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DOMINATION IN GENERALIZED PETERSEN GRAPHS

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Abstract. Generalized Petersen graphs are certain graphs consisting of one quadratic factor. For these graphs some numerical invariants concerning the domination are studied, namely the domatic number $d(G)$, the total domatic number $d_t(G)$ and the k-ply domatic number $d^k(G)$ for $k = 2$ and $k = 3$. Some exact values and some inequalities are stated.

Keywords: domatic number, total domatic number, k-ply domatic number, generalized Petersen graph

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In this paper we will study three numerical invariants of graphs which concern the domination, namely the domatic number $d(G)$, total domatic number $d_t(G)$ and k-ply domatic number $d^k(G)$ of a graph G. We will investigate them for generalized Petersen graphs. The vertex set of a graph G will be denoted by $V(G)$. For a vertex $v \in V(G)$ the symbol $N_G[v]$ denotes the closed neighbourhood of v in G, i.e. the set consisting of v and of all vertices adjacent to v in G.

A subset D of $V(G)$ is called dominating (or total dominating) in G, if for each $x \in V(G) \setminus D$ (or for each $x \in V(G)$ respectively) there exists a vertex $y \in D$ adjacent to x. The set D is called k-ply dominating for a positive integer k, if for each $x \in V(G) \setminus D$ there exist k distinct vertices $y_1, \ldots, y_k$ of D which are all adjacent to x.

A domatic (or total domatic, or k-ply domatic) partition of G is a partition of $V(G)$, all of whose classes are dominating (or total dominating, or k-ply dominating respectively) sets in G. The maximum number of classes of a domatic (or total domatic, of k-ply domatic) partition of G is the domatic (or total domatic, or k-ply domatic respectively) number of G. The domatic number of G is denoted by $d(G)$, the total domatic number by $d_t(G)$, the k-ply domatic number by $d^k(G)$.

In this paper we will consider $d^k(G)$ for $k = 2$ and $k = 3$ and we will speak about the doubly domatic number and the triply domatic number.

The domatic number was introduced by E. J. Cockayne and S. T. Hedetniemi in [2], the total domatic number by the same authors and R. M. Dawes in [3], the k-ply domatic number by the author of this paper in [6].

Sometimes it is convenient to speak about the domatic colouring. The domatic number of G can be alternatively defined as the maximum number of colours by which the vertices of G can be coloured in such a way that each vertex is adjacent to vertices of all colours different from its own. Evidently this definition is equivalent to that written above. Similarly by means of colourings, also $d_t(G)$ and $d^k(G)$ may be defined.

As was mentioned, the number $d^k(G)$ will be used only for the concrete values $k = 2$ and $k = 3$. Thus in the sequel the symbol k will be used in another sense.

In the whole paper the symbols n, k will denote relatively prime positive integers such that $k < n$, $n \geqslant 3$. The generalized Petersen graph $\mathrm{GP}(n,k)$ is defined as follows. Let C_n, C'_n be two disjoint circuits of length n. Let the vertices of C_n be $u_1, \dots, u_n$ and edges $u_i u_{i+1}$ for $i = 1, \dots, n-1$ and $u_n u_i$. Let the vertices of C'_n be $v_1, \dots, v_n$ and edges $v_i v_{i+k}$ for $i = 1, \dots, n$, the sum $i + k$ being taken modulo n. The graph $\mathrm{GP}(n,k)$ is obtained from the union of C_n and C'_n by adding the edges $u_i v_i$ for $i = 1, \dots, n$.

The graph $\mathrm{GP}(5,2)$ is the well-known Petersen graph. The generalized Petersen graphs were studied e.g. in [1], [4], [5].

For integers n, k fulfilling the above stated conditions we define the numbers $f(n,k)$, $g(n,k)$. They are positive integers such that $f(n,k) \leqslant n-1$, $g(n,k) \leqslant n-1$, $kf(n,k) \equiv 1 \pmod{n}$, $kg(n,k) \equiv -1 \pmod{n}$. It is easy to see that

$$f(n,k) + g(n,k) = n,$$
$$\mathrm{GP}(n,k) \cong \mathrm{GP}(n, n-k) \cong \mathrm{GP}(n, f(n,k)) \cong \mathrm{GP}(n, g(n,k)).$$

Theorem 1. *Let* $\mathrm{GP}(n,k)$ *be a generalized Petersen graph. Then*

$$d(\mathrm{GP}(n,k)) = 4$$

if and only if $n \equiv 0 \pmod{4}$.

