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STIEFEL-WHITNEY CLASSES OF THE FLAG MANIFOLD $\mathbb{R}F(1,1,n-2)$

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1. Introduction

We give explicit expressions for several Stiefel-Whitney classes of the real flag manifold

$$\mathbb{R}F(1,1,n-2) = \frac{O(n)}{O(1)\times O(1)\times O(n-2)}, \quad n \geqslant 3,$$

which is a smooth connected compact homogeneous manifold of dimension $2n-3$.

Then we deduce upper bounds for the span of $\mathbb{R}F(1,1,n-2)$, where the span of a manifold M is the maximal number of linearly independent tangent vector fields of M. The upper bounds are found by using the fact that if the k-th Stiefel-Whitney class $w_k(M)\neq 0$, then span $M \leqslant m-k$, where m is the dimension of M (cf. [9]). This was used in [3] to obtain upper bounds for the span of the real Grassmannians.

The only known result on the span of $\mathbb{R}F(1,1,n-2)$, $n>4$ is the lower bound obtained for the general flag manifold in Theorem 1.3 of [2] in which it is proved that provided $n=(2a+1)2^{c+4d}$ is even with $a,c,d\geqslant 0$, $c\leqslant 3$ and $\nu(n)=2^c+8d-1$,

$$\operatorname{span}\mathbb{R}F(1,1,n-2)\geqslant \nu(n).$$

Let γ_1 and γ_2 be the canonical line bundles over $F=\mathbb{R}F(1,1,n-2)$ and let $\omega_1(\gamma_1)$ and $\omega_1(\gamma_2)$ be their first Stiefel-Whitney classes. According to [1], $H^*(F;\mathbb{Z}_2)$

is generated by $x = \omega_1(\gamma_1)$ and $y = \omega_1(\gamma_2)$ subject to the relations $\bar{\sigma}_{n-1} = 0 = \bar{\sigma}_n$ so that $x^n = 0 = y^n$, where

$$\bar{\sigma}_i = \bar{\sigma}_i(x, y) = \sum_{k=0}^{i} x^{i-k} y^k, \quad i \geqslant 1$$

denotes the i-th complete symmetric function in x and y.

We shall prove

Theorem 1. *We have the following Stiefel-Whitney classes for* $F = \mathbb{R}F(1, 1, n-2)$, *where we put* $\sigma_1 = x + y$, $\sigma_2 = xy$ *and* $\omega_k = \omega_k(F)$:

(i) $\omega(F) = 1 + \sigma_1 + \sigma_1^2 + \ldots + \sigma_1^{n-2}$, *if* $n = 2^r$, $r \geqslant 2$.

(ii) $\omega_{2^r+s} = \sigma_1^{2^r+s}$, *if* $0 \leqslant s < 2^r$, $n \equiv 0 \bmod 2^{r+1}$ *and* $r \geqslant 0$.

(iii) $\omega_{2^r+s} = 0$, *if* $0 \leqslant s < 2^r$, $n \equiv 2^r \bmod 2^{r+1}$ *and* $r \geqslant 0$.

(iv) $\omega_{2^r+s} = \sigma_1^{2^r+s-2^{p+1}} \sigma_2^{2^p}$, *if* $0 \leqslant s < 2^r$, $n \equiv 2^p \bmod 2^{r+1}$, $0 \leqslant p < r$ *and* $r \geqslant 1$.

(v) $\omega_{2^r+2s} = \sigma_2^{2^{r-1}+s}$, *if* $0 \leqslant s < 2^{r-1}$, $n \equiv 2^{r-1} + s \bmod 2^{r+1}$ *and* $r \geqslant 1$.

Theorem 2. *The following are upper bounds for the span of* $\mathbb{R}F(1, 1, n-2)$:

(i) $\operatorname{span} \mathbb{R}F(1, 1, n-2) \leqslant n - 1$, *if* n *is even or* $n \equiv 1 \bmod 4$.

(ii) $\operatorname{span} \mathbb{R}F(1, 1, n-2) \leqslant n$ *if* $n \equiv 3 \bmod 4$.

Theorem 3.

(i) $\operatorname{span} \mathbb{R}F(1, 1, 4) = 1$.

(ii) $\operatorname{span} \mathbb{R}F(1, 1, 6) = 7$.

2. Proof of Theorem 1

If γ_1 and γ_2 are the two canonical line bundles, ξ is the complementary $(n-2)$-plane bundle and $\gamma_1 \oplus \gamma_2 \oplus \xi$ is an n-plane trivial bundle, all over $F = \mathbb{R}F(1, 1, n-2)$, then by [6], the tangent bundle of F is given by

$$\tau(F) = (\gamma_1 \otimes \gamma_2) \oplus (\gamma_1 \otimes \xi) \oplus (\gamma_2 \otimes \xi).$$

If $n\xi$ stands for the n-fold Whitney sum of ξ, we have that

$$\tau(F) \oplus (\gamma_1 \otimes \gamma_1) \oplus n\xi \oplus (\gamma_1 \otimes \gamma_2) \oplus (\gamma_2 \otimes \gamma_2)$$

is an n^2-plane trivial bundle.

