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A BOREL EXTENSION APPROACH TO WEAKLY COMPACT OPERATORS ON $C_0(T)$

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Dedicated to Professor K. S. Padmanabhan on the occasion of his seventieth birthday.

Abstract. Let X be a quasicomplete locally convex Hausdorff space. Let T be a locally compact Hausdorff space and let $C_0(T) = \{f\colon T \to \mathbb{C},\ f$ is continuous and vanishes at infinity$\}$ be endowed with the supremum norm. Starting with the Borel extension theorem for X-valued σ-additive Baire measures on T, an alternative proof is given to obtain all the characterizations given in [13] for a continuous linear map $u\colon C_0(T) \to X$ to be weakly compact.

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1. Introduction

Let T be a locally compact Hausdorff space and let $C_0(T)$ be the Banach space of all complex valued continuous functions vanishing at infinity in T, endowed with the supremum norm. Then its dual $M(T)$ is the Banach space of all bounded complex Radon measures μ on T with the norm given by $\|\mu\| = \mathrm{var}(\mu, \mathscr{B}(T))(T)$. Let X be a locally convex Hausdorff space (briefly, an lcHs) which is quasicomplete and let $u\colon C_0(T) \to X$ be a continuous linear map. When X is complete and T is compact, Grothendieck gave in Theorem 6 of [6] some necessary and sufficient conditions for u to be weakly compact. As observed in [14], Grothendieck's techniques, contrary to

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Remark 2 on p. 161 of [6], are not powerful enough to extend his characterizations when T is a non σ-compact locally compact Hausdorff space.

In [13], using the Baire and σ-Borel characterizations of weakly compact subsets of $M(T)$ as given in [12], we obtained 35 characterizations for the continuous linear map $u\colon C_0(T) \to X$ to be weakly compact, where X is a quasicomplete lcHs. These include the characterizations mentioned in Remark 2 on p. 161 of Grothendieck [6] and in Theorem 9.4.10 of [5], whose proof as given in [5] is incorrect without the hypothesis of σ-compactness of T (see [14]). In [13] we also obtained a theorem on regular Borel and σ-Borel extensions of X-valued σ-additive Baire measures on T (briefly, the Borel extension theorem) and Theorem 5.3 of Thomas [16] (dispensing with the technique of reduction to the metrizable compact case) as a consequence of these characterizations.

The Riesz representation theorem was used in [9], [10] to obtain the regular Borel and σ-Borel extensions of a complex Baire measure on T. The paper [13] can be considered to be its analogue for X-valued Baire measures on T with the Riesz representation theorem being replaced by the Bartle-Dunford-Schwartz representation of weakly compact operators, since the Borel extension theorem for such Baire measures was deduced there from the characterizations of weakly compact operators on $C_0(T)$.

On the other hand, the regular σ-Borel extension of positive Baire measures on T was used in Halmos [7] to derive the Riesz representation theorem for positive linear forms on $C_0(T)$. In this context the following question arises: Is it possible to obtain all the characterizations given in [13] for a continuous linear map $u\colon C_0(T) \to X$ to be weakly compact, starting with the Borel extension theorem for X-valued Baire measures on T? Recently, in our joint work with Dobrakov ([4]), combining the Borel extension theorem with the first part of Theorem 1 of [13] and Lemma 1 and Theorem 2 of [6], we answered the question in the affirmative when $c_0 \not\subset X$ and X is a quasicomplete lcHs (namely, Theorem 5.3 of [16]). In the present paper, we also answer the question in the affirmative for arbitrary quasicomplete lcHs X and for this, along with the Borel extension theorem, we use the quoted results of [13] and [6], Lemmas 1–7 of Section 2 below and Theorem 1 of [11]. Thus the present paper can be considered to be the vector analogue of the treatment of Halmos [7].

In this section we fix the notation and terminology. For the convenience of the reader we also give some definitions and results from literature.