P r o o f. According to [2], $d(G) \leqslant \delta(G) + 1$, where $\delta(G)$ is the minimum degree of a vertex in G. Every graph $\mathrm{GP}(n,k)$ is regular of degree 3, therefore $d(\mathrm{GP}(n,k)) \leqslant 4$. Suppose that $n \equiv 0 \pmod{4}$. We construct a domatic colouring c such that c: $V(\mathrm{GP}(n,k)) \to \{1,2,3,4\}$. For $i = 1, \dots, n$ we define c by $c(u_i) \equiv i \pmod{4}$,

$c(v_i) \equiv i + 2 \pmod 4$) The reader may verify himself that c is a domatic colouring of $\mathrm{GP}(n,k)$ by four colours and therefore $d(\mathrm{GP}(n,k)) = 4$.

On the other hand, suppose that $d(\mathrm{GP}(n,k)) = 4$. Let $\mathscr{D} = \{D_1, D_2, D_3, D_4\}$ be a domatic partition of $\mathrm{GP}(n,k)$. Evidently for any $i \in \{1,2,3,4\}$ no two vertices of D_i are adjacent and each vertex not belonging to D_i is adjacent to exactly one vertex of D_i. We will say that x dominates y, if either $y = x$, or y is adjacent to x. Let $a = |D_1 \cap V(C_n)|$, $b = |D_1 \cap V(C'_n)|$. Each vertex of $D_1 \cap V(C_n)$ dominates three vertices of C_n and one vertex of C'_n, while each vertex of $D_1 \cap V(C'_n)$ dominates three vertices of C'_n and one vertex of C_n. Therefore $3a + b = n$, $a + 3b = n$. These two equations imply $a = b = n/4$ and therefore $n \equiv 0 \pmod 4$. □

Remark. Let $n \equiv 0 \pmod 3$, let $\mathrm{GP}(n,k)$ be a generalized Petersen graph. Since it is easy to construct a domatic colouring of $\mathrm{GP}(n,k)$ by three colours, we have $d(\mathrm{GP}(n,k)) \geqslant 3$.

Theorem 2. *Let* $\mathrm{GP}(n,k)$ *be a generalized Petersen graph. If* $n \not\equiv 0 \pmod 3$ *and either* $k \equiv f(n,k) \equiv 0 \pmod 3$, *or* $k \equiv f(n,k) \equiv n \pmod 3$, *then the inequality* $d(\mathrm{GP}(n,k)) \geqslant 3$ *holds.*

P r o o f. First let $n \equiv 1 \pmod 3$, $k \equiv 1 \pmod 3$, $f(n,k) \equiv 1 \pmod 3$. Consider the Hamiltonian path P in $\mathrm{GP}(n,k)$ having subsequent vertices $u_1, u_2, \ldots, u_n, v_n, v_{n+k}, \ldots, v_{n-k}$, where the subscripts are taken modulo n. We colour its vertices subsequently by $1, 2, 3, 1, 2, 3, \ldots$ The last vertex $v_{(n-1)k} = v_{n-k}$ is coloured by 2 and is adjacent to $v_{(n-2)k}$ coloured by 1 and to u_{n-1} coloured by 3. The first vertex u_1 is coloured by 1 and is adjacent to u_n coloured by 2 and to v_1 coloured by 3. For any other vertex it is evident that it is adjacent to vertices of all colours different from its own. Therefore the described colouring is a domatic colouring of $\mathrm{GP}(n,k)$ by three colours.

Now let $n \equiv 2 \pmod 3$, $k \equiv 0 \pmod 3$, $f(n,k) \equiv 0 \pmod 3$. We construct the domatic colouring of $\mathrm{GP}(n,k)$ in the same way. The last vertex v_{n-k} is coloured by 1 and is adjacent to v_n coloured by 3 and to u_{n-k} coloured by 2. The first vertex u_1 is coloured by 1 and is adjacent to u_n coloured by 2 and to v_1 coloured by 3. Again the described colouring is domatic.

