

Ladislav Nebeský

New proof of a characterization of geodetic graphs

Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 1, 33–39

Persistent URL: <http://dml.cz/dmlcz/127700>

Terms of use:

© Institute of Mathematics AS CR, 2002

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

NEW PROOF OF A CHARACTERIZATION
OF GEODETIC GRAPHS

LADISLAV NEBESKÝ, Praha

(Received January 20, 1999)

Abstract. In [3], the present author used a binary operation as a tool for characterizing geodetic graphs. In this paper a new proof of the main result of the paper cited above is presented. The new proof is shorter and simpler.

Keywords: geodetic graphs, shortest paths, binary operations

MSC 2000: 05C75, 05C12, 20N02

By a graph we will mean a graph in the sense of [1], i.e. a finite undirected graph without loops or multiple edges. Let G be a graph (with vertex set $V(G)$ and edge set $E(G)$). Then G is said to be *geodetic* if it is connected and there exists exactly one shortest $u - v$ path for each ordered pair of $u, v \in V(G)$.

Let G be a geodetic graph. Following [3], we say that $*$ is the *proper operation* of G if $*$ is the binary operation on $V(G)$ defined as follows:

$$u * v = u \text{ if } u = v,$$

$$u * v \text{ is the second vertex on the shortest } u - v \text{ path if } u \neq v.$$

for all $u, v \in V(G)$. Thus, if $*$ is the proper operation of G , then $\{x, x * y\} \in E(G)$ for all ordered pairs of distinct $x, y \in V(G)$.

Let G be a graph. Following [3], we say that $*$ is a binary operation *associated with* G if $*$ is a binary operation on $V(G)$ and

$$E(G) = \{\{u, v\}; u, v \in V(G), u \neq v, u * v = v \text{ and } v * u = u\}.$$

It is easy to see that if G is a geodetic graph, then the proper operation of G is associated with G .

The following theorem, which was proved in [3], gives a characterization of geodetic graphs and their proper operations:

Theorem. *Let G be a graph, and let $*$ be a binary operation associated with G . Put $U = V(G)$. Then G is geodetic and $*$ is the proper operation of G if and only if G is connected and $*$ satisfies the following Axioms (A)–(D):*

- (A) *if $u, v \in U$, then $(u * v) * u = u$;*
- (B) *if $u, v \in U$, then $u = v$ or $(u * v) * v \neq u$;*
- (C) *if $u, v \in U$, then $v * u = u$ or $u * (v * u) = u * v$;*
- (D) *if $u, v, w \in U$ and $w * v = v$, then $u * v = u * w$ or $w * (u * v) = v$.*

As was shown in [3], the condition that G is connected cannot be omitted in this theorem.

The proof of this theorem given in [3] is rather long and complicated. In the present paper we will give a new proof. This proof (including the proofs of the lemmas) is shorter and simpler.

The following lemma was presented in [3] without proof (note that only Axioms (A) and (B) are utilized in the proof):

Lemma 1. *Let $*$ be a binary operation on a nonempty set U , and let $*$ satisfy Axioms (A)–(D). Then*

$$u * v = u \quad \text{if and only if} \quad u = v$$

and

$$u * v = v \quad \text{if and only if} \quad v * u = u$$

for all $u, v \in U$.

The next lemma was proved in [3]:

Lemma 2. *Let $*$ be a binary operation on a nonempty set U , and let $*$ satisfy Axioms (A)–(D). Let $u_1, \dots, u_h, u_{h+1}, v, w \in U$, where $h \geq 1$. Assume that*

$$u_1 \neq u_2, \dots, u_h \neq u_{h+1}$$

and

$$u_2 = u_1 * v, \dots, u_{h+1} = u_h * v.$$

If $w * u_1 = v$, then

$$w * u_2 = \dots = w * u_{h+1} = v$$

and

$$u_2 = u_1 * w, \dots, u_{h+1} = u_h * w.$$

Proof (outlined). Consider g , $1 \leq g \leq h$, and assume that $u_g * v = u_{g+1}$ and $w * u_g = v$. Since $u_g \neq u_{g+1}$, Lemma 1 implies that $u_g \neq v$ and therefore, $w * u_g \neq u_g$. By Axiom (C), $u_g * (w * u_g) = u_g * w$. Hence $u_g * w = u_g * v = u_{g+1}$. Since $w * u_g \neq u_g$, Lemma 1 implies that $u_g * w \neq w$. By Axiom (C), $w * (u_g * w) = w * u_g$. Hence $w * u_{g+1} = w * u_g = v$.