In the sequel T will denote a locally compact Hausdorff space and $C_0(T)$ the Banach space of all complex valued continuous functions vanishing at infinity in T, endowed with the supremum norm $\|f\|_T = \sup_{t\in T} |f(t)|$.

Let $\mathscr{K}$ ($\mathscr{K}_0$) be the family of all compacts (compact $G_\delta s$) in T. $\mathscr{B}_0(T)$, $\mathscr{B}_c(T)$ and $\mathscr{B}(T)$ are the σ-rings generated by $\mathscr{K}_0$, $\mathscr{K}$ and the class of all open sets in T, respectively. The members of $\mathscr{B}_0(T)$ ($\mathscr{B}_c(T)$, $\mathscr{B}(T)$) are called *Baire sets* (*σ-Borel sets*, *Borel sets*, respectively) of T. Since a subset E of T belongs to $\mathscr{B}_c(T)$ if and only if E is a σ-bounded Borel set, the members of $\mathscr{B}_c(T)$ are called σ-Borel sets.

$M(T)$ is the Banach space of all bounded complex Radon measures on T with their domain restricted to $\mathscr{B}(T)$. Thus each $\mu \in M(T)$ is a Borel regular (bounded) complex measure on $\mathscr{B}(T)$ and has the norm given by $\|\mu\| = \operatorname{var}(\mu, \mathscr{B}(T))(T)$. For $\mu \in M(T)$, $|\mu|(E) = \operatorname{var}(\mu, \mathscr{B}(T))(E)$, $E \in \mathscr{B}(T)$.

We recall the following result from [12, Lemma 1].

Proposition 1. *For $\mu \in M(T)$,*

$$|\mu|\big|_{\mathscr{B}_0(T)}(\cdot) = \operatorname{var}(\mu\big|_{\mathscr{B}_0(T)}, \mathscr{B}_0(T))(\cdot)$$

and

$$|\mu|\big|_{\mathscr{B}_c(T)}(\cdot) = \operatorname{var}(\mu\big|_{\mathscr{B}_c(T)}, \mathscr{B}_c(T))(\cdot).$$

A vector measure is an additive set function defined on a ring of sets with values in an lcHs. In the sequel X will denote an lcHs with a topology τ. Let Γ be the set of all τ-continuous seminorms on X. The dual of X is denoted by X^*.

The strong topology $\beta(X^*, X)$ of X^* is the locally convex topology induced by the seminorms $\{p_B\colon B \text{ bounded in } X\}$, where $p_B(x^*) = \sup_{x\in B} |x^*(x)|$. X^{**} denotes the dual of $(X^*, \beta(X^*, X))$ and is endowed with the locally convex topology τ_e of uniform convergence in equicontinuous subsets of X^*. Note that $(X^*, \beta(X^*, X))$ and (X^{**}, τ_e) are lcHs.

It is well known that the canonical injection $J\colon X \to X^{**}$ given by $\langle Jx, x^*\rangle = \langle x, x^*\rangle$ for all $x \in X$ and $x^* \in X^*$, is linear. Identifying X with $JX \subset X^{**}$ one has $\tau_e\big|_{JX} = \tau_e\big|_X = \tau$.

Definition 1. A linear map $u\colon C_0(T) \to X$ is called a *weakly compact operator* on $C_0(T)$ if $\{uf\colon \|f\|_T \leqslant 1\}$ is relatively weakly compact in X.

The following result (Corollary 9.3.2 of [5], which is essentially a consequence of Lemma 1 of [6]) plays a key role in Section 4.

Proposition 2. *Let E and F be lcHs with F quasicomplete. If $u\colon E \to F$ is linear and continuous, then the following conditions are equivalent.*

(i) *u maps bounded subsets of E into relatively weakly compact subsets of F.*
(ii) *$u^*(A)$ is relatively $\sigma(E^*, E^{**})$-compact for each equicontinuous subset A of F^*.*
(iii) *$u^{**}(E^{**}) \subset F$.*

The following result is due to Theorem 2 of [6], which is the same as Theorem 4.22.1 of [5].