If $n \equiv 1 \pmod 3$, $k \equiv 0 \pmod 3$, $f(n,k) \equiv 0 \pmod 3$, then $n - k \equiv 1 \pmod 3$, $f(n, n-k) = g(n,k) = n - f(n,k) \equiv 1 \pmod 3$ and $\mathrm{GP}(n, n-k) \cong \mathrm{GP}(n,k)$; therefore the assertion also holds. Similarly if $n \equiv 2 \pmod 3$, $k \equiv 2 \pmod 3$, $f(n,k) \equiv 2 \pmod 3$, then $n - k \equiv 0 \pmod 3$, $f(n, n-k) \equiv 0 \pmod 3$ and the assertion holds. □

The following theorem concerns the graphs $\mathrm{GP}(n,1)$, i.e., graphs of n-side prisms.

Theorem 3. *For any integer $n \geqslant 3$ the inequality $d(\mathrm{GP}(n,1)) \geqslant 3$ holds.*

P r o o f. If $n \equiv 0 \pmod 3$, the assertion follows from Remark. If $n \equiv 1 \pmod 3$, then it follows from Theorem 2, because $f(n,1) = 1$. If $n \equiv 2 \pmod 3$, we define the colouring of vertices of $\mathrm{GP}(n,1)$ as follows. If $t \leqslant n-2$, then $c(u_t) \equiv t \pmod 3$, $c(v_t) \equiv 1 - t \pmod 3$. Then we put $c(u_{n-1}) = 2$, $c(u_n) = 1$, $c(v_{n-1}) = 2$, $c(v_n) = 2$. The colouring by 3 colours obtained is this way is domatic and $d(\mathrm{GP}(n,1)) \geqslant 3$. □

Example. The domatic number of the original Petersen graph $\mathrm{GP}(5,2)$ is 2.

P r o o f. The domatic number of a graph without isolated vertices is always at least 2. Suppose that there exists a domatic partition $\mathscr{D} = \{D_1, D_2, D_3\}$ of $\mathrm{GP}(5,2)$ with three classes. As the graph has ten vertices and no dominating set with less than three vertices, at least two classes of $\mathscr{D}$ must consist of three vertices. Without loss of generality let $|D_1| = 3$. It is easy to verify that then there exists a vertex v such that D_1 is its open neighbourhood. Without loss of generality suppose $v \in D_2$. Then $v \notin D_3$ and v is adjacent to no vertex of D_3, therefore D_3 is not dominating in $\mathrm{GP}(5,2)$, which is a contradiction. Therefore $d(\mathrm{GP}(5,2)) = 2$. □

Now we shall study total domatic numbers. According to [3] we have $d_t(G) \leqslant \delta(G)$. As $\mathrm{GP}(n,k)$ is regular of degree 3, we have always $d_t(\mathrm{GP}(n,k)) \leqslant 3$.

Theorem 4. *Let $\mathrm{GP}(n,k)$ be a generalized Petersen graph. Then*

$$d_t(\mathrm{GP}(n,k)) = 3$$

if and only if $n \equiv 0 \pmod 3$.

P r o o f. Suppose that $d(\mathrm{GP}(n,k)) = 3$ and let $\{D_1, D_2, D_3\}$ be the corresponding total domatic partition. Evidently no vertex is adjacent to exactly one vertex of any class of this partition. Let u, v be two adjacent vertices from D_1. Then $M(u,v) = N_G[u] \cup N_G[v]$ has six elements. The sets $M(u,v)$ for different pairs $\{u,v\}$ of adjacent vertices from D_1 must be disjoint and therefore they form a partition of $V(\mathrm{GP}(n,k))$. This implies that the number $2n$ of vertices of $\mathrm{GP}(n,k)$ is divisible by 6 and therefore $n \equiv 0 \pmod 3$.

