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TWO EXTENSION THEOREMS. MODULAR FUNCTIONS ON COMPLEMENTED LATTICES

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Abstract. We prove an extension theorem for modular functions on arbitrary lattices and an extension theorem for measures on orthomodular lattices. The first is used to obtain a representation of modular vector-valued functions defined on complemented lattices by measures on Boolean algebras. With the aid of this representation theorem we transfer control measure theorems, Vitali-Hahn-Saks and Nikodým theorems and the Liapunoff theorem about the range of measures to the setting of modular functions on complemented lattices.

Keywords: complemented lattices, orthomodular lattices, exhaustive modular functions, measures, extension, Vitali-Hahn-Saks theorem, Nikodým theorems, Liapunoff theorem

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Introduction

The basic result of this paper is the extension theorem 2.1 for modular functions defined on an arbitrary lattice with values in a complete Hausdorff locally convex linear space E. This result is used in Section 3.1 to obtain an isomorphism between the space of all E-valued exhaustive modular functions defined on a complemented (or sectionally complemented or relatively complemented) lattice L and the space of all E-valued order continuous measures on a complete Boolean algebra, namely on the center $C(\tilde{L})$ of a uniform completion $\tilde{L}$ of L. (Observe that any orthomodular lattice is complemented and relatively complemented and therefore sectionally complemented). This isomorphism allows us to transfer results known for measures on Boolean algebras to the case of modular functions on complemented lattices. This is done for a decomposition theorem (Section 3.2), control measure theorems (Section 3.3), Vitali-Hahn-Saks and Nikodým theorems (Section 3.4) and Liapunoff type theorems about the range of measures (Section 3.5). Versions of some results of Sections 3.3, 3.4 and 3.5 are already contained in [1], [2] [3] and [5]; these results are there obtained—in contrast to our method indicated above—with similar methods as in the Boolean case.

The proof of the extension Theorem 2.1 is organized in a way that it yields at the same time an extension theorem for measures with values in complete Hausdorff locally convex linear spaces on orthomodular lattices (Theorem 2.2). This extension theorem generalizes an extension theorem [4, 2.5] of Avallone and Hamhalter for measures of bounded variation with values in Banach spaces with the Radon-Nikodým property.

1. Preliminaries

Throughout the paper let L be a lattice, (E,τ) a complete Hausdorff locally convex linear space and E' its continuous dual.

If L is *bounded*, i.e. if L has the smallest and the greatest element, we denote these elements, respectively, by 0 and 1. The *center* $C(L)$ of a bounded lattice L is the set of the elements $c \in L$ for which there is an element $c' \in L$ such that $\varphi(x) = (x \wedge c, x \wedge c')$ defines a lattice isomorphism from L onto $[0,c] \times [0,c']$. $C(L)$ is a Boolean sublattice of L.

A *lattice uniformity* is a uniformity on a lattice which makes the lattice operations $\vee$ and $\wedge$ uniformly continuous; a lattice endowed with a lattice uniformity is called a *uniform lattice*. A lattice uniformity u on L is called (σ-)*order continuous* if order convergence of a monotone net (sequence) implies topological convergence in (L,u), and *exhaustive* if every monotone sequence is Cauchy in (L,u).

Any Hausdorff uniform lattice (L,u) is a sublattice and a dense subspace of a Hausdorff uniform lattice $(\tilde{L},\tilde{u})$ which is complete as a uniform space; $(\tilde{L},\tilde{u})$ is called the *completion* of (L,u).

Theorem 1.1. *Let u be a Hausdorff exhaustive lattice uniformity on L and $(\tilde{L},\tilde{u})$ the completion of (L,u).*

(a) *Then $(\tilde{L},\leqslant)$ is a complete lattice and $\tilde{u}$ is order continuous. (See* [17, 6.15] *or the Russian paper of Kiseleva cited there.)*

(b) *$C(\tilde{L})$ is a complete Boolean sublattice of $\tilde{L}$, i.e.* $\sup M$, $\inf M \in C(\tilde{L})$ *for $M \subset C(\tilde{L})$* [21, 3.4].

(c) *If L is a complemented or a sectionally complemented or a relatively complemented modular lattice, then $\tilde{L}$ is a complemented modular lattice* [21, 4.2].

Theorem 1.2 [17, 6.3]**.** *Let u be a Hausdorff lattice uniformity on L. Then (L,u) is a complete uniform space and u is exhaustive iff $(L,\leqslant)$ is a complete lattice and u is order continuous.*

Let $\mu\colon L \to E$ be a function which is *modular*, i.e. $\mu(x \vee y) + \mu(x \wedge y) = \mu(x) + \mu(y)$ for $x, y \in L$. If $(p_\alpha)_{\alpha \in A}$ is a system of seminorms generating the topology of E, then

$$d_\alpha(a,b) = \sup\{p_\alpha\big(\mu(x) - \mu(y)\big)\colon\ x, y \in L, \quad a \wedge b \leqslant x \leqslant y \leqslant a \vee b\}, \quad \alpha \in A,$$

defines a family of pseudometrics on L generating the *μ-uniformity*, i.e. the weakest lattice uniformity making μ uniformly continuous (see [10], [19, 3.1]). The topology induced by the μ-uniformity is called the *μ-topology*. If μ is an increasing (= isotone) real-valued modular function on L, then the μ-uniformity is generated by the pseudometric $(a,b) \mapsto d(a,b) = \mu(a \vee b) - \mu(a \wedge b)$ introduced in [7, section X.1]. If Λ is a set of modular functions on L, then the supremum of the λ-uniformities, $\lambda \in \Lambda$, is called the *Λ-uniformity* or the *uniformity generated by* Λ.

From the description of the μ-uniformity given above one easily obtains:

Proposition 1.3 [19, 3.2]**.** *Let u be a lattice uniformity on L. Then a modular function $\mu\colon (L,u) \to E$ is continuous iff the μ-topology is coarser than the u-topology (i.e. the topology induced by u).*

Proposition 1.4 (see [17, section 1.2] and [19, 2.5])**.** *Let u be a lattice uniformity on L.*

(a) *Then $N(u) := \bigcap\limits_{U \in u} U$ is a congruence relation on L and the quotient $(\hat{L}, \hat{u}) := (L,u)/N(u)$ is a Hausdorff uniform lattice.*

(b) *If $\mu\colon L \to E$ is a modular function and u the μ-uniformity, then $\hat{L}$ is modular and $N(u) = N(\mu)$ where*

$$N(\mu) = \{(x,y) \in L^2\colon\ \mu \text{ is constant on } [x \wedge y, x \vee y]\}.$$

(c) *If $\mu\colon L \to E$ is a modular function and $N(u) \subset N(\mu)$, then $\mu(\hat{x}) = \mu(x)$ $(x \in \hat{x} \in \hat{L})$ defines a modular function $\hat{\mu}$ on $\hat{L}$.*

The modularity of $\hat{L}$ in 1.4 (b) was proved in [10] generalizing [7, Theorem X.2.2].

A modular function $\mu\colon L \to E$ is (σ-)*order continuous* if $\lim \mu(x_\alpha) = \mu(x)$ for every monotone net (sequence) (x_α) with order limit x, and *exhaustive* if $\big(\mu(x_n)\big)$ is a Cauchy sequence for every monotone sequence (x_n) in L. By [19, 3.5 and 3.6], μ is σ-order continuous, order continuous or exhaustive, respectively, iff the μ-uniformity is σ-order continuous, order continuous or exhaustive.

Theorem 1.5 [20, 1.1.4]**.** *If $\mu\colon L \to E$ is a σ-order continuous modular function, L σ-complete and E metrizable, then L is complete with respect to the μ-uniformity.*

Theorem 1.6 [19, 6.3]**.** *Let $\mu\colon L \to (E,\tau)$ be an exhaustive modular function and u a lattice uniformity on L. Then $\mu\colon (L,u) \to (E,\tau)$ is continuous iff*

$\mu\colon (L,u)\to(E,\sigma(E,E'))$ *is continuous iff* $x'\circ\mu\colon (L,u)\to\mathbb{R}$ *is continuous for every* $x'\in E'$.

The *total variation* $|\mu|\colon L\to[0,+\infty]$ of a Banach space valued function μ on L is defined by

$$|\mu|(a) := \sup\left\{\sum_{i=1}^{n}\|\mu(x_i)-\mu(x_{i-1})\|\colon\ n\in N,\quad x_i\in L,\quad x_0\leqslant x_1\leqslant\ldots\leqslant x_n=a\right\}.$$

Proposition 1.7. *Let E be a Banach space, $(\mu_\gamma)_{\gamma\in\Gamma}$ a net of E-valued functions on L converging pointwise to a function μ with respect to the weak topology $\sigma(E,E')$, and $(\nu_\gamma)_{\gamma\in\Gamma}$ a net of monotone $[0,+\infty]$-valued functions on L converging pointwise to a function ν. If $\|\mu_\gamma(q)-\mu_\gamma(p)\|+\nu_\gamma(p)\leqslant\nu_\gamma(q)$ holds for every $\gamma\in\Gamma$ and $p,q\in L$ with $p\leqslant q$, then $|\mu|\leqslant\nu$.*

P r o o f. We first show that

$$(*)\qquad \|\mu(q)-\mu(p)\|+\nu(p)\leqslant\nu(q)\quad\text{for any}\quad p,q\in L\quad\text{with}\quad p\leqslant q.$$

For $x'\in E'$ with $\|x'\|=1$ we have

$$x'(\mu_\gamma(q)-\mu_\gamma(p))+\nu_\gamma(p)\leqslant\|\mu_\gamma(q)-\mu_\gamma(p)\|+\nu_\gamma(p)\leqslant\nu_\gamma(q).$$

For the limit functions we therefore get

$$x'(\mu(q)-\mu(p))+\nu(p)\leqslant\nu(q).$$

Now choosing x' with $x'(\mu(q)-\mu(p))=\|\mu(q)-\mu(p)\|$ we obtain the desired inequality.