Proposition 3. *Let A be a bounded set in $M(T)$. Then the following assertions are equivalent.*

(i) *A is relatively weakly compact.*
(ii) *For each disjoint sequence $(U_n)_1^\infty$ of open sets in T,*

$$\lim_n \sup_{\mu \in A} |\mu(U_n)| = 0.$$

(iii) *For (U_n) as in* (ii), $\lim_n \sup_{\mu \in A} |\mu|(U_n) = 0$.
(iv) *Let $\varepsilon > 0$.*
   (a) *For each compact K in T, there exists an open set U in T such that $K \subset U$ and $\sup_{\mu \in A} |\mu|(U \setminus K) < \varepsilon$; and*
   (b) *there exists a compact C such that $\sup_{\mu \in A} |\mu|(T \setminus C) < \varepsilon$.*

For each τ-continuous seminorm p on X, let $p(x) = \|x\|_p$, $x \in X$, and let $X_p = (X, \|\cdot\|_p)$ be the associated seminormed space. The completion of the quotient normed space $X_p/p^{-1}(0)$ is denoted by $\tilde{X}_p$. Let $\Pi_p\colon X_p \to X_p/p^{-1}(0) \subset \tilde{X}_p$ be the canonical quotient map.

Let $\mathscr{S}$ be a σ-ring of subsets of a non empty set Ω. Given a vector measure $m\colon \mathscr{S} \to X$, for each τ-continuous seminorm p on X, let $m_p\colon \mathscr{S} \to \tilde{X}_p$ be given by $m_p(E) = (\Pi_p \circ m)(E)$ for $E \in \mathscr{S}$. Then m_p is a Banach space valued vector measure on $\mathscr{S}$. We define the p-semivariation $\|m\|_p$ of m by

$$\|m\|_p(E) = \|m_p\|(E) \ \text{ for } \ E \in \mathscr{S}$$

and

$$\|m\|_p(\Omega) = \|m_p\|(\Omega) = \sup_{E \in \mathscr{S}} \|m_p\|(E)$$

where $\|m_p\|$ is the semivariation of the vector measure m_p and is given by $\|m_p\|(E) = \sup\{|x^* \circ m|(E)\colon\ x^* \in \tilde{X}_p^*,\ \|x^*\| \leqslant 1\}$ (see p. 2 of [1]).

An X-valued vector measure m on a σ-ring $\mathscr{S}$ of subsets of Ω is said to be *bounded* if $\{m(E)\colon\ E \in \mathscr{S}\}$ is bounded in X and equivalently, if $\|m\|_p(\Omega) < \infty$ for each τ-continuous seminorm p on X. When m is σ-additive, then m_p is a Banach space valued σ-additive vector measure on the σ-ring $\mathscr{S}$ and hence by Corollary I.1.19 of [1], $\|m\|_p(\Omega) = \|m_p\|(\Omega) \leqslant 4\sup_{E\in\mathscr{S}}\|m(E)\|_p < \infty$.

For the theory of integration of bounded $\mathscr{S}$-measurable scalar functions with respect to a bounded quasicomplete lcHs-valued vector measure on the σ-ring $\mathscr{S}$, the reader is referred to [11] or [13]. We need the following results from Lemma 6 of [11] and Proposition 7 of [13].

Proposition 4. *Let X be a quasicomplete lcHs and let $\mathscr{S}$ be a σ-ring of subsets of Ω. Then:*

(i) *If f is a bounded $\mathscr{S}$-measurable scalar function and m is an X-valued bounded vector measure on $\mathscr{S}$, then f is m-integrable and*

$$x^*\left(\int_\Omega f\,\mathrm{d}m\right) = \int_\Omega f d(x^* \circ m)$$

for each $x^* \in X^*$.