Proceeding by the induction on g , we will prove the lemma. □

We will need three more lemmas.

Lemma 3. *Let $*$ be a binary operation on a nonempty set U , and let $*$ satisfy Axioms (A)–(D). Consider $u_1, \dots, u_{h+1} \in U$, where $h \geq 1$, such that*

$$u_1 \neq u_2, \dots, u_h \neq u_{h+1}$$

and

$$u_h = u_{h+1} * u_1, \dots, u_1 = u_2 * u_1.$$

Then

$$(1) \quad u_2 = u_1 * u_{h+1}, \dots, u_{h+1} = u_h * u_{h+1}.$$

Proof. We proceed by induction on h . If $h = 1$, the result follows from Lemma 1. Let $h \geq 2$. Since

$$u_{h-1} = u_h * u_1, \dots, u_1 = u_2 * u_1,$$

it follows from the induction hypothesis that

$$u_2 = u_1 * u_h, \dots, u_h = u_{h-1} * u_h.$$

Since $u_{h+1} * u_1 = u_h$, Lemma 2 implies that

$$u_2 = u_1 * u_{h+1}, \dots, u_h = u_{h-1} * u_{h+1}.$$

By Axiom (A), $u_{h+1} = (u_{h+1} * u_1) * u_{h+1}$. We get $u_{h+1} = u_h * u_{h+1}$. Hence (1) holds. □

The next lemma is similar to Lemma 5 of [3], but our proof will be different and shorter.

Lemma 4. *Let G be a connected graph, let $*$ be a binary operation associated with a connected graph G , and let $*$ satisfy Axioms (A)–(D). Consider arbitrary*

distinct $u, v \in V(G)$. Then there exist pairwise distinct $u_1, \dots, u_{m+1} \in V(G)$, $m \geq 1$, such that $u_1 = u$, $u_{m+1} = v$ and

$$u_2 = u_1 * v, \dots, u_{m+1} = u_m * v.$$

P r o o f. Suppose, to the contrary, that the lemma is false. Since $V(G)$ is finite, it is easy to see that there exist $v_1, \dots, v_{k+1} \in V(G)$, where $k \geq 1$, such that $v_1 = u$,

$$(2) \quad v_2 = v_1 * v \neq v, \dots, v_{k+1} = v_k * v \neq v$$

and there exists j , $1 \leq j \leq k$, with the property that $v_j = v_{k+1}$ and the vertices v_j, v_{j+1}, \dots, v_k are pairwise distinct. By virtue of Lemma 1, $j < k$. Let d denote the distance function of G . For each $w \in V(G)$, we denote

$$e(w) = \min\{d(w, v_i); j \leq i \leq k\}.$$

Moreover, we denote by Z the set of all $z \in V(G)$ such that

$$v_{j+1} = v_j * z, \dots, v_{k+1} = v_k * z.$$

As follows from (2), $v \in Z$. Consider an arbitrary $x \in Z$ such that

$$(3) \quad e(x) \leq e(z) \quad \text{for all } z \in Z.$$

Since $x \in Z$, Lemma 1 implies that $e(x) \geq 1$. Therefore, there exists $y \in V(G)$ such that $e(y) = e(x) - 1$ and $\{x, y\} \in E(G)$. By Lemma 1, $y * x = x$. By (3), $y \notin Z$. Recall that $v_{k+1} = v_j$. Without loss of generality, we may assume that $v_{k+1} \neq v_k * y$. Thus $v_k * x \neq v_k * y$. By Axiom (D),

$$x = y * (v_k * x) = y * v_{k+1} = y * v_j.$$

Since $v_j \neq v_{j+1}, \dots, v_k \neq v_{k+1}$, Lemma 2 implies that

$$v_{j+1} = v_j * y, \dots, v_k * y = v_{k+1}.$$

Thus $y \in Z$, which is a contradiction. □

Let G be a graph, let $*$ be a binary operation associated with G , and let $*$ satisfy Axioms (A)–(D). We will say that

$$(u_1, \dots, u_k, u_{k+1})$$

is a $(*)$ -path in G , if $k \geq 1$, u_1, \dots, u_k, u_{k+1} are pairwise distinct vertices of G , and

$$u_2 = u_1 * u_{k+1}, \dots, u_{k+1} = u_k * u_{k+1}.$$

Obviously, every $(*)$ -path in G is a path in G .