Let $a\in L$ and $x_i\in L$ with $x_0\leqslant x_1\leqslant\ldots\leqslant x_n=a$. Then we have by $(*)$

$$\sum_{i=1}^{n}\|\mu(x_i)-\mu(x_{i-1})\|+\sum_{i=1}^{n}\nu(x_{i-1})\leqslant\sum_{i=1}^{n}\nu(x_i)$$

and therefore $\sum\limits_{i=1}^{n}\|\mu(x_i)-\mu(x_{i-1})\|\leqslant\nu(x_n)=\nu(a)$; hereby observe that $\nu(x_i)$ is finite if $\nu(a)$ is finite by the monotonicity of ν. It follows that $|\mu|(a)\leqslant\nu(a)$. □

Proposition 1.7 contains in particular the following statement about the total variation of measures on orthomodular lattices. Recall that a function μ on an orthomodular lattice L is a *measure* if $\mu(x+y)=\mu(x)+\mu(y)$ for orthogonal elements $x,y\in L$.

If $(\mu_\gamma)_{\gamma\in\Gamma}$ is a net of Banach space valued measures on an orthomodular lattice L converging pointwise to a function μ with respect to the weak topology $\sigma(E,E')$ and $\nu\colon L\to[0,+\infty]$ is a measure such that $|\mu_\gamma|\leqslant\nu$ for every $\gamma\in\Gamma$, then $|\mu|\leqslant\nu$.

Proposition 1.8. *Let E be a Banach space and $\mu\colon L \to E$ a modular function.*

(a) *Then $|\mu|$ is a modular function* [20, 1.3.10].

(b) *If $|\mu|$ is bounded, then μ is exhaustive. If μ is exhaustive, then μ is bounded* [19, section 2].

2. Extension of modular functions

The main result of this section is the following extension theorem for modular functions.

Theorem 2.1. *Let L be a bounded lattice, A a Boolean sublattice of $C(L)$, Λ a set of increasing real-valued modular functions on L, u the Λ-uniformity and $\mu\colon (A,u) \to (E,\tau)$ a continuous measure. Then μ has an extension to an exhaustive continuous modular function $\overline{\mu}\colon (L,u) \to (E,\tau)$ with $\overline{\mu}(L) \subset \overline{\mathrm{co}}\,\mu(A)$.*

A result of this type was presented in [18, 3.2] for L being complemented and μ σ-additive and real-valued.

We organize the proof of 2.1 in a way that it yields at the same time a similar extension theorem (Theorem 2.2) for measures on orthomodular lattices.

Theorem 2.2. *Let L be an orthomodular lattice, A a Boolean sublattice of $C(L)$, Λ a set of positive real-valued measures on L, $\Lambda_0 := \{\lambda|A\colon \lambda \in \Lambda\}$ the set of the restrictions on A of the measures $\lambda \in \Lambda$, u_0 the Λ_0-uniformity and $\mu\colon (A,u_0) \to (E,\tau)$ a continuous measure. Then μ has an extension to a measure $\overline{\mu}\colon L \to E$ with $\overline{\mu}(L) \subset \overline{\mathrm{co}}\,\mu(A)$.*

Theorem 2.2 generalizes the extension theorem [4, 2.5] of Avallone and Hamhalter; in [4] it is additionally assumed that E is a Banach space with the Radon-Nikodým property and μ has bounded variation. [4, 2.5] is for its part a generalization of the extension theorem [18, 3.3] for σ-additive real-valued measures. What is new here in 2.2 (and also in 2.1) is in particular the statement about the range of $\overline{\mu}$, namely that $\overline{\mu}(L) \subset \overline{\mathrm{co}}\,\mu(A)$. This becomes important in Section 3.5. In particular, if in 2.1 or in 2.2 the range of μ is closed and convex, then $\overline{\mu}(L) = \mu(A)$; if E is a locally convex-solid Riesz space and μ is positive, then $\overline{\mu}$ is positive as well.

In the proof of 2.1 and 2.2 we use the following facts.

Proposition 2.3. *Let C be a convex subset of E and $0 \in C$. If $x_1,\dots,x_n \in E$ and $\sum\limits_{i\in I} x \in C$ for any $I \subset \{1,\dots,n\}$, then $\sum\limits_{i=1}^{n} \alpha_i x_i \in C$ for $0 \leqslant \alpha_i \leqslant 1$.*

Proof. $\sum\limits_{i=1}^{n} \alpha_i x_i \in \sum\limits_{i=1}^{n} \mathrm{co}\{0,x_i\} = \mathrm{co}\sum\limits_{i=1}^{n}\{0,x_i\} \subset C.$ □

Proposition 2.4. *Let $\lambda, \nu\colon B \to \mathbb{R}$ be σ-additive real-valued measures on a σ-complete Boolean algebra B, λ strictly positive and $\varepsilon > 0$. Then there is a finite decomposition $d_0, \ldots, d_n \in B$ of the maximal element of B and $k \in \mathbb{N}$ such that $|\nu|(d_0) \leqslant \varepsilon$ and $(-k + i\varepsilon - \varepsilon)\lambda(x) \leqslant \nu(x) \leqslant (-k + i\varepsilon)\lambda(x)$ for $B \ni x \leqslant d_i$ and $i = 1, \ldots, n$.*

Sketch of the proof. Let $\alpha \in \mathbb{R}$; the Hahn decomposition theorem for the measures $\alpha\lambda - \nu$ yields an element $x_\alpha \in B$ such that $\nu(x) \leqslant \alpha\lambda(x)$ for $B \ni x \leqslant x_\alpha$ and $\nu(x) \geqslant \alpha\lambda(x)$ for $B \ni x \leqslant x'_\alpha$ where x'_α denotes the complement of x_α. Let $k, n \in \mathbb{N}$ with $n \geqslant 2k/\varepsilon$, $d_i = x_{-k+i\varepsilon} \wedge x'_{-k+i\varepsilon-\varepsilon}$ for $i = 1, \ldots, n$ and let d_0 be the complement of $d_1 \vee \ldots \vee d_n$. If n is large enough, then $|\nu|(d_0) \leqslant \varepsilon$ and the d_i's have the desired properties. $\square$

Proposition 2.5 (See [19, 4.1])**.** *Let $\mu\colon L \to E$ be a modular function.*

(a) *If μ is exhaustive, then $\mu(L)$ is relatively weakly compact.*

(b) *If K is a weakly compact subset of E such that $\sum\limits_{i \in I} \mu(x_i) - \mu(x_{i-1}) \in K$ for any finite chain $x_0 < \ldots < x_n$ in L and $I \subset \{1, \ldots, n\}$, then μ is exhaustive.*

2.5 (b) is slightly stronger than [19, 4.1 (b)]; but the proof of [19, 4.1 (b)] works without any change also in the situation of 2.5 (b). 2.5 (a) generalizes the well known fact—also used in the following proof—that any E-valued σ-additive measure on a σ-complete Boolean algebra has a relatively weakly compact range.

Proof of 2.1 and 2.2. Replacing in 2.1 any $\lambda \in \Lambda$ by $\lambda - \lambda(0)$, we may assume that $\lambda(0) = 0$ for $\lambda \in \Lambda$. To unify the proof of 2.1 and 2.2, we first assume that L is an arbitrary lattice and Λ a set of increasing functions $\lambda\colon L \to [0, +\infty[$ such that $\lambda(x_1 \vee x_2) = \lambda(x_1) + \lambda(x_2)$ whenever $x_i \in L$, $z_i \in A$, $x_i \leqslant z_i$ and $z_1 \wedge z_2 = 0$. This is satisfied under the assumptions of 2.1 or 2.2.

Let u_0 be the uniformity on A generated by the restrictions $\lambda\big|A$ $(\lambda \in \Lambda)$, $N := \overline{\{0\}}$ the closure of $\{0\}$ in (A, u_0), $(\tilde{A}, \tilde{u}_0)$ the completion of the quotient $(\hat{A}, \hat{u}_0) := (A, u_0)/N$ and $\pi\colon A \to \hat{A}$ the quotient map. Set $\hat{a} := \pi(a)$ for $a \in A$. Then $\hat{\lambda}(\hat{a}) := \lambda(a)$ defines for $\lambda \in \Lambda$ a $\hat{u}_0$-continuous measure on $\hat{A}$. Let $\tilde{\lambda}$ be the unique $\tilde{u}_0$-continuous extension of $\hat{\lambda}$ to $\tilde{A}$ and, analogously, let $\tilde{\mu}\colon \tilde{A} \to E$ be the unique $\tilde{u}_0$-continuous measure with $\tilde{\mu} \circ \pi = \mu$.

