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APPROXIMATIONS TO MILD SOLUTIONS OF STOCHASTIC SEMILINEAR EQUATIONS WITH NON-LIPSCHITZ COEFFICIENTS

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Abstract. In the present paper, using a Picard type method of approximation, we investigate the global existence of mild solutions for a class of Ito type stochastic differential equations whose coefficients satisfy conditions more general than the Lipschitz and linear growth ones.

Keywords: mild solution, Picard approximations

MSC 2000: 60H15

1. Introduction

Let us consider a stochastic differential equation of Ito type

$$\begin{cases} \mathrm{d}X(t) = \big(AX(t) + F(t, X(t))\big)\,\mathrm{d}t + B(t, X(t))\,\mathrm{d}W(t), \\ X(0) = \xi. \end{cases} \tag{1}$$

We will assume that a probability space $(\Omega, \mathcal{F}, P)$ together with a complete right continuous filtration $\mathcal{F}_t$, $t \geqslant 0$ are given. We denote by $\mathcal{P}_T$ the predictable σ-fields on $\Omega_T = [0, T] \times \Omega$.

Let U and H be two separable Hilbert spaces and W a Wiener process on U with the covariance operator Q, positive, linear and bounded on U with $\operatorname{Tr} Q < \infty$. Let $U_0 = Q^{1/2}(U)$ with the induced norm $\|u\|_0 = \|Q^{-1/2}u\|$. Denote by L_2^0 the separable Hilbert space of all Hilbert-Schmidt operators from U_0 to H equipped with the norm

$$\|D\|_{L_2^0} = \left(\sum_{j=1}^{\infty} \|DQ^{1/2}e_j\|^2\right)^{1/2}, \quad D \in L_2^0$$

where $\{e_j\}$ is a complete orthonormal basis on U. The spaces H and L_2^0 are equipped with Borel σ-fields $\mathcal{B}(H)$ and $\mathcal{B}(L_2^0)$. Moreover, ξ is an H-valued random variable, $\mathcal{F}_0$-measurable.

We fix $T > 0$ and impose the following conditions on the coefficients A, F and B of the equation (1):

(i) A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, $t \geqslant 0$ in H.

(ii) The mapping $F\colon [0,T] \times \Omega \times H \to H$, $(t,\omega,x) \to F(t,\omega,x)$ is measurable from $(\Omega_T \times H, \mathcal{P}_T \times \mathcal{B}(H))$ into $(H, \mathcal{B}(H))$.

(iii) The mapping $B\colon [0,T] \times \Omega \times H \to L_2^0$, $(t,\omega,x) \to B(t,\omega,x)$ is measurable from $(\Omega_T \times H, \mathcal{P}_T \times \mathcal{B}(H))$ into $(L_2^0, \mathcal{B}(L_2^0))$.

A mapping $X\colon [0,T] \times \Omega \to H$ which is measurable from $(\Omega_T, \mathcal{P}_T)$ into $(H, \mathcal{B}(H))$ is said to be a *mild solution* of (1), if for arbitrary $t \in [0,T]$ we have

$$P\biggl(\int_0^t (\|S(t-s)F(s,X(s))\| + \|S(t-s)B(s,X(s))\|_{L_2^0}^2)\,\mathrm{d}s < +\infty\biggr) = 1$$

and

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s,X(s))\,\mathrm{d}s + \int_0^t S(t-s)B(s,X(s))\,\mathrm{d}W(s) \ \ P\text{-a.s.}$$

Existence and uniqueness theorems for solutions of the equation (1) under Lipschitz conditions on the coefficients were studied by A. Ichikawa at the beginning of the eighties (see [8]). Since then much more general results have been established, most of them concerning equations with a non-Lipschitz drift satisfying some dissipativity type conditions (see [5], Chapter 7, [6], Chapter 5, and the reference therein). Many general theorems on existence of mild solutions of (1) were obtained by R. Manthey and his coworkers (see e.g. [10]), and I. Gyöngy, E. Pardoux et al. (see e.g. [2]). A remarkable early attempt at proving the existence of mild solutions to a stochastic semilinear heat equations with an additive (but cylindrical) Wiener process using Picard approximations under Yamada type assumptions upon the drift may be found in a paper of R. Manthey (see [9]). Recently Eddahbi and Erraoui have proved in [7] the existence and uniqueness result for a quasi-linear parabolic stochastic differential equations with non-Lipschitz coefficients.