Let G be a connected graph, and let d denote its distance function. For every ordered pair $x, y \in V(G)$, we denote

$$A_G(x, y) = \{x\}, \quad \text{if } x = y$$

and

$$A_G(x, y) = \{z \in V(G); d(x, z) = 1 \text{ and } d(z, y) = d(x, y) - 1\}, \text{ if } x \neq y.$$

(Note that if $x \neq y$, then $A_G(x, y)$ is $N_1(x, y)$ in the sense of [2].)

The next lemma is the main one:

Lemma 5. *Let G be a connected graph, let $*$ be a binary operation associated with G , and let $*$ satisfy Axioms (A)–(D). Consider arbitrary $u, v \in V(G)$. Then*

$$(4) \quad A_G(u, v) = \{u * v\}.$$

Proof. Let P_* denote the set of all $(*)$ -paths in G , and let d denote the distance function of G . Put $n = d(u, v)$. We proceed by induction on n . Clearly, if $n \leq 1$, then (4) holds. Let $n \geq 2$. We assume that

$$(5) \quad A_G(x, y) = \{x * y\} \text{ for all } x, y \in V(G) \quad \text{such that } d(x, y) < n.$$

Suppose, to the contrary, that $A_G(u, v) \neq \{u * v\}$. Then there exists $w \in A_G(u, v)$ such that $w \neq u * v$. By Lemma 4, there exist $u_1, \dots, u_m, u_{m+1} \in V(G)$ such that $m \geq n$, $u_1 = u$, $u_{m+1} = v$, and $(u_1, \dots, u_m, u_{m+1}) \in P_*$. Since $w \in A_G(u, v)$, there exist $u_{m+2}, \dots, u_{m+n}, u_{m+n+1} \in V(G)$ such that $u_{m+n} = w$, $u_{m+n+1} = u$ and $(u_{m+n+1}, u_{m+n}, \dots, u_{m+1})$ is a shortest $u - v$ path in G .

Obviously, $u_{m+n+1} = u_1$. Put $u_{m+n+2} = u_2, u_{m+n+3} = u_3, \dots, u_{m+2n+1} = u_{n+1}$. Define

$$\alpha_h = (u_h, \dots, u_{h+m-1}, u_{h+m}) \quad \text{and} \quad \beta_h = (u_{h+m+n}, u_{h+m+n-1}, \dots, u_{h+m})$$

for each $h = 1, \dots, n+1$.

Obviously, $\alpha_1 \in P_*$. Since $w \neq u * v$ and $u_2 = u * v$, we have $u_2 \neq u_{m+n}$ and $\beta_1 \notin P_*$.

Since $\alpha_1 \in P_*$, Lemma 3 implies that

$$(u_{m+1}, u_m, \dots, u_2, u_1) \in P_*.$$

Therefore, $(u_m, \dots, u_2, u_1) \in P_*$. Applying Lemma 3 again, we have

$$(u_1, u_2, \dots, u_m) \in P_*.$$

Hence $u_2 = u_1 * u_m$.

Clearly, $w \in A_G(u, u_{m+2})$. Since $d(u, u_{m+2}) = n - 1$, it follows from (5) that $A_G(u, u_{m+2}) = \{u * u_{m+2}\}$ and thus $w = u * u_{m+2}$. Since $w \neq u_2 = u * u_m$, we get $u_m \neq u_{m+2}$.