(ii) *(Lebesgue bounded convergence theorem) If m is an X-valued σ-additive vector measure on $\mathscr{S}$ and (f_n) is a bounded sequence of $\mathscr{S}$-measurable scalar functions with $\lim_n f_n(w) = f(w)$ for each $w \in \Omega$, then f is m-integrable and*

$$\int_E f\,\mathrm{d}m = \lim_n \int_E f_n\,\mathrm{d}m$$

for each $E \in \mathscr{S}$.

The following result follows from the first part of Theorem 1 of [13], and is analogous to Theorem VI.2.1 of [1] for lcHs-valued continuous linear maps on $C_0(T)$. It plays a key role in Sections 3 and 4.

Proposition 5. *Let X be an lcHs. Let $u\colon C_0(T) \to X$ be a continuous linear map. Then there exists a unique X^{**}-valued vector measure m on $\mathscr{B}(T)$ possessing the following properties:*

(i) *$x^* \circ m \in M(T)$ for each $x^* \in X^*$ and consequently, $m\colon \mathscr{B}(T) \to X^{**}$ is σ-additive in the $\sigma(X^{**}, X^*)$-topology.*

(ii) *The mapping $x^* \to x^* \circ m$ of X^* into $M(T)$ is weak*-weak* continuous. Moreover, $u^* x^* = x^* \circ m$, $x^* \in X^*$.*

(iii) $x^*uf = \int_T f \, \mathrm{d}(x^* \circ m)$ *for each* $f \in C_0(T)$ *and* $x^* \in X^*$.
(iv) $\{m(E)\colon E \in \mathscr{B}(T)\}$ *is* τ_e*-bounded in* X^{**}.
(v) $m(E) = u^{**}(\chi_E)$ *for* $E \in \mathscr{B}(T)$.

Definition 2. Let $u\colon C_0(T) \to X$ be a continuous linear map. Then the vector measure m as given in Proposition 5 is called the *representing measure* of u.

Definition 3. A σ-additive vector measure $m\colon \mathscr{B}_0(T) \to X$ ($\mathscr{B}(T) \to X$, $\mathscr{B}_c(T) \to X$) is called an *X-valued Baire* (*Borel, σ-Borel*) *measure* on T.

Definition 4. Let $\mathscr{S}$ be a σ-ring of sets in T with $\mathscr{S} \supset \mathscr{K}$ or $\mathscr{K}_0$. Let $m\colon \mathscr{S} \to X$ be a vector measure. Then m is said to be *$\mathscr{S}$-regular* (*$\mathscr{S}$-outer regular, $\mathscr{S}$-inner regular*) in $E \in \mathscr{S}$ if, given a p in Γ and $\varepsilon > 0$, there exist a compact $K \in \mathscr{S}$ and an open set $U \in \mathscr{S}$ with $K \subset E \subset U$ (an open set $U \in \mathscr{S}$ with $E \subset U$, a compact $K \in \mathscr{S}$ with $K \subset E$) such that $\|m\|_p(U \setminus K) < \varepsilon$ ($\|m\|_p(U \setminus E) < \varepsilon$, $\|m\|_p(E \setminus K) < \varepsilon$, respectively). Even though T does not belong to $\mathscr{S}$ one can define *$\mathscr{S}$-inner regularity* of m in T as follows. Given $p \in \Gamma$ and $\varepsilon > 0$, there exists a compact $K \in \mathscr{S}$ such that $\|m\|_p(B) < \varepsilon$ for all $B \in \mathscr{S}$ with $B \subset T \setminus K$. The vector measure m is said to be *$\mathscr{S}$-regular* (*$\mathscr{S}$-outer regular, $\mathscr{S}$-inner regular*) if it is so in each $E \in \mathscr{S}$. When $\mathscr{S} = \mathscr{B}(T)$ ($\mathscr{B}_0(T)$, $\mathscr{B}_c(T)$), we use the term *Borel* (*Baire, σ-Borel*) *regularity* or *outer regularity* or *inner regularity*.

Remark 1. In the above definition one can replace Γ by any other family of