If $\bar{\omega}$ is the dual total Stiefel-Whitney class to ω, taking the total Stiefel-Whitney classes and using the Whitney product formula, we have $\omega(F) = \bar{\omega}(n\xi)\bar{\omega}(\gamma_1 \otimes \gamma_2)$. Then

$$\omega(F) = (1+\sigma_1+\sigma_2)^n(1+\sigma_1)^{-1}. \tag{1}$$

(i) If $n = 2^r$, then $(1+\sigma_1+\sigma_2)^n = 1+\sigma_1^n+\sigma_2^n = 1+x^n+y^n+x^ny^n = 1$, since $x^n = 0 = y^n$. Hence

$$\omega(F) = (1+\sigma_1)^{-1} = 1+\sigma_1+\sigma_1^2+\ldots+\sigma_1^{n-2},$$

since $\sigma_1^{n-1} = \bar{\sigma}_{n-1} = 0$.

(ii) If $0 \leqslant s < 2^r$, then $2^r + s < 2^{r+1}$. Let $n = 2^{r+1}m$, $m \in \mathbb{N}$. Then

$$\omega(F) = (1+\sigma_1^{2^{r+1}}+\sigma_2^{2^{r+1}})^m(1+\sigma_1+\sigma_1^2+\sigma_1^3+\ldots).$$

Hence $\omega_{2^r+s} = \sigma_1^{2^r+s}$, if $0 \leqslant s < 2^r$, $r \geqslant 0$.

(iii) Let $n = 2^r + 2^{r+1}m$, $m \in \mathbb{N}$. Then

$$\omega(F) = (1+\sigma_1^{2^r}+\sigma_2^{2^r})(1+\sigma_1^{2^{r+1}}+\sigma_2^{2^{r+1}})^m(1+\sigma_1+\sigma_1^2+\sigma_1^3+\ldots).$$

Hence $\omega_{2^r+s} = \sigma_1^{2^r+s}+\sigma_1^{2^r+s} = 0$, if $0 \leqslant s < 2^r$.

(iv) Let $n = 2^p + 2^{r+1}m$, $m \in \mathbb{N}$, $0 \leqslant p < r$. Then

$$\omega(F) = (1+\sigma_1^{2^{r+1}}+\sigma_2^{2^{r+1}})^m(1+\sigma_1^{2^p}+\sigma_2^{2^p})(1+\sigma_1+\sigma_1^2+\ldots).$$

Hence if $0 \leqslant s < 2^r$, the result follows.

(v) If $0 \leqslant s < 2^{r-1}$, then $2^r + 2s < 2^{r+1}$. Let $n = 2^{r-1} + s + 2^{r+1}m$, $m \in \mathbb{N}$, $0 \leqslant s < 2^{r-1}$. Then

$$\begin{aligned}\omega(F) &= (1+\sigma_1+\sigma_2)^s(1+\sigma_1^{2^{r-1}}+\sigma_2^{2^{r-1}})(1+\sigma_1^{2^{r+1}}+\sigma_2^{2^{r+1}})^m(1+\sigma_1)^{-1}\\ &= (1+\sigma_1^{2^{r+1}}+\sigma_2^{2^{r+1}})^m(1+\sigma_1^{2^{r-1}}+\sigma_2^{2^{r-1}})\sum_{i=0}^{s}\binom{s}{i}(1+\sigma_1)^{i-1}\sigma_2^{s-i}.\end{aligned}$$

Hence $\omega_{2^r+2s} = \sigma_2^{2^{r-1}+s}$, if $0 \leqslant s < 2^{r-1}$.

3. Proof of Theorem 2

Note that according to [1], an additive basis for $H^*(F;\mathbb{Z}_2)$ is $\{x^iy^j \mid 0 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant n-2\}$, so that $\sigma_1^s \neq 0$, $1 \leqslant s \leqslant n-2$ and $\sigma_2^k \neq 0$, $1 \leqslant k \leqslant n-2$.