For ordinary stochastic differential equations there are some articles which have dealt with existence and uniqueness of solution under non-Lipschitz coefficients. Results on the convergence of the Picard approximations under assumptions closely related to those used in our article may be found in a paper by T. Yamada (see [16]) and in a paper by T. Taniguchi (see [14]).

In [3] the first author extended the results of Taniguchi [14] to the infinite dimensional case using the technique of measure of noncompactness. In this paper we show that similar results can be obtained without using measures of noncompactness.

The following proposition ([5], Proposition 7.3) is an important estimation concerning stochastic convolution.

Proposition 1.1. *Let $p > 2$, $T > 0$ and let Φ be an L_2^0-valued, predictable process such that $E\bigl(\int_0^T \|\Phi(s)\|^p_{L_2^0}\,\mathrm{d}s\bigr) < +\infty$. Then there exists a constant C_T such that*

$$E\biggl(\sup_{t\in[0,T]}\biggl\|\int_0^t S(t-s)\Phi(s)\,\mathrm{d}W(s)\biggr\|^p\biggr) \leqslant C_T E\biggl(\int_0^T \|\Phi(s)\|^p_{L_2^0}\,\mathrm{d}s\biggr).$$

Moreover, $W_A^{\Phi}(t) = \int_0^t S(t-s)\Phi(s)\,\mathrm{d}W(s)$ has a continuous modification.

Remark 1.1. (i) If A generates a contraction semigroup, then Proposition 1.1 is true for $p \geqslant 2$ (see [15]).

(ii) A generalization of Proposition 1.1 to evolution systems can be found in [12].

2. Existence and uniqueness of solutions

Let us fix a real number p, $p > 2$ and denote by B_T the space of all H-valued predictable processes $X(t,\omega)$ defined on $[0,T]\times\Omega$ which are continuous in t for a.e. fixed $\omega\in\Omega$ and satisfy

$$\|X(\cdot,\cdot)\|_{B_T} \stackrel{\text{def}}{=} \Bigl\{E\Bigl(\sup_{0\leqslant t\leqslant T}\|X(t,\omega)\|^p\Bigr)\Bigr\}^{1/p} < \infty.$$

The space B_T is a Banach space (see [1] for $p = 2$, the case $p > 2$ has a similar proof).

In the following we shall impose Taniguchi conditions on F and B (see [14]), which are:

(a1) The functions $F(t,\omega,x)$ and $B(t,\omega,x)$ are continuous in x for each fixed $(t,\omega)\in\Omega_T$ and there exists a function $H\colon\ [0,T]\times[0,\infty)\to[0,\infty)$, $(t,u)\to H(t,u)$ such that

$$E(\|F(t,X)\|^p) + E(\|B(t,X)\|^p_{L_2^0}) \leqslant H(t, E(\|X\|)^p)$$

for all $t\in[0,T]$ and all $X\in L^p(\Omega,\mathcal{F},H)$.

(a2) $H(t,u)$ is locally integrable in t for each fixed $u\in[0,\infty)$, it is continuous and nondecreasing in u for each fixed $t\in[0,T]$ and for all $\alpha>0$, $u_0\geqslant 0$ the integral equation $u(t) = u_0 + \alpha\int_0^t H(s,u(s))$ has a global solution on $[0,T]$.

(a3) There exists a function $K\colon [0,T]\times[0,\infty)\to[0,\infty)$ which is locally integrable in t for each fixed $u\in[0,\infty)$ and continuous, monotone nondecreasing in u for each fixed $t\in[0,T]$. Moreover, $K(t,0)\equiv 0$ and

$$E(\|F(t,X)-F(t,Y)\|^p)+E(\|B(t,X)-B(t,Y)\|^p_{L_2^0})\leqslant K(t,E(\|X-Y\|^p))$$

for all $t\in[0,T]$ and $X,Y\in L^p(\Omega,\mathcal{F},H)$.

(a4) If a nonnegative, continuous function z satisfies

$$\begin{cases} z(t)\leqslant \alpha\int_0^t K(s,z(s))\,\mathrm{d}s, & t\in[0,T]\\ z(0)=0 \end{cases}$$

for some $\alpha>0$, then $z(t)=0$ for all $t\in[0,T]$.