For simplicity we suppose in the first step that there is a $z \in \tilde{A}$ and a $\lambda \in \Lambda$ such that $\tilde{\lambda}(x) > 0$ for $x \in \tilde{A}$ with $0 < x \leqslant z$ and $\tilde{\mu}(x) = 0$ for $x \in \tilde{A}$ with $x \wedge z = 0$.

For any $p \in L$, $\lambda_p(a) := \lambda(p \wedge a)$ defines a measure on A; λ_p is u_0-continuous since $0 \leqslant \lambda_p \leqslant \lambda\big|A$. Let $\tilde{\lambda}_p\colon \tilde{A} \to [0, +\infty[$ be the unique $\tilde{u}_0$-continuous measure such that $\tilde{\lambda}_p \circ \pi = \lambda_p$. For any finite decomposition $D \subset \tilde{A} \setminus \{0\}$ of z and $p \in L$ we define

$$\mu_D(p) := \sum_{d \in D} (\tilde{\lambda}_p(d)/\tilde{\lambda}(d)) \cdot \tilde{\mu}(d).$$

By 2.3, $\mu_D(L) \subset \operatorname{co} \tilde{\mu}(\tilde{A}) \subset \overline{\operatorname{co}}\, \mu(A)$ since $0 \leqslant \tilde{\lambda}_p(d)/\tilde{\lambda}(d) \leqslant 1$.

We show that $(\mu_D(p))_D$ is weakly Cauchy uniform in $p \in L$, where D runs in the system of all finite decompositions of z contained in $\tilde{A} \setminus \{0\}$ directed by refinement. Let $x' \in E'$ and $\varepsilon > 0$. For $\tilde{\nu} = x' \circ \tilde{\mu}$ and $\tilde{\lambda}$ choose a decomposition $d_0, \dots, d_n \in \tilde{A}$ of z and k according to 2.4; ν, λ and B of 2.4 correspond here to $\tilde{\nu}$, $\tilde{\lambda}$ and $\{x \in \tilde{A} \colon x \leqslant z\}$. Let $D \subset \tilde{A} \setminus \{0\}$ be a finite decomposition of z which is a refinement of $D_0 := \{d_0, \dots, d_n\} \setminus \{0\}$. To simplify the notation we agree for the following calculation that $0/0 := 0$. For any $p \in L$ we have

$$\begin{aligned}
&|x'\mu_D(p) - x'\mu_{D_0}(p)| \\
&\quad = \Big| \sum_{i=0}^{n} \sum_{d_i \geqslant h \in D} \tilde{\lambda}_p(h) \cdot x'\tilde{\mu}(h)/\tilde{\lambda}(h) - \sum_{i=0}^{n} \tilde{\lambda}_p(d_i) \cdot x'\tilde{\mu}(d_i)/\tilde{\lambda}(d_i) \Big| \\
&\quad \leqslant \Big| \sum_{i=1}^{n} \sum_{d_i \geqslant h \in D} \tilde{\lambda}_p(h) \cdot [x'\tilde{\mu}(h)/\tilde{\lambda}(h) - x'\tilde{\mu}(d_i)/\tilde{\lambda}(d_i)] \Big| \\
&\qquad + \sum_{d_0 \geqslant h \in D} |x'\tilde{\mu}(h)| \cdot \tilde{\lambda}_p(h)/\tilde{\lambda}(h) + |x'\tilde{\mu}(d_0)| \cdot \tilde{\lambda}_p(d_0)/\tilde{\lambda}(d_0).
\end{aligned}$$

The number in the brackets $[\dots]$ is the difference of two numbers of the interval $[-k + i\varepsilon - \varepsilon, -k + i\varepsilon]$; therefore its absolute value is $\leqslant \varepsilon$. It follows that

$$\begin{aligned}
|x'\mu_D(p) - x'\mu_{D_0}(p)| &\leqslant \sum_{i=1}^{n} \sum_{d_i \geqslant h \in D} \tilde{\lambda}(h) \cdot \varepsilon + \sum_{d_0 \geqslant h \in D} |x'\tilde{\mu}|(h) + |x'\tilde{\mu}|(d_0) \\
&\leqslant \tilde{\lambda}(1) \cdot \varepsilon + 2|x'\tilde{\mu}|(d_0) \leqslant \big(\tilde{\lambda}(1) + 2\big) \cdot \varepsilon.
\end{aligned}$$

We have proved that, for $p \in L$, $(\mu_D(p))$ is a weak Cauchy net in $\overline{\operatorname{co}}\, \mu(A)$. Since $\tilde{\mu}(\tilde{A})$ is relatively weakly compact and therefore by a theorem of Krein $K := \overline{\operatorname{co}}\, \mu(A) = \overline{\operatorname{co}}\, \tilde{\mu}(\tilde{A})$ is weakly compact, $(\mu_D(p))_D$ has a weak limit $\overline{\mu}(p)$ in $\overline{\operatorname{co}}\, \mu(A)$.

We now show that $\overline{\mu}$ is an extension of μ. We need here that $\tilde{\lambda}_p(d) = \tilde{\lambda}(\hat{p} \wedge d)$ for $p \in A$ and $d \in \tilde{A}$: In fact, for $p \in A$ the measures $\tilde{\lambda}_p$ and $\tilde{A} \ni d \mapsto \tilde{\lambda}(\hat{p} \wedge d)$ are equal since they are $\tilde{u}_0$-continuous and coincide on the dense subalgebra $\hat{A}$. In particular, for $d \in \tilde{A}$ and $p \in A$, we have $\tilde{\lambda}_p(d) = \tilde{\lambda}(\hat{p} \wedge d) = \tilde{\lambda}(d)$ if $d \leqslant \hat{p}$, and $\tilde{\lambda}_p(d) = \tilde{\lambda}(0) = 0$ if $\hat{p} \wedge d = 0$. Let $p \in A$ and let D be a finite decomposition of z in $\tilde{A} \setminus \{0\}$ such that for each $d \in D$ we have $d \leqslant z \wedge \hat{p}$ or $d \leqslant z \setminus \hat{p}$. Then

$$\mu_D(p) = \sum_{z \wedge \hat{p} \geqslant d \in D} \tilde{\mu}(d) = \tilde{\mu}(\hat{p} \wedge z) = \tilde{\mu}(\hat{p}) = \mu(p)$$

and therefore $\overline{\mu}(p) = \mu(p)$.

If L is an orthomodular lattice and λ a measure, then $\tilde{\lambda}_p(d) + \tilde{\lambda}_q(d) = \tilde{\lambda}_{p\vee q}(d)$ for orthogonal elements $p, q \in L$ and $d \in \hat{A}$. By the continuity of the functions $\tilde{\lambda}_p$, $\tilde{\lambda}_q$, $\tilde{\lambda}_{p\vee q}$, this equality holds also for any $d \in \tilde{A}$. Using this fact, one immediately sees that the μ_D's are measures and therefore the limit $\overline{\mu}$ is a measure.

Now suppose that the functions in Λ are modular. Let $(\tilde{L}, \tilde{u})$ be the completion of $(\hat{L}, \hat{u}) := (L, u)/N(u)$. We may assume that $(\tilde{A}, \tilde{u}_0)$ is a subspace of $(\tilde{L}, \tilde{u})$. The continuous extension on $(\tilde{L}, \tilde{u})$ of the map $(\hat{L}, \hat{u}) \ni \hat{x} \mapsto \lambda(x)$, where $x \in \hat{x}$, extends $\tilde{\lambda}$. If we denote this extension also by $\tilde{\lambda}$, then $\tilde{\lambda}_p(d) = \tilde{\lambda}(\hat{p} \wedge d)$ for $p \in L$ and $d \in \tilde{A}$. Using this fact one easily sees that the μ_D's are u-continuous modular functions. Since $(\mu_D(a))$ converges to $\overline{\mu}(a)$ uniformly in $a \in L$ with respect to the weak topology $\sigma(E, E')$, also the limit $\overline{\mu}$ is a u-continuous modular function with respect to $\sigma(E, E')$. We will show that $\overline{\mu}\colon (L, u) \to (E, \tau)$ is exhaustive; then we can conclude by 1.6 that $\overline{\mu}\colon (L, u) \to (E, \tau)$ is u-continuous with respect to the topology τ. The exhaustivity of $\overline{\mu}$ will be proved with the aid of 2.5 (b). We show that the assumption of 2.5 (b) is satisfied for $K = \overline{\mathrm{co}}\, \mu(A)$: Let $x_0 < \ldots < x_n$ be a finite chain in L and $I \subset \{1, \ldots, n\}$. For any finite decomposition $D \subset \tilde{A} \setminus \{0\}$ of z we have

$$\sum_{i\in I} \mu_D(x_i) - \mu_D(x_{i-1}) = \sum_{d\in D} \beta_d \cdot \tilde{\mu}(d),$$

where

$$0 \leqslant \beta_d := \sum_{i\in I} (\tilde{\lambda}_{x_i}(d) - \tilde{\lambda}_{x_{i-1}}(d))/\tilde{\lambda}(d) \leqslant \sum_{i=1}^{n} (\tilde{\lambda}_{x_i}(d) - \tilde{\lambda}_{x_{i-1}}(d))/\tilde{\lambda}(d) \leqslant 1,$$

hence $\sum\limits_{i\in I} \mu_D(x_i) - \mu_D(x_{i-1}) \in \overline{\mathrm{co}}\, \mu(A)$ by 2.3 and therefore the limit $\sum\limits_{i\in I} \overline{\mu}(x_i) - \overline{\mu}(x_{i-1})$ belongs to $\overline{\mathrm{co}}\, \mu(A)$. Since $\overline{\mathrm{co}}\, \mu(A)$ is weakly compact, $\overline{\mu}$ is exhaustive by 2.5. Hence $\overline{\mu}\colon (L, u) \to (E, \tau)$ is continuous by 1.6.