Now suppose that $n \equiv 0 \pmod 3$. For each vertex x of $\mathrm{GP}(n,k)$ we determine its colour $c(x) \in \{1,2,3\}$ in such a way that $c(u_i) = c(v_i) \equiv i \pmod 3$ for $i = 1, \ldots, n$. As k is relatively prime with n, it is also non-divisible by 3 and the colouring thus defined is total domatic. This implies $d(\mathrm{GP}(n,k)) = 3$. □

Theorem 5. *Let $\mathrm{GP}(n,k)$ be a generalized Petersen graph. Then the inequality $d_t(\mathrm{GP}(n,k)) \geqslant 2$ holds.*

P r o o f. The partition $\{V(C_n), V(C'_n)\}$ is evidently a total domatic partition of $\mathrm{GP}(n,k)$. □

At the end we turn to k-ply domatic numbers for $k = 2$ and $k = 3$. In [6] the inequality $d^k(G) \leqslant \lfloor \delta(G)/k \rfloor + 1$ is found, where again $\delta(G)$ is the minimum degree of a vertex in G. This implies $d^2(\mathrm{GP}(n,k)) \leqslant 2$, $d^3(\mathrm{GP}(n,k)) \leqslant 2$, $d^m(\mathrm{GP}(n,k)) = 1$ for $m \geqslant 4$. We prove two theorems.

Theorem 6. *Let* $\mathrm{GP}(n,k)$ *be a generalized Petersen graph. Then*

$$d^3(\mathrm{GP}(n,k)) = 2$$

if and only if n *is even.*

Remark. As n, k must be relatively prime, in this case k is odd.

P r o o f. If and only if n is even, the graph $\mathrm{GP}(n,k)$ contains no circuit of odd length and thus it is a bipartite graph. Its bipartition classes are classes of a triply domatic partition and the assertion holds. On the other hand, if $\{D_1, D_2\}$ is a triply domatic partition of $\mathrm{GP}(n,k)$, then each edge joins a vertex of D_1 with a vertex of D_2, the graph is bipartite and n is even, because otherwise the graph $\mathrm{GP}(n,k)$ would contain circuits C_n, C'_n of odd lengths. Thus the assertion is proved. □

Theorem 7. *Let* $\mathrm{GP}(n,k)$ *be a generalized Petersen graph. Then*

$$d^2(\mathrm{GP}(n,k)) = 2.$$

P r o o f. If n is even, then by Theorem 7 there exists a triply domatic partition of $\mathrm{GP}(n,k)$ with two classes. This partition is also doubly domatic and thus $d^2(\mathrm{GP}(n,k)) = 2$. Suppose that n is odd. As $\mathrm{GP}(n,k) \cong \mathrm{GP}(n, n-k)$, we may suppose that $k \leqslant (n-1)/2$. We put $c(u_i) = 1$ for i odd and $c(u_i) = 2$ for i even. Further, $c(v_1) = c(v_n) = 2$. The circuit C'_n consists of two paths, both with the end vertices v_1, v_n. One of them has an odd length and the other has an even length; let the former be R_1 and the latter R_2. The vertices of R_2 can be coloured alternately by 1 and 2, starting in v_1 of colour 2 and ending in v_n of colour 2. If R_1 contains the edge $v_n v_k$, then it contains also the edge $v_k v_{2k}$. We put $c(v_k) = c(v_{2k}) = 1$ and colour the vertices of the rest of R_1 alternately by 1 and 2, starting in v_{2k} of colour 1 and ending in v_1 of colour 2. If R_1 does not contain $v_n v_k$, it contains the edge $v_{n-k} v_n$. We put $c(v_{n-k}) = 2$ and colour the vertices of the rest of R_1 alternately by 1 and 2, starting in v_1 of colour 2 and ending in v_{n-k} of colour 2. Now suppose that k is odd. If R_1 contains the edge $v_n v_k$, we put $c(v_k) = 2$ and colour the vertices of the rest of R_1 alternately by 1 and 2, starting in v_k of colour 2 and ending in v_1 of colour 2. If R_1 does not contain $v_n v_k$, then it contains the edge $v_{n-k} v_k$ and the edge

$v_{n-2k}v_{n-k}$. We put $c(v_{n-k}) = c(v_{n-2k}) = 1$ and colour the vertices of the rest of R_1 alternately by 1 and 2, starting in v_1 of colour 2 and ending in v_{n-2k} of colour 1. In all the cases we obtain a doubly domatic colouring of $\mathrm{GP}(n,k)$, which proves the assertion. □

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