Remark 2.1. (i) The inequality from (a3) is satisfied if the function K is concave with respect to u for each fixed $t\geqslant 0$ and

$$\|F(t,x)-F(t,y)\|^p+\|B(t,x)-B(t,y)\|^p_{L_2^0}\leqslant K(t,\|x-y\|^p)$$

for all $x,y\in H$ and $t\geqslant 0$. This follows immediately from Jensen's inequality.

(ii) The function $K(t,u)=\lambda(t)\alpha(u)$, $t\geqslant 0$, $u\geqslant 0$, where $\lambda(t)\geqslant 0$ is locally integrable and $\alpha\colon R_+\to R_+$ is a continuous, monotone nondecreasing and concave function with $\alpha(0)=0$, $\alpha(u)>0$ for $u>0$ and $\int_{0+}1/\alpha(u)\,\mathrm{d}u=\infty$, is an example for (a3) (see [14]).

In the following we shall consider Picard type approximations to (1):

$$\begin{cases} X_0(t)=S(t)\xi,\\ X_{n+1}(t)=S(t)\xi+\int_0^t S(t-s)F(s,X_n(s))\,\mathrm{d}s\\ \qquad\qquad +\int_0^t S(t-s)B(s,X_n(s))\,\mathrm{d}W(s), \quad t\in[0,T], \quad n\geqslant 0. \end{cases}$$

The main result of this paper is

Theorem 2.1. *Under the conditions* (a1) *through* (a4), *assume that*

$$\xi\in L^p(\Omega,\mathcal{F}_0,P).$$

Then the sequence $\{X_n\}_{n\geqslant 0}$ *converges in* B_T *to the unique solution of* (1) *in* B_T.

For the proof of theorem we shall state some lemmas.

Lemma 2.1. *Under the conditions* (a1) *through* (a3) *the operator* $G\colon B_T \to B_T$,

$$GX(t) = S(t)\xi + \int_0^t S(t-s)F(s,X(s))\,\mathrm{d}s + \int_0^t S(t-s)B(s,X(s))\,\mathrm{d}W(s),$$

$t \in [0,T]$ *is well defined and continuous.*

P r o o f. If $X \in B_T$ then $E(\|X(s)\|^p) \leqslant E\Big(\sup_{0\leqslant t\leqslant T} \|X(s)\|^p\Big) = \|X\|^p_{B_T}$. We have

$$\begin{aligned} E\Big(\sup_{0\leqslant t\leqslant T}\|GX(t)\|^p\Big) &\leqslant 3^p E\Big(\sup_{t\in[0,T]}\|S(t)\xi\|^p\Big) \\ &\quad + 3^p E\bigg(\sup_{t\in[0,T]}\bigg\|\int_0^t S(t-s)F(s,X(s))\,\mathrm{d}s\bigg\|^p\bigg) \\ &\quad + 3^p E\bigg(\sup_{t\in[0,T]}\bigg\|\int_0^t S(t-s)B(s,X(s))\,\mathrm{d}W(s)\bigg\|^p\bigg) \\ &\leqslant 3^p M^p E(\|\xi\|^p) + 3^p M^p T^{p-1}\int_0^T E(\|F(s,X(s))\|^p)\,\mathrm{d}s \\ &\quad + 3^p C_T \int_0^T E(\|B(s,X(s))\|^p_{L_2^0})\,\mathrm{d}s \\ &\leqslant 3^p M^p E(\|\xi\|^p) + C'_T\int_0^T H(s,\|X\|^p_{B_T})\,\mathrm{d}s < \infty. \end{aligned}$$

We have denoted $M = \sup_{t\in[0,T]}\|S(t)\|_{L(H)}$, $C'_T = 3^p M^p T^{p-1} + 3^p C_T$ and applied the Hölder inequality for the first integral and Proposition 1.1 for the second integral.