Now, in the second step, we do not assume any more that there are a z and $\lambda \in \Lambda$ as stated above. Let D be a maximal disjoint set in $\tilde{A} \setminus \{0\}$ with the property that for each $d \in D$ the restriction of $\tilde{\lambda}$ to $\tilde{A} \wedge d$ is strictly positive for some $\lambda \in \Lambda$. Then $\sup D = 1$ since $(\tilde{A}, \leqslant)$ is complete and $\tilde{\lambda}$ is order continuous for any $\lambda \in \Lambda$. As proved in the first step, the measure $\mu_d\colon A \to E$ defined by $\mu_d(x) := \tilde{\mu}(\hat{x} \wedge d)$ has an extension to a function $\overline{\mu}_d\colon L \to E$ with $\overline{\mu}_d(L) \subset \overline{\mathrm{co}}\, \mu_d(A) \subset \overline{\mathrm{co}}\, \tilde{\mu}(\tilde{A} \wedge d)$ which is, respectively, exhaustive, u-continuous and modular (in 2.1) or a measure (in 2.2). We show that $(\overline{\mu}_d(x))_{d\in D}$ is summable uniformly in $x \in L$. Let U be a closed convex 0-neighbourhood in (E, τ). Since $\tilde{u}$ is order continuous, the net $(\sup F)_{F\in\mathfrak{F}}$ where $\mathfrak{F}$ is the system of all finite subsets of D, converges to 1 in $(\tilde{A}, \tilde{u})$. Since $\tilde{\mu}$ is $\tilde{u}$-continuous, D contains a finite subset F_0 such that $\tilde{\mu}(\tilde{A} \wedge z_0) \subset U$ where z_0 denotes the complement of $\sup F_0$ in $\tilde{A}$. Let F be a finite subset of $D \setminus F_0$. Then we get for $x \in L$

$$\sum_{d\in F} \overline{\mu}_d(x) \in \sum_{d\in F} \overline{\mathrm{co}}\, \tilde{\mu}(\tilde{A}\wedge d) \subset \overline{\mathrm{co}} \sum_{d\in F} \tilde{\mu}(\tilde{A}\wedge d) \subset \overline{\mathrm{co}}\, \tilde{\mu}(\tilde{A}\wedge \sup F) \subset \overline{\mathrm{co}}\, \tilde{\mu}(\tilde{A}\wedge z_0) \subset U.$$

By Cauchy's criterion, $(\overline{\mu}_d(x))_{d\in D}$ is summable uniformly in $x \in L$. Therefore $\overline{\mu} := \sum\limits_{d\in D} \overline{\mu}_d$ is an exhaustive u-continuous modular function (in 2.1) or a measure (in 2.2) extending μ. Since $\sum\limits_{d\in F} \overline{\mu}_d(x) \in \overline{\mathrm{co}}\, \tilde{\mu}(A \wedge \sup F) \subset \overline{\mathrm{co}}\, \mu(A)$ for any finite subset F of D and $x \in L$, we have $\overline{\mu}(L) \subset \overline{\mathrm{co}}\, \mu(A)$. □

In the proof of 2.1 and 2.2 we have used the completion of a quotient of (A, u_0). For the proof of 2.1 it would also be possible to work with the completion of a quotient of (L, u).

Remark 2.6. If E is a Banach space, then in 2.1 and 2.2 the extension $\overline{\mu}$ of μ can be chosen with the additional property that the total variation of $\overline{\mu}$ extends the total variation of μ.

Proof. We use the same notation as in the proof of 2.1 and 2.2. Since E is a Banach space, the μ-uniformity has a countable base. Therefore Λ contains a countable subset $\Lambda_1 = \{\lambda_n \colon n \in \mathbb{N}\}$ such that the μ-uniformity is coarser than the uniformity generated by $\{\lambda_n\big|A \colon n \in \mathbb{N}\}$ on A. Replacing Λ by Λ_1 we may assume that $\Lambda = \{\lambda_n \colon n \in \mathbb{N}\}$. Since $\lambda := \sum\limits_{n=1}^{\infty} \varepsilon_n \lambda_n$ for suitable small real numbers $\varepsilon_n > 0$ induces on A (and in the case of 2.1 also on L) the same uniformity as Λ_1, we may assume that Λ contains only a single function λ. Then the additional assumption of the first step of the proof of 2.1 and 2.2 is satisfied (with $z = 1$).

Let $a \in A$. Obviously $|\overline{\mu}|(a) \geqslant |\mu|(a)$. To prove the other inequality, we may assume that $|\mu|(a)$ is finite. Then $\nu(x) := |\mu|(x \wedge a)$ defines a bounded measure on A. Analogously to $\tilde{\mu}$, μ_D, $\overline{\mu}$ we define $\tilde{\nu}$, ν_d, $\overline{\nu}$; i.e. $\tilde{\nu}$ is the continuous measure on $(\tilde{A}, \tilde{u}_0)$ with $\tilde{\nu}\circ\pi = \nu$, $\nu_D(p) = \sum\limits_{d\in D} (\tilde{\lambda}_p(d)/\tilde{\lambda}(d))\cdot\tilde{\nu}(d)$ for any finite decomposition D of z $(= 1)$ in $\tilde{A}\setminus\{0\}$ and $\overline{\nu}(p)$ is the limit of $\nu_D(p)$ for $p \in L$. Put $\mu_{D,a}(p) := \mu_D(p\wedge a)$ and $\overline{\mu}_a(p) := \overline{\mu}(p \wedge a)$ for $p \in L$. We will apply 1.7. Let $p, q \in L$ with $p \leqslant q$ and let D be a finite decomposition of z in $\tilde{A} \setminus \{0\}$ such that $d \leqslant \hat{a}$ or $d \wedge \hat{a} = 0$ for all $d \in D$. In the following estimation we use that $\|\tilde{\mu}(d)\| \leqslant \tilde{\nu}(d)$ and $\tilde{\lambda}_p(d) = \tilde{\lambda}_{p\wedge a}(d)$ for $d \in \tilde{A}$ with $d \leqslant \hat{a}$, and $\tilde{\nu}(d) = \tilde{\lambda}_{p\wedge a}(d) = 0$ for $d \in \tilde{A}$ with $d \wedge \hat{a} = 0$; this is obviously true for $d \in \hat{A}$ and by continuity also for $d \in \tilde{A}$. So we have

$$\begin{aligned}
&\|\mu_{D,a}(q) - \mu_{D,a}(p)\| + \nu_D(p) \\
&\quad\leqslant \sum_{\hat{a}\geqslant d\in D} (\tilde{\lambda}_{q\wedge a}(d) - \tilde{\lambda}_{p\wedge a}(d))/\tilde{\lambda}(d))\cdot\|\tilde{\mu}(d)\| + \sum_{a\geqslant d\in D} (\tilde{\lambda}_p(d)/\tilde{\lambda}(d))\cdot\tilde{\nu}(d) \\
&\quad\leqslant \sum_{\hat{a}\geqslant d\in D} (\tilde{\lambda}_q(d) - \tilde{\lambda}_p(d)/\tilde{\lambda}(d))\cdot\tilde{\nu}(d) + \sum_{a\geqslant d\in D} (\tilde{\lambda}_p(d)/\tilde{\lambda}(d))\cdot\tilde{\nu}(d) = \nu_D(q).
\end{aligned}$$

By 1.7 we therefore get $|\overline{\mu}_a|(a) \leqslant \overline{\nu}(a)$, hence

$$|\overline{\mu}|(a) = |\overline{\mu}_a|(a) \leqslant \overline{\nu}(a) = \nu(a) = |\mu|(a).$$

$\square$

3. Modular functions on complemented or sectionally complemented or relatively complemented lattices

3.1. Representation of modular functions.

The theorem which we prove in this section gives in particular an isomorphism between the space of all exhaustive E-valued modular functions on a complemented (c.) or sectionally complemented (s.c.) or relatively complemented (r.c.) lattice and the space of all order continuous E-valued measures on a suitable complete Boolean algebra. In the subsequent sections we show how this theorem can be used to transfer results known for measures on Boolean algebras to the case of modular functions on c. or s.c. or r.c. lattices.

We first select some properties of lattice uniformities and of modular functions on c. or s.c. or r.c. lattices.