Recall that α_1 and β_1 are paths in G . Thus

$$(6) \quad u_3 \neq u_1, u_4 \neq u_2, \dots, u_{m+n+2} \neq u_{m+n}.$$

Assume that $\alpha_{n+1} \in P_*$. Then also $(u_{m+1}, \dots, u_{m+n}, u_{m+n+1}) \in P_*$ and, by Lemma 3, $\beta_1 \in P_*$; a contradiction. Thus $\alpha_{n+1} \notin P_*$. Since $\alpha_1 \in P_*$ and $\beta_1 \notin P_*$, there exists i , $1 \leq i \leq n$ such that

$$(7) \quad \alpha_i \in P_* \text{ and } \beta_i \notin P_*,$$

and

$$(8) \quad \alpha_{i+1} \notin P_* \text{ or } \beta_{i+1} \in P_*.$$

Let $\alpha_{i+1} \in P_*$. By (8), $\beta_{i+1} \in P_*$. Since $u_{i+1} = u_{i+m+n+1}$ and $u_{i+2} = u_{i+m+n+2}$, we have $u_{i+m+n+2} = u_{i+2} = u_{i+1} * u_{i+m+1} = u_{i+m+n+1} * u_{i+m+1} = u_{i+m+n}$, which contradicts (6). Thus $\alpha_{i+1} \notin P_*$.

By (7), $\alpha_i \in P_*$. This implies that $u_{i+1} = u_i * u_{i+m}$ and

$$u_{i+2} = u_{i+1} * u_{i+m}, \dots, u_{i+m} = u_{i+m-1} * u_{i+m}.$$

Let $u_{i+m} = u_{i+m+1} * u_{i+1}$. By Lemma 2,

$$u_{i+2} = u_{i+1} * u_{i+m+1}, \dots, u_{i+m} = u_{i+m-1} * u_{i+m+1}.$$

Since $u_{i+m+1} = u_{i+m} * u_{i+m+1}$, we get $\alpha_{i+1} \in P_*$, which is a contradiction. Thus $u_{i+m} \neq u_{i+m+1} * u_{i+1}$.

Since $u_{i+1} = u_i * u_{i+m}$, we get $u_{i+m} \neq u_{i+m+1} * (u_i * u_{i+m})$. By Axiom (D),

$$(9) \quad u_i * u_{i+m+1} = u_i * u_{i+m}.$$

Since $u_i = u_{i+m+n}$, we have $d(u_i, u_{i+m+1}) \leq n - 1$.

First, let $d(u_i, u_{i+m+1}) = n - 1$. By (5), $A_G(u_{i+m+n}, u_{i+m+1}) = \{u_{i+m+n} * u_{i+m+1}\}$ and thus $u_{i+m+n} * u_{i+m+1} = u_{i+m+n-1}$. It follows from (9) that $u_{i+m+n+1} = u_{i+m+n-1}$, which contradicts (6).

Now, let $d(u_i, u_{i+m+1}) < n - 1$. Thus $d(u_i, u_{i+m}) < n$. Applying (5) step by step, we see that α_i is a shortest $u_i - u_{i+m}$ path in G . We get $n \leq m = d(u_i, u_{i+m}) < n$; a contradiction.

Thus (4) holds. □

P r o o f of the theorem. Put $U = V(G)$. Let G be geodetic and let $*$ be its proper operation. Then G is connected. It is easy to see that $*$ satisfies Axioms (A), (B) and (C). Moreover, it is not difficult to show that $*$ satisfies also Axiom (D); this verification can be found in [3].

Conversely, let G be connected and let $*$ satisfy Axioms (A)–(D). By Lemma 5, $|A_G(x, y)| = 1$, for all $x, y \in U$. It is easy to prove, by induction on $d(x, y)$, that G is geodetic. By Lemma 5, $A_G(x, y) = \{x * y\}$ for all $x, y \in U$. This implies that $*$ is the proper operation of G , which completes the proof of the theorem. □

Remark. Obviously, every tree is a geodetic graph. For trees, a stronger result can be found in [4].

References

- [1] *G. Chartrand and L. Lesniak*: Graphs & Digraphs. Third edition. Chapman & Hall, London, 1996.
- [2] *H. M. Mulder*: The Interval Function of a Graph. Mathematisch Centrum, Amsterdam, 1980.
- [3] *L. Nebeský*: An algebraic characterization of geodetic graphs. Czechoslovak Math. J. 48 (123) (1998), 701–710.
- [4] *L. Nebeský*: A tree as a finite nonempty set with a binary operation. Math. Bohem. 125 (2000), 455–458.

Author's address: Univerzita Karlova v Praze, Filozofická fakulta, nám. J. Palacha 2, 116 38 Praha 1, Czech Republic.