The continuity of the operator G follows easily. In fact, for $X, X_1, \ldots$ in B_T we have

$$\begin{aligned} \|GX - GX_n\|^p_{B_T} &= E\Big(\sup_{t\in[0,T]}\|GX(t) - GX_n(t)\|^p\Big) \\ &\leqslant 2^p M^p T^{p-1}\int_0^T E\big(\|F(s,X(s)) - F(s,X_n(s))\|^p\big)\,\mathrm{d}s \\ &\quad + 2^p C_T\int_0^T E\big(\|B(s,X(s)) - B(s,X_n(s))\|^p_{L_2^0}\big)\,\mathrm{d}s \\ &\leqslant C'_T\int_0^T K\big(s, E(\|X(s) - X_n(s)\|^p)\big)\,\mathrm{d}s \\ &\leqslant C'_T\int_0^T K(s,\|X - X_n\|^p_{B_T})\,\mathrm{d}s \end{aligned}$$

from which we get $\|GX - GX_n\|^p_{B_T} \to 0$ as $\|X - X_n\|_{B_T} \to 0$. □

Lemma 2.2. *Under the condition* (a1) *through* (a3), *there exists* $C'_T > 0$ *such that, if* X *and* Y *are in* B_T, *then*

$$\|GX - GY\|^p_{B_t} \leqslant C'_T \int_0^t K(s, \|X - Y\|^p_{B_s})\,\mathrm{d}s$$

for each $t \in [0, T]$.

P r o o f. The proof is contained in the proof of Lemma 2.1. □

Lemma 2.3. *Under the conditions* (a1) *and* (a2) *the sequence* $\{X_n\}_{n \geqslant 0}$ *is bounded in the space* B_T.

P r o o f. For $n \geqslant 0$ we have, by the same argument as in Lemma 2.1,

$$\|X_{n+1}\|^p_{B_t} \leqslant k_1 + k_2 \int_0^t H(s, \|X_n\|^p_{B_s})\,\mathrm{d}s \tag{2}$$

where k_1, k_2 are positive constants independent of n. Let $u(t)$, $t \in [0, T]$, be a global solution of the equation

$$u(t) = u_0 + k_2 \int_0^t H(s, u(s))\,\mathrm{d}s, \quad t \in [0, T]$$

with an initial condition $u_0 > \max(k_1, M^p E(\|\xi\|^p))$. We shall prove by mathematical induction that

$$\|X_n(t)\|^p_{B_t} \leqslant u(t) \quad \text{for } t \in [0, T]. \tag{3}$$

For $n = 0$ the inequality (3) holds by the definition of u. Let us suppose that

$$\|X_n(t)\|^p_{B_t} \leqslant u(t) \quad \text{for } t \in [0, T].$$

Then by (2) we obtain that

$$u(t) - \|X_{n+1}\|^p_{B_t} \geqslant k_2 \int_0^t \big(H(s, u(s)) - H(s, \|X_n\|^p_{B_s})\big)\,\mathrm{d}s \geqslant 0.$$

The inequalities follow from the assumption of the mathematical induction and (a2). □

Lemma 2.4. *Under the conditions* (a1) *through* (a4) *the sequence* $\{X_n\}_{n\geqslant 0}$ *is a Cauchy sequence in* B_T *and the limit is a mild solution for equation* (1).

Proof. Let

$$r_n(t) = \sup_{m\geqslant n} (\|X_m - X_n\|^p_{B_t}), \quad t \in [0,T], \quad n \geqslant 0.$$

The functions r_n, $n \geqslant 0$, are well defined, uniformly bounded (by Lemma 2.3) and, evidently, monotone nondecreasing. Since $\{r_n(t)\}_{n\geqslant 0}$ is a monotone nonincreasing sequence for each $t \in [0,T]$, there exists a monotone nondecreasing function r such that

$$\lim_{n\to\infty} r_n(t) = r(t). \tag{4}$$

By an argument similar to that in Lemma 2.2, we find

$$\|X_m - X_n\|^p_{B_t} \leqslant k \int_0^t K(s, \|X_{m-1} - X_{n-1}\|^p_{B_s}) \,\mathrm{d}s$$

for some positive constant k, from which it follows that

$$r(t) \leqslant r_n(t) \leqslant k \int_0^t K(s, r_{n-1}(s)) \,\mathrm{d}s.$$

Taking into account (4) and the Lebesgue convergence theorem, we obtain

$$r(t) \leqslant k \int_0^t K(s, r(s)) \,\mathrm{d}s. \tag{5}$$

Now it follows from (a4) that $r \equiv 0$ provided r is continuous. The case of a non-negative, monotone nondecreasing function r which satisfies (5) is the object of Lemma 2.2 in [3]. But $\|X_m - X_n\|^p_{B_T} \leqslant r_n(T)$ and $r_n(T) \overset{n\to\infty}{\longrightarrow} r(T) = 0$. Therefore $\|X_m - X_n\|_{B_T} \overset{n,m\to\infty}{\longrightarrow} 0$. The last part of the lemma is a consequence of continuity of the operator G. □

Lemma 2.5. *Equation* (1) *has at most one solution in* B_T.