Proposition 3.1.1. *Let L be a modular r.c. lattice and u and v lattice uniformities on L. Then u is coarser than v iff the u-topology is coarser than the v-topology.*

Proof. For s.c. lattices the equivalence is proved in [17, 6.10]. Let now $e \in L$, $L_1 = \{x \in L\colon\ x \geqslant e\}$ and $L_2 = \{x \in L\colon\ x \leqslant e\}$ and suppose that the u-topology is coarser than the v-topology. Since L_1 is s.c., we have $u\big|L_1 \subset v\big|L_1$ by [17, 6.10], and dually $u\big|L_2 \subset v\big|L_2$. Now apply the following lemma. $\square$

Lemma 3.1.2. *Let L be modular, $e \in L$, $L_1 = \{x \in L\colon\ x \geqslant e\}$, $L_2 = \{x \in L\colon\ x \leqslant e\}$ and let u, v be lattice uniformities such that $u\big|L_1 \subset v\big|L_1$ and $u\big|L_2 \subset v\big|L_2$. Then $u \subset v$.*

Proof. Let $U \in u$, $\Delta = \{(x,x)\colon\ x \in L\}$ and $U_1, U_2 \in u$ with $U_1 \circ U_2 \subset U$ and $U_2 \vee \Delta$, $U_2 \wedge \Delta \subset U_1$. By assumption, there is a $V_0 \in v$ such that $V_0 \cap (L_1 \times L_1) \subset U_2$ and $V_0 \cap (L_2 \times L_2) \subset U_2$. Let $V \in v$ with $V \vee \Delta$, $V \wedge \Delta \subset V_0$ and $(a,b) \in V$. We shall show that $(a,b) \in U$. Since by [17, 1.1.3] any lattice uniformity has a base of sets W such that for all pairs $(x,y) \in W$ the rectangle $[x \wedge y, x \vee y]^2$ is contained in W, we may assume that $a \leqslant b$ and that U_1 is symmetric. Now $(a \vee e, b \vee e) \in V_0 \cap (L_1 \times L_1) \subset U_2$, hence $((a \vee e) \wedge b, b) \in U_2 \wedge \Delta \subset U_1$ and dually $((b \wedge e) \vee a, a) \in U_2 \vee \Delta \subset U_1$. Since $(a \vee e) \wedge b = (b \wedge e) \vee a$ by the modularity of L, we obtain $(a,b) \in U_1 \circ U_1 \subset U$. $\square$

Proposition 3.1.3. *Let L be c. or s.c. or r.c. and let $\mu\colon L \to E$ be a modular function.*

(a) *If u is a lattice uniformity on L and $\mu\colon (L,u) \to E$ is continuous, then μ is uniformly continuous with respect to u.*

(b) *If E is a Banach space and $|\mu|$ is bounded, then the μ-uniformity agrees with the $|\mu|$-uniformity.*

(c) *If μ is real-valued, then μ is bounded iff the total variation $|\mu|$ is bounded iff μ is exhaustive.*

Proof. (a) Passing to the quotient $L/N(\mu)$, we may assume that L is modular (see 1.4) and therefore r.c. (see [7, I.14]). Since by 1.3 the μ-topology is coarser than the u-topology, we obtain by 3.1.1 that the μ-uniformity is coarser than u, i.e. $\mu\colon (L,u) \to E$ is uniformly continuous.

(b) If $|\mu|$ is bounded, then the μ-topology agrees with the $|\mu|$-topology by [20, 1.3.11]. Therefore μ and $|\mu|$ induce the same uniformity by (a) (or by 3.1.1 observing that we may assume as in (a) that L is modular and r.c.).

(c) holds by [19, 2.7 and 2.8]. □

Proposition 3.1.4. *Let L be c. or s.c. or r.c. and let $\mathscr{L}U_0(L)$ be the set of all lattice uniformities each of which is generated by a set of real-valued bounded increasing modular functions.*

(a) *For any set Λ of exhaustive E-valued modular functions on L, the Λ-uniformity belongs to $\mathscr{L}U_0(L)$.*

(b) *A modular function $\mu\colon L \to E$ is exhaustive iff $\mu\colon (L,u) \to E$ is continuous for some $u \in \mathscr{L}U_0(L)$.*

Proof. (a) Since the supremum of a set of uniformities of $\mathscr{L}U_0(L)$ belongs to $\mathscr{L}U_0(L)$, we may assume that Λ contains only one element μ. Let $\Lambda' := \{x' \circ \mu\colon x' \in E'\}$ and let u be the Λ'-uniformity. Obviously, u is coarser than the μ-uniformity. By 1.6, $\mu\colon (L,u) \to E$ is continuous and therefore uniformly continuous by 3.1.3 (a). It follows that u is the μ-uniformity. On the other hand, u is also generated by $\Lambda_0 := \{|\nu| : \nu \in \Lambda'\}$ (see 3.1.3 (b), (c)). Therefore $u \in \mathscr{L}U_0(L)$.

(b) The implication $\Leftarrow$ follows from the facts that every $u \in \mathscr{L}U_0(L)$ is exhaustive and that any continuous modular function $\mu\colon (L,u) \to E$ is uniformly continuous by 3.1.3. For $\Rightarrow$ observe that the μ-uniformity belongs to $\mathscr{L}U_0(L)$ by (a). □

Proposition 3.1.5 [18]**.** *Let L be a complete r.c. lattice and $\nu\colon L \to \mathbb{R}$ an order continuous modular function. Then ν attains its supremum and infimum on $C(L)$.*

The proof is essentially contained in [18, 4.1]: The assertion follows from [18, 2.1] if the ν-topology is Hausdorff, i.e. if $N(\nu) = \{(x,x)\colon x \in L\}$. But the assertion can be easily reduced to the Hausdorff case: By the next lemma

$s := \sup\{x \in L : |\nu|(x) = 0\} \in C(L)$. Let t be the (unique) complement of s. Then the restriction $\nu\big|[0,t]$ (and therefore ν) attains its supremum and infimum on $C([0,t])$ by [18, 2.1]. Now observe that $C([0,t]) \subset C(L)$ since $t \in C(L)$. $\square$

Lemma 3.1.6 [11, section III.3]**.** *Let L be a r.c. bounded lattice and $\simeq$ a congruence relation of L. If $\{x \in L\colon\ x \simeq 0\}$ has a maximal element s, then $s \in C(L)$.*

P r o o f. By [11, III.3.10], any congruence relation of L is standard. Therefore s is by [11, III.3.3] a standard element, hence neutral by [11, exercise III.2.19]. It follows that $s \in C(L)$, since an element of a bounded lattice belongs to its center iff it is neutral and has a complement, see [7, Theorem III.9.12]. (In [11; section III.4, p. 156], the center is *defined* as the set of complemented neutral elements.) $\square$

Theorem 3.1.7. *Let L be c. or s.c. or r.c., $u \in \mathscr{L}U_0(L)$ where $\mathscr{L}U_0(L)$ is defined as in 3.1.4 and $(\tilde{L}, \tilde{u})$ the uniform completion of the quotient $(\hat{L}, \hat{u}) := (L, u)/N(u)$.*

(a) *Then, for any continuous modular function $\mu\colon (L,u) \to (E,\tau)$, the function $\hat{\mu}\colon (\hat{L} \to E)$ defined by $\hat{\mu}(\hat{x}) = \mu(x)$, $x \in \hat{x} \in \hat{L}$, has a unique continuous extension $\tilde{\mu}(\tilde{L}, \tilde{u}) \to E$. Denote by $\overline{\mu}\colon C(\tilde{L}) \to E$ the restriction of $\tilde{\mu}$ to the center $C(\tilde{L})$ of $\tilde{L}$. Then*

$$\overline{\mu(L)} = \overline{\tilde{\mu}(\tilde{L})} \quad \text{and} \quad \overline{\mathrm{co}}\,\mu(L) = \overline{\mathrm{co}}\,\overline{\mu}(C(\tilde{L})).$$

(b) *$\mu \mapsto \overline{\mu}$ defines an isomorphism from the linear space of all continuous modular functions $\mu\colon (L,u) \to (E,\tau)$ with $\overline{\mu}(0) = 0$ onto the linear space of all continuous measures $\overline{\mu}\colon (C(\tilde{L}), \tilde{u}) \to (E,\tau)$. In particular, if u is generated by the set of all real-valued bounded modular functions on L, then $\mu \mapsto \overline{\mu}$ defines an isomorphism from the linear space of all exhaustive modular functions $\mu\colon L \to E$ with $\overline{\mu}(0) = 0$ onto the linear space of all order continuous measures $\overline{\mu}\colon C(\tilde{L}) \to E$.*

(c) *If F is a complete locally convex Hausdorff linear space and $\mu\colon (L,u) \to E$ and $\nu\colon (L,u) \to F$ are continuous modular functions, we have $\mu \ll \nu$ iff $\overline{\mu} \ll \overline{\nu}$ and $\mu \perp \nu$ iff $\overline{\mu} \perp \overline{\nu}$. Moreover, if $T\colon E \to F$ is a continuous additive map and $\nu = T \circ \mu$, then $\hat{\nu} = T \circ \hat{\mu}$, $\tilde{\nu} = T \circ \tilde{\mu}$ and $\overline{\nu} = T \circ \overline{\mu}$.*

(d) *Let E be a Banach space and $\mu\colon (L,u) \to E$ a continuous modular function. Then μ has bounded variation iff $\tilde{\mu}$ has bounded variation iff $\overline{\mu}$ has bounded variation. Moreover, if $\nu := |\mu|$ is bounded, then $\tilde{\nu} = |\tilde{\mu}|$ and $\overline{\nu} = |\overline{\mu}|$.*

Here $\mu \perp \nu$ means that the infimum of the μ-uniformity and the ν-uniformity is trivial. $\mu \ll \nu$ means that the μ-uniformity is coarser than the ν-uniformity; in this case we also say that μ is *ν-continuous*.