(i) From (1) in Section 2 above we have

$$\omega(F) = \sum_{i=0}^{n} \binom{n}{i} (1+\sigma_1)^{n-1-i}\sigma_2^i,$$

$$\omega_{n-2}(F) = \begin{cases} \displaystyle\sum_{i=0}^{m-1} \binom{2m}{m-1-i}\binom{m+i}{2i}\sigma_1^{2i}\sigma_2^{m-1-i}, & \text{if } n = 2m \text{ is even,} \\ \displaystyle\sum_{i=0}^{2m-1} \binom{4m+1}{2m-1-i}\binom{2m+1+i}{2i+1}\sigma_1^{2i+1}\sigma_2^{2m-1-i}, & \text{if } n = 4m+1. \end{cases}$$

Also $\omega_{n-2}(F) = \sum_{k=0}^{2m-2} a_k x^{2m-2-k}y^k$, if $n = 2m$ where a_k is either 0 or 1 and

$$a_0 = \text{coefficient of } x^{2m-2} = \binom{2m-1}{2m-2} = 1 \bmod 2.$$

Hence $\omega_{n-2}(F) \neq 0$, if n is even and so

$$\operatorname{span} \mathbb{R}F(1,1,n-2) \leqslant (2n-3)-(n-2) = n-1, \quad \text{if } n \text{ is even.}$$

If we put $\omega_{n-2}(F) = \sum_{k=0}^{4m-1} b_k x^{4m-1-k}y^k$ where $n = 4m+1$, then $b_1 =$ coefficient of $x^{4m-2}y$ in

$$\binom{4m}{4m-1}\sigma_1^{4m-1} + \binom{4m+1}{1}\binom{4m-1}{4m-3}\sigma_1^{4m-3}\sigma_2$$

is $0 + (4m+1)(4m-1)(4m-2)/2 = 1 \bmod 2$.

Hence $\omega_{n-2}(F) \neq 0$, if $n \equiv 1 \bmod 4$, and so $\operatorname{span} \mathbb{R}F(1,1,n-2) \leqslant n-1$, if $n \equiv 1 \bmod 4$. This completes the proof of (i).

(ii) $\omega_{n-3}(F) = \sum_{i=0}^{2m} \binom{4m+3}{2m-i}\binom{2m+2+i}{2i}\sigma_1^{2i}\sigma_2^{2m-i}$, if $n = 4m+3$.

If $\omega_{n-3}(F) = \sum_{k=0}^{4m} c_k x^{4m-k}y^k$, then

$$c_0 = \text{coefficient of } x^{4m} = \binom{4m+2}{4m} = (4m+2)(4m+1)/2 \equiv 1 \bmod 2.$$

Hence $\omega_{n-3}(F) \neq 0$, if $n \equiv 3 \bmod 4$, and so

$$\operatorname{span} \mathbb{R}F(1,1,n-2) \leqslant (2n-3)-(n-3)=n,$$

if $n \equiv 3 \bmod 4$. This proves (ii).

4. Proof of Theorem 3

(i) $\omega(\mathbb{R}F(1,1,4)) = (1+\sigma_1+\sigma_2)^6(1+\sigma_1+\sigma_1^2+\sigma_1^3+\ldots)$. Then $\omega_8(\mathbb{R}F(1,1,4)) = \sigma_2^4 \neq 0$, since $n=6$. Thus $\operatorname{span} \mathbb{R}F(1,1,4) \leqslant 1$. But by Theorem 1.3 in [2], $\operatorname{span} \mathbb{R}F(1,1,4) \geqslant 1$. Hence the result follows.

(ii) From Theorem 2 (i), $\operatorname{span} \mathbb{R}F(1,1,6) \leqslant 7$, when $n=8$. The result now follows since by Theorem 1.3 in [2], $\operatorname{span} \mathbb{R}F(1,1,6) \geqslant 7$.

Remark. Korbaš in [2] obtained $\operatorname{span} \mathbb{R}F(1,1,2)$ to be 3.

References

[1] *A. Borel*: La cohomologie mod 2 de certains espaces homogénes. Comment. Math. Helvetici *27* (1953), 165–197.

[2] *J. Korbaš*: Vector fields on real flag manifolds. Ann. Global Anal. Geom. *3* (1985), 173–184.

[3] *J. Korbaš*: Some partial formulae for Stiefel-Whitney classes of Grassmannians. Czechoslovak Math. J. *36 (111)* (1986), 535–540.

[4] *J. Korbaš*: Note on Stiefel-Whitney classes of flag manifolds. Rend. Circ. Mat. Palermo *2 (Suppl. 16)* (1987), 109–111.

[5] *J. Korbaš and P. Zvengrowski*: The vector field problem: A survey with emphasis on specific manifolds. Expo. Math. *12* (1994), 1–30.

[6] *K. Y. Lam*: A formula for the tangent bundle of flag manifolds and related manifolds. Trans. Amer. Math. Soc. *213* (1975), 305–314.

[7] *J. Milnor and J. Stasheff*: Characteristic Classes. Annals of Mathematics Studies vol. 76. Princeton Univ. Press, Princeton, 1974.

[8] *E. Thomas*: On tensor products of n-plane bundles. Arch. Math. (Basel) *X* (1959), 174–179.

[9] *E. Thomas*: Vector fields on manifolds. Bull. Amer. Math. Soc. *75* (1969), 643–683.

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