Proof. If $X, Y \in B_T$ were two fixed points of G, then we would have

$$\begin{aligned} E\Big(\sup_{0\leqslant s\leqslant t} \|X(s) - Y(s)\|^p\Big) &\leqslant 2^p M^p t^{p-1} E\Big(\int_0^t \|F(s, X(s)) - F(s, Y(s))\|^p \,\mathrm{d}s\Big) \\ &\quad + 2^p C_T E\Big(\int_0^t \|B(s, X(s)) - B(s, Y(s))\|^p_{L_2^0} \,\mathrm{d}s\Big) \\ &\leqslant (2^p M^p t^{p-1} + 2^p C_T) \int_0^t K(s, E(\|X(s) - Y(s)\|^p)) \,\mathrm{d}s. \end{aligned}$$

Therefore

$$\|X - \|^p_{B_t} \leqslant (2^p M^p T^{p-1} + 2^p C_T) \int_0^t K(s, \|X - Y\|^p_{B_t})\,\mathrm{d}s.$$

Condition (a4) yields that $\|X - Y\|^p_{B_T} \equiv 0$, that is $X \equiv Y$. □

Remark 2.2. To obtain the existence of mild solutions to equation (1) under the conditions (a1) through (a4), the assumption $E(|\xi|^p) < \infty$ can be omitted. Indeed, it can be shown that if ξ and η are two initial conditions satisfying $E(|\xi|^p) < \infty$, $E(|\eta|^p) < \infty$ and $X, Y \in B_T$ are the corresponding solutions of equation (1) then

$$I_\Gamma X = I_\Gamma Y \quad P\text{-a.s.}$$

where $\Gamma = \{\omega \in \Omega\colon\ \xi(\omega) = \eta(\omega)\}$. The argument is the same as in [5], Theorem 7.4. Now if $E(|\xi|^p) = \infty$ then we define, for $n = 1, 2, \ldots$,

$$\xi_n = \begin{cases} \xi, & \text{if } |\xi| \leqslant n, \\ 0, & \text{if } |\xi| > n \end{cases}$$

and denote by $X_n \in B_T$ the corresponding solution of (1). By the previous argument we have

$$X_n(t) = X_{n+1}(t) \quad \text{on } \{\omega \in \Omega\colon\ |\xi| \leqslant n\}.$$

Therefore the process

$$X(t) = \lim_{n\to\infty} X_n(t)$$

is P-a.s. well defined and satisfies equation (1).

The following corollary is an immediate consequence of our Theorem 2.1 and Remark 2.1.

Corollary 2.1. *For the stochastic differential equation* (1), *suppose that the following conditions are satisfied:*

(i) $\|F(t,x) - F(t,y)\|^p + \|B(t,x) - B(t,y)\|^p_{L_2^0} \leqslant \lambda(t)\alpha(\|X - Y\|^p)$,

(ii) $E(\|F(t,0)\|)$, $E(\|B(t,0)\|_{L_2^0}) \in L^p_{\mathrm{loc}}([0,\infty), R^+)$ *for all* $t \in [0,\infty)$ *and* $x, y \in H$, *where* $\lambda(t) \geqslant 0$ *is locally integrable and* $\alpha\colon R_+ \to R_+$ *is a continuous, monotone nondecreasing and concave function with* $\alpha(0) = 0$ *and* $\int_{0+} 1/\alpha(u)\,\mathrm{d}u = \infty$.

Let $E(\|\xi\|^p) < \infty$. *Then on any finite interval* $[0,T]$ *the equation* (1) *has a unique solution which can be found by Picard approximations given in Theorem* 2.1.

Remark 2.2. (i) If $\lambda(t) \equiv L$ ($L > 0$) and $\alpha(u) = u$, $u \geqslant 0$ then condition (a3) implies a global Lipschitz condition.

(ii) Another example: $\alpha(u) = u\ln(1/u)$ for $0 < u < u_0$ (u_0 sufficiently small), $\alpha(0) = 0$ and $\alpha(u) = (au + b)$ for $u \geqslant u_0$, where $au + b$ is the tangent line of the function $u\ln(1/u)$ at the point u_0.

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