Part of this theorem was already proved in [20, 3.2.4] in the slightly more special case that u is generated by the set of *all* real-valued increasing bounded modular functions on L. What is new here in 3.1.7 (a) is in particular the information on the convex hull of $\mu(L)$ and in (b) the surjectivity of the map $\mu \mapsto \overline{\mu}$. These two facts are important for the applications given below.

P r o o f. (a), (b): First observe that by 1.1 (c) and 1.4 $\tilde{u}$ is order continuous, $\tilde{L}$ is a complete modular complemented (and therefore by [7, I.14] a r.c.) lattice and $C(\tilde{L})$ is a complete Boolean sublattice of $\tilde{L}$.

Obviously, $\hat{\mu}$, $\tilde{\mu}$ and $\overline{\mu}$ in (a) are well defined and $\tilde{u}$-continuous modular functions (see 1.4 and 3.1.3 (a)), and the map $\mu \mapsto \overline{\mu}$ in (b) is well defined and linear. To prove the injectivity of this map, let $\mu\colon (L,u) \to E$ be a continuous modular function such that $\overline{\mu} = 0$. Since for any $x' \in E'$ the real-valued modular function $x' \circ \tilde{\mu}$ attains by 3.1.5 its supremum and infimum on $C(\tilde{L})$ and $x' \circ \overline{\mu} = 0$, we obtain $x' \circ \tilde{\mu} = 0$. Therefore $\tilde{\mu} = 0$ and $\mu = 0$.

We now prove the surjectivity of the map $\mu \mapsto \overline{\mu}$. Let Λ be a set of real-valued bounded increasing modular functions generating the uniformity u and $\tilde{\Lambda} := \{\tilde{\lambda}\colon \lambda \in \Lambda\}$. It is easy to see that then $\tilde{u}$ is the $\tilde{\Lambda}$-uniformity (cf. [19, 3.8]). Let $\overline{\mu}\colon (C(\tilde{L}), \tilde{u}) \to (E,\tau)$ be a continuous measure. By 2.1, $\overline{\mu}$ has a continuous extension $\tilde{\mu}\colon (\tilde{L}, \tilde{u}) \to (E,\tau)$ to a modular function. Define $\mu\colon L \to E$ by $\mu(x) := \tilde{\mu}(\hat{x})$, $x \in \hat{x} \in \hat{L}$. Then $\overline{\mu}$ is the image of μ under the map defined in (b).

In the proof of the last statement in (a) we may assume, replacing μ by $\mu - \overline{\mu}(0)$, that $\overline{\mu}(0) = 0$. Since $\tilde{\mu}$ is the unique $\tilde{u}$-continuous E-valued modular function on $\tilde{L}$ extending $\overline{\mu}$, we have by 2.1 that $\tilde{\mu}(\tilde{L}) \subset \overline{\mathrm{co}}\,\overline{\mu}(C(\tilde{L}))$, hence $\overline{\mathrm{co}}\,\tilde{\mu}(\tilde{L}) = \overline{\mathrm{co}}\,\overline{\mu}(C(\tilde{L}))$. Obviously, $\mu(L)$ is dense in $\tilde{\mu}(\tilde{L})$. Therefore $\overline{\mu(L)} = \overline{\tilde{\mu}(\tilde{L})}$ and $\overline{\mathrm{co}}\,\mu(L) = \overline{\mathrm{co}}\,\tilde{\mu}(\tilde{L})$.

To prove the second statement in (b), let u be the uniformity generated by the set of all real-valued bounded modular functions on L. By 3.1.4, $u \in \mathscr{L}U_0(L)$. Using the fact that a complete r.c. modular lattice admits by [17, 5.10] and 3.1.1 at most one Hausdorff order continuous lattice uniformity, one easily sees that $\tilde{u}$ is generated by the set of all order continuous real-valued modular functions on $\tilde{L}$, and the restriction $\tilde{u}\big|C(\tilde{L})$ by the set of all order continuous real-valued measures on $C(\tilde{L})$. Now it is sufficient to observe that a modular function $\mu\colon L \to E$ is exhaustive iff μ is u-continuous, and a measure $\overline{\mu}\colon C(\tilde{L}) \to E$ is order continuous iff $\overline{\mu}$ is $\tilde{u}$-continuous (see 3.1.4).

(c) For the first statement, see [20, 3.2.4]. The second statement is obvious.

(d) (i) If $\nu := |\mu|$ is bounded, then ν is u-continuous by 3.1.3 (b). It is easy to see (and contained in the proof of [20, 1.3.11]) that $|\tilde{\mu}|$ is the continuous extension of $|\hat{\mu}|$, i.e. $\tilde{\mu}$ has bounded variation and $\tilde{\nu} = |\tilde{\mu}|$.

(ii) If $\tilde{\mu}$ is of bounded variation, then obviously also the restriction $\overline{\mu}$ is so.

(iii) If $\overline{\mu}$ has bounded variation, then $|\tilde{\mu}|$ extends $|\overline{\mu}|$ by 2.6 and the injectivity of the isomorphism of (b), i.e. $\tilde{\mu}$ has bounded variation and $\overline{\nu} = |\overline{\mu}|$. □

Proposition 3.1.8. *With the assumption and notation of* 3.1.7, *let* $u \in \mathscr{L}U_0(L)$ *and let* M *be a set of* E*-valued* u*-continuous modular functions on* L.

(a) *If* M *is uniformly exhaustive (i.e.* $(\mu(a_n))_{n\in\mathbb{N}}$ *converge uniformly in* $\mu \in M$ *for any monotone sequence* $(a_n)_{n\in\mathbb{N}}$ *in* L*), then* $\{\overline{\mu}\colon\ \mu \in M\}$ *is uniformly exhaustive.*

(b) *If* $\{\overline{\mu} : \mu \in M\}$ *is uniformly exhaustive and* M *is pointwise bounded, then* M *is uniformly exhaustive.*

P r o o f. Replacing μ by $\mu - \overline{\mu}(0)$, we may assume that $\overline{\mu}(0) = 0$ for all $\mu \in M$. Let τ_∞ be the topology of uniform convergence on E^M and $\ell_\infty[M, E]$ the subspace of (E^M, τ_∞) consisting of all bounded functions from M into E.

(a) If M is uniformly exhaustive, then M is equicontinuous with respect to u by [9] or [19, 6.2]. Hence $\hat{\nu} := (\hat{\mu})_{\mu\in M}\colon (\hat{L}, \hat{u}) \to (E^M, \tau_\infty)$ is continuous. Let $\tilde{\nu}\colon (\tilde{L}, \tilde{u}) \to (E^M, \tau_\infty)$ be the continuous extension of $\hat{\nu}$. Then $\tilde{\nu} = (\tilde{\mu})_{\mu\in M}$. Since $\tilde{\nu}$ is exhaustive, the set $\{\tilde{\mu}\colon\ \mu \in M\}$ is uniformly exhaustive and therefore $\{\overline{\mu}\colon\ \mu \in M\}$ is uniformly exhaustive.

(b) Since $\{\overline{\mu}\colon\ \mu \in M\}$ is uniformly exhaustive and any $\overline{\mu}$, for $\mu \in M$, is $\tilde{u}$-continuous, $\{\overline{\mu}\colon\ \mu \in M\}$ is even equicontinuous with respect to $\tilde{u}$, i.e. $(\overline{\mu})_{\mu\in M}\colon (C(\tilde{L}), \tilde{u}) \to (\ell_\infty[M, E], \tau_\infty)$ is continuous. Let $\nu\colon (L, u) \to \ell_\infty[M, E]$ be the modular function which corresponds to $(\overline{\mu})_{\mu\in M}$ according to 3.1.7 (b), i.e. $\overline{\nu} = (\overline{\mu})_{\mu\in M}$. Applying the last statement of 3.1.7 (c) with the projections $(x_\mu)_{\mu\in M} \mapsto x_\mu$ from $\ell_\infty[M, E]$ onto E one obtains that $\nu = (\mu)_{\mu\in M}$. Since $\nu\colon (L, u) \to \ell_\infty[M, E]$ is continuous and therefore exhaustive, M is uniformly exhaustive. □

3.2. A decomposition theorem.

We illustrate the method of transferring results for measures from the Boolean case to the case of c. or s.c. or r.c. lattices by means of Lebesgue decomposition theorem.

Theorem 3.2.1. *Let* L *be c. or s.c. or r.c. and let* $\mu\colon L \to E$ *and* $\nu\colon L \to F$ *be exhaustive modular functions where* F *is a locally convex Hausdorff linear space. Then there are exhaustive modular functions* μ_1 *and* μ_2 *such that* $\mu = \mu_1 + \mu_2$, $\mu_1 \ll \nu$ *and* $\mu_2 \perp \nu$.

