x(\gamma \circ \alpha^t) = (x\gamma)\alpha^t = (x\gamma)\beta = x(\gamma \circ \beta)$.

Case 4. $x \in \operatorname{dom} \lambda_j$ for some $j \in \{1, \dots, m\}$ and $x \in \operatorname{dom}(\beta \circ \gamma)$.

Then $x \in \operatorname{dom} \beta$ and $y = x\beta \in \operatorname{dom} \gamma$. Since, by (2), $\beta|\operatorname{dom} \lambda_j = \alpha^{t_j}|\operatorname{dom} \lambda_j$, $t_j \geqslant 0$ and $y \in \operatorname{dom} \lambda_j$. By Theorem 1, one of the following two cases holds.

Case 4.1. $y\gamma \in \operatorname{dom} \lambda_i$ for some $i \in \{1, \dots, m\}$.

Then, by Theorem 1, $\ell(\varrho_i)$ divides $\ell(\varrho_j)$, $\operatorname{dom} \lambda_j \subseteq \operatorname{dom} \gamma$, and $x\gamma \in \operatorname{dom} \lambda_i$. Since $t_j \neq -\infty$, $t_i \neq -\infty$ by (4b). Thus, since $\beta|\operatorname{dom} \lambda_i = \alpha^{t_i}|\operatorname{dom} \lambda_i$, $\operatorname{dom} \lambda_i \subseteq \operatorname{dom} \beta$.

Hence $x \in \operatorname{dom} \lambda_j \subseteq \operatorname{dom} \gamma$ and $x\gamma \in \operatorname{dom} \lambda_i \subseteq \operatorname{dom} \beta$, which implies $x \in \operatorname{dom}(\gamma \circ \beta)$. It follows by Case 3 that $x(\beta \circ \gamma) = x(\gamma \circ \beta)$.

Case 4.2. $y\gamma \in N$.

Then, by Theorem 1, $y \notin \operatorname{dom} \varrho_j$. Thus, since $y = x\beta = x\alpha^{t_j}$, there is a cilium $\xi = (m_1 \ldots m_v y_r\rangle$ in λ_j such that for some $p \in \{1, \ldots, v\}$, $y = m_p$, $p - t_j \geqslant 1$, and $x = m_{p-t_j}$. Since $y\gamma \in N$, it follows by Theorem 1 that there is a maximal chain $\eta_i = (k_1 \ldots k_u]$ in α such that γ maps an initial segment $(m_1 \ldots m_{p-t_j} \ldots m_p \ldots]$ of $(m_1 \ldots m_v]$ onto a terminal segment of η_i. Let $m_p\gamma = k_q$ ($q \in \{1, \ldots, u\}$). Then, since γ maps an initial segment of $(m_1 \ldots m_v]$ onto a terminal segment of η_i, $q - t_j \geqslant 1$ and $m_{p-t_j}\gamma = k_{q-t_j}$. Since $q - t_j \geqslant 1$ and $p - t_j \geqslant 1$, $t_j < \min\{q, p\} \leqslant \min\{u, v\} \leqslant \min\{d(N), r(\lambda_j)\}$. Thus, by (3), $t_j = t$ and so $k_{q-t_j} = k_{q-t} \in \operatorname{dom} \beta$ (since $k_{q-t}\alpha^t = k_q$ and $\operatorname{dom} \beta \big| N = \operatorname{dom} \alpha^t \big| N$). Hence $x = m_{p-t_j} \in \operatorname{dom} \gamma$ and $x\gamma = m_{p-t_j}\gamma = k_{q-t_j} \in \operatorname{dom} \beta$, which implies $x \in \operatorname{dom}(\gamma \circ \beta)$. It follows by Case 3 that $x(\beta \circ \gamma) = x(\gamma \circ \beta)$.

This concludes the proof. □

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