P r o o f. We use the notation of 3.1.7 where u is generated by the set of all real-valued bounded modular functions on L. Replacing μ by $\mu - \overline{\mu}(0)$ we may assume that $\overline{\mu}(0) = 0$. $\overline{\mu}$ has by [15] or [16, 5.1] a unique decomposition of the form $\overline{\mu} = \lambda_1 + \lambda_2$ where λ_1 and λ_2 are E-valued $\overline{\mu}$-continuous measures on $C(\tilde{L})$, $\lambda_1 \ll \overline{\nu}$ and $\lambda_2 \perp \overline{\nu}$. By 3.1.7 there are exhaustive modular functions $\mu_1\colon L \to E$ with $\overline{\mu}_i = \lambda_i$ decomposing μ according to 3.2.1. □

3.3. Controls.

A modular function ν is called a *control* for a modular function Λ or for a set of modular functions Λ if the ν-uniformity agrees with the Λ-uniformity.

In [20, 3.2.5] it was already explained how to transfer Rybakov's theorem to the case of modular functions on c. or s.c. or r.c. lattices:

Theorem 3.3.1 [20, 3.2.5]**.** *Let L be c. or s.c. or r.c., let E be a Banach space and $\mu\colon L \to E$ an exhaustive modular function. Then there is an $x' \in E'$ such that $x' \circ \mu$ is a control of μ.*

P r o o f. With the notation of 3.1.7, by the Rybakov theorem there is an $x' \in E'$ such that $x' \circ \overline{\mu}$ is a control of $\overline{\mu}$; then, by 3.1.7 (c), $x' \circ \mu$ is a control of μ. $\square$

Using [8, Corollary 1] instead of the Rybakov theorem for measures on Boolean algebras, one obtains by the same argument that for any exhaustive modular function $\mu\colon L \to E$ there is a $x' \in E'$ such that $x' \circ \mu$ is a control of μ if E is a dual nuclear space.

Theorem 3.3.2. *Let L be c. or s.c. or r.c. and let (μ_n) be a sequence of uniformly exhaustive E-valued modular functions on L. Then there is a modular function $\nu\colon L \to E$ which is a control of $\{\mu_n\colon n \in \mathbb{N}\}$.*

In the case of L being a Boolean algebra, 3.3.2 is proved in [6, Theorem 2]. In [3, Theorem 2.11], the result 3.3.2 was obtained by transferring the *proof* of [6, Theorem 2] from the Boolean case to the case of c. (or s.c.) lattices. We now show that with the aid of 3.1.7 one can transfer the *result* [6, Theorem 2] to the case of c. (or s.c. or r.c.) lattices.

P r o o f of 3.3.2. We use the notation of 3.1.7 where—as in the second statement of 3.1.7 (b)—u is generated by the set of all real-valued bounded modular functions on L. By 3.1.8, $\{\overline{\mu}_n\colon n \in \mathbb{N}\}$ is uniformly exhaustive and has therefore a control $\overline{\nu}\colon C(\tilde{L}) \to E$ by [6, Theorem 2]. By 3.1.7, the corresponding function $\nu\colon L \to E$ is a control for $\{\mu_n\colon n \in \mathbb{N}\}$. $\square$

3.4. Vitali-Hahn-Saks and Nikodým theorems.

Here we transfer the Vitali-Hahn-Saks and Nikodým theorems to the setting of modular functions on c. or s.c. or r.c. lattices.

Theorem 3.4.1. *Let L be a σ-complete c. or s.c. or r.c. lattice and $\mu_n\colon L \to E$, $n \in \mathbb{N}$, a sequence of σ-order continuous modular functions, which converges pointwise to $\mu\colon L \to E$.*

(a) *Then the sequence $(\mu_n)_{n\in\mathbb{N}}$ is uniformly exhaustive and μ is a σ-order continuous modular function.*

(b) *If, for every $n \in \mathbb{N}$, μ_n is continuous with respect to a lattice uniformity v on L, then $\{\mu_n\colon\ n \in \mathbb{N}\} \cup \{\mu\}$ is equicontinuous with respect to v.*

P r o o f. Since E is a subspace of a product of Banach spaces, we may assume that already E is a Banach space. Let u be the uniformity generated by $\{\mu_n\colon\ n \in \mathbb{N}\}$. Then (L, u) is complete by 1.5. Passing to the quotient $(\hat{L}, \hat{u}) := (L, u)/N(u)$ we may assume for (b) and for the first statement of (a) that u is Hausdorff. So $(L, u) = (\hat{L}, \hat{u}) = (\tilde{L}, \tilde{u})$ with the notation of 3.1.7. (Observe that $u \in \mathscr{L}U_0(L)$ by 3.1.4 as required in 3.1.7.) Replacing μ_n by $\mu_n - \mu_n(0)$, we may assume that $\mu_n(0) = 0$. By the Vitali-Hahn-Saks theorem (sometimes also called the Brooks-Jewett theorem) for measures on Boolean algebras the restrictions $\overline{\mu}_n := \mu_n \big| C(L)$, $n \in \mathbb{N}$, are uniformly exhaustive. Therefore μ_n, $n \in \mathbb{N}$, are uniformly exhaustive by 3.1.8. From this fact (b) follows by [9] or [19, 6.2].

Applying (b) with v being the finest σ-order continuous lattice uniformity on L we obtain that μ is σ-order continuous. $\square$

The proof shows that in 3.4.1 we do not need to assume the convergence of $(\mu_n(a))_{n\in\mathbb{N}}$ for *all* $a \in L$.

Theorem 3.4.2. *Let L be a σ-complete c. or s.c. or r.c. lattice and M a pointwise bounded set of σ-order continuous E-valued modular functions on L. Then M is uniformly bounded.*

P r o o f. Since a subset A of a locally convex linear space is bounded iff all countable subsets of A are bounded, we may assume that M is countable, i.e. that $M = \{\mu_n\colon\ n \in \mathbb{N}\}$. We use the same notation as in the proof of 3.4.1. As there we may assume that E is a Banach space, u is Hausdorff and complete and $L = \hat{L} = \tilde{L}$. By the classical Nikodým boundedness theorem, $\{\overline{\mu}_n\colon\ n \in \mathbb{N}\}$ is uniformly bounded, i.e. for some positive real number r we have $\overline{\mu}_n(C(\tilde{L})) \subset B := \{x \in E : \|x\| \leqslant r\}$ for all $n \in \mathbb{N}$. By 3.1.7 (a), $\mu_n(L) \subset \overline{\mathrm{co}}\, \overline{\mu}_n(C(\tilde{L})) \subset B$. Hence $\{\mu_n\colon\ n \in \mathbb{N}\}$ is uniformly bounded. $\square$

Results related to 3.4.1 and 3.4.2, but with quite different proofs are contained in [5].

3.5. The range of modular functions.

Proposition 3.5.1. *Let L be a r.c. irreducible complete lattice. Suppose that there exists an order continuous modular function $\nu\colon\ L \to E$ which is not constant.*

(a) *Then there is a unique order continuous modular function $\lambda\colon\ L \to \mathbb{R}$ with $\lambda(0) = 0$ and $\lambda(1) = 1$. λ is strictly increasing and every order continuous modular function $\mu\colon\ L \to E$ has the form $\mu = \lambda \cdot \mu(1) + (1 - \lambda) \cdot \mu(0)$.*

(b) *If L is atomless, then the range of λ is the closed real unit interval I.*

(c) *If L is not atomless, then L is a geometric lattice of finite length n and $\lambda = h/n$ where h is the height function.*

P r o o f. (a) Let $\mu\colon L \to E$ be an order continuous modular function which is not constant. Then for some $x' \in E'$, $x' \circ \mu$ is not constant and has bounded variation by 3.1.3 (c). Therefore $\lambda := |x' \circ \mu|/|x' \circ \mu|(1)$ is an increasing modular function with $\lambda(0) = 0$ and $\lambda(1) = 1$. Since $\sup\{x \in L\colon \lambda(x) = 0\} \in C(L) = \{0, 1\}$ by 3.1.6, λ is strictly positive and therefore strictly increasing. Moreover, λ is order continuous since the λ-uniformity agrees with the $x' \circ \mu$-uniformity by 3.1.3 (b). We now will apply 3.1.7 for u being the λ-uniformity. Observe that (L, u) is complete by 1.5 and therefore $L = \hat{L} = \tilde{L}$. Since the modular functions $\mu - (1 - \lambda) \cdot \mu(0)$ and $\lambda \cdot \mu(1)$ agree on $C(L)$, they are equal by 3.1.7. Hence $\mu = \lambda \cdot \mu(1) + (1 - \lambda) \cdot \mu(0)$. So μ is uniquely determined by the values $\mu(0)$ and $\mu(1)$. This implies also (for $E = \mathbb{R}$) the uniqueness of λ.

(b) If L is atomless, then (L, u) is connected by [21, 5.4 (a)]. Therefore the continuous image $\lambda(L)$ is an interval. Since λ is increasing, $\lambda(0) = 0$ and $\lambda(1) = 1$, we get $\lambda(L) = I$.

(c) First observe that L is modular by 1.4 since the λ-uniformity is Hausdorff. It now follows from [21, 5.13] that L is a geometric lattice of finite length if L is not atomless. By the uniqueness statement in (a) we obtain $\lambda = h/n$. (Here we use that the height function is modular, see [7, p. 40].) □

Theorem 3.5.2. *Let L be a complete complemented lattice and $\mu\colon L \to E$ an order continuous modular function with $\mu(0) = 0$ and $N(\mu) = \{(x, x)\colon x \in L\}$. Then there is a μ-continuous modular function $\nu\colon L \to E$ and there are μ-continuous increasing modular functions $\varrho_a\colon L \to \mathbb{R}$ and $\sigma_b\colon L \to \mathbb{R}$ and elements $y_a, z_b \in E$ ($a \in A$, $b \in B$) with the following properties:*

(1) *$(\varrho_a(x) \cdot y_a)_{a \in A}$ and $(\sigma_b(x) \cdot z_b)_{b \in B}$ are summable uniformly in $x \in L$; $\mu = \nu + \varrho + \sigma$ where $\varrho := \sum\limits_{a \in A} \varrho_a(x) \cdot y_a$ and $\sigma := \sum\limits_{b \in B} \sigma_b \cdot z_b$.*

(2) *$\varrho_a(L) = I$ (:=closed real unit interval) for $a \in A$; $\sigma_b(L) = \{i/n_b\colon i = 0, \ldots, n_b\}$ for $b \in B$ and some $n_b \in \mathbb{N}$; $\varrho(L)$ is convex and compact; $\sigma(L)$ is compact; $\overline{\mathrm{co}}\, \nu(L) = \overline{\mathrm{co}}\, \nu(C(L))$; $\mu(L) = \nu(L) + \varrho(L) + \sigma(L)$.*

(3) *The restriction $\nu | C(L)$ is an atomless measure.*

(4) *$\sigma = 0$ iff L is atomless.*

P r o o f. First observe that L is modular by 1.4 and therefore r.c. by [7, I.14]. We may assume that $L \neq \{0\}$. Let A be the set of all atoms a of $C(L)$ for which $[0, a]$ is atomless, and let B be the set of all the other atoms of $C(L)$. For $p \in A \cup B$, the interval $[0, p]$ is an irreducible lattice. Therefore there is by 3.5.1 an increasing modular function $\lambda_p\colon [0, p] \to \mathbb{R}$ with $\lambda_p(0) = 0$, $\lambda_p(p) = 1$ and $\mu(x) = \lambda_p(x) \cdot \mu(p)$ ($x \in [0, p]$); $\lambda_p([0, p]) = I$ if $p \in A$, and $\lambda_p([0, p]) = \{i/n_p\colon i = 0, \ldots, n_p\}$ for some

$n_p \in \mathbb{N}$ if $p \in B$. Let t be the (unique) complement of $\sup(A \cup B)$ in $C(L)$. We put $\nu(x) = \mu(x \wedge t)$, $\varrho_a(x) = \lambda_a(x \wedge a)$, $\sigma_b(x) = \lambda_b(x \wedge b)$, $y_a = \mu(a)$, $z_b = \mu(b)$ for $a \in A$, $b \in B$ and $x \in L$. We shall verify the properties (1) through (4). (1) holds by [21, 6.3]. By 3.1.7 we have $\overline{\text{co}}\, \nu(L) = \overline{\text{co}}\, \nu(C(L))$ observing that $(\tilde{L}, \tilde{u}) = (\hat{L}, \hat{u}) = (L, u)$ for u being the μ-uniformity. $\varrho(L)$ is the image of the compact convex set I^A under the continuous affine map $(t_a)_{a \in A} \mapsto \sum\limits_{a \in A} t_a \cdot y_a$ and therefore it is compact and convex. Similarly we obtain that $\sigma(L)$ is compact. (3), (4) and the last statement of (2) hold obviously. □

The last result allows us to transfer Liapunoff's theorem to the setting of modular functions on c. or s.c. or r.c. lattices.

We call a modular function $\mu\colon L \to E$ on a c. (or s.c.) lattice *atomless* if the quotient $L/N(\mu)$ has no atom, or equivalently, if for every $a \in L$ with $(0, a) \notin N(\mu)$ there is an element $b \in [0, a]$ with $\mu(b) \notin \{\mu(0), \mu(a)\}$.

Theorem 3.5.3. *Let $n \in \mathbb{N}$ and let $\mu\colon L \to \mathbb{R}^n$ be a σ-order continuous modular function on a complemented σ-complete lattice. Then $\mu(L)$ is compact. If μ is atomless, then $\mu(L)$ is convex.*

P r o o f. Replacing μ by $\mu - \mu(0)$ and passing then to the quotient $L/N(\mu)$, we may assume that $\mu(0) = 0$ and that the μ-uniformity is Hausdorff. Then the assumptions of 3.5.2 are satisfied by 1.2 and 1.5. With the notation of 3.5.2 we obtain from Liapunoff's theorem that $\nu(C(L))$ is compact and convex. Therefore $\nu(L) = \nu(C(L))$ by 3.5.2 (2) and $\mu(L)$ is the sum of three compact sets $\nu(L)$, $\varrho(L)$ and $\sigma(L)$ and consequently it is compact. If μ is atomless, then $\sigma = 0$ and $\mu(L)$ is the sum of two convex sets $\nu(L)$ and $\varrho(L)$ and therefore it is convex. □

The convexity assertion of 3.5.3 was already proved by Avallone [1]. More precisely, the main result of [1] gives a generalization of the finitely additive version of Liapunoff's convexity theorem to modular functions on complemented lattices.

We now give a condition under which $\overline{\mu(L)}$ is compact and convex. Let $\mu\colon L \to E$ be a modular function. L is called *μ-chained* (see [19, p. 50]) if for every 0-neighbourhood U in E and every $a, b \in L$ with $a < b$ there is a finite chain $a = x_0 < x_1 < \ldots < x_n = b$ in L such that $\mu(x) - \mu(y) \in U$ for $x, y \in [x_{i-1}, x_i]$ and $i = 1, \ldots, n$. If μ is exhaustive and L is c. (or s.c.) and complete with respect to the μ-uniformity, then L is μ-chained iff μ is atomless (see [20, section 2.2]).

Theorem 3.5.4. *Let L be c. or s.c. or r.c.*

(a) *Assume that $\overline{\mu(A)}$ is compact (convex) for every order continuous atomless measure $\mu\colon A \to E$ defined on a complete Boolean algebra. Then $\overline{\mu(L)}$ is compact (or convex, respectively) for every exhaustive modular function $\mu\colon L \to E$ (with L being μ-chained).*

(b) *Assume that E is a Banach space and $\overline{\mu(A)}$ is compact (convex) for every order continuous atomless measure $\mu\colon A \to E$ of bounded variation defined on a complete Boolean algebra. Then $\overline{\mu(L)}$ is compact (or convex, respectively) for every modular function $\mu\colon L \to E$ of bounded variation (with L being μ-chained).*

P r o o f. We prove only the convexity assertion in (b). The other three assertions can be proved in a similar way. Let $\mu\colon L \to E$ be a modular function of bounded variation such that L is μ-chained. μ is exhaustive since μ has bounded variation. We apply 3.1.7 with u being the μ-uniformity. Since $\overline{\mu(L)} = \overline{\tilde{\mu}(\tilde{L})}$, it is enough to show that $\overline{\tilde{\mu}(\tilde{L})}$ is convex. Replacing μ and L by $\tilde{\mu}$ and $\tilde{L}$ we may assume that μ is order continuous and L complete. Replacing μ by $\mu - \mu(0)$ we may further assume that $\mu(0) = 0$. So all assumptions of 3.5.2 are satisfied. With the notation of 3.5.2, $\sigma = 0$ since L is μ-chained and therefore atomless, $\varrho(L)$ is compact and therefore $\overline{\mu(L)} = \overline{\nu(L)} + \varrho(L)$. $\overline{\nu(C(L))}$ is convex by the assumption on E and therefore $\overline{\nu(L)} = \overline{\nu(C(L))}$. Hence $\overline{\nu(L)}$ is convex. Moreover, $\varrho(L)$ is convex. It follows that $\overline{\mu(L)} = \overline{\nu(L)} + \varrho(L)$ is convex. □

In the assumption of 3.5.4 one can replace the order continuous measures on complete Boolean algebras by σ-additive measures on σ-fields of sets. This follows from the fact that every σ-complete Boolean algebra is isomorphic to a factor algebra $\mathfrak{F}/\mathfrak{N}$ where $\mathfrak{F}$ is a σ-field of sets and $\mathfrak{N}$ a σ-ideal in $\mathfrak{F}$.

Theorem 3.5.4 allows us to transfer e.g. the following theorems of Uhl, of Kadets and of Kadets and Shekhtman to the setting of modular functions on c. or s.c. or r.c. lattices:

Theorem. *Let $\mathfrak{F}$ be a σ-field, E a Banach space and $\mu\colon \mathfrak{F} \to E$ a σ-additive measure.*

If E has the Radon-Nikodým property and μ has bounded variation, then $\overline{\mu(\mathfrak{F})}$ is compact [14].

If E has the Radon-Nikodým property and μ has bounded variation [14] *or if E is B-convex and μ has bounded variation* [12] *or if $E = c_0$ or $E = \ell_p$ for some $p \in [1, \infty[\setminus\{2\}$* [13], *then $\overline{\mu(\mathfrak{F})}$ is convex.*

With quite another method, Avallone [2] generalized these theorems of Uhl, Kadets and of Kadets-Shekhtman (for $E = \ell_1$) to modular functions on complemented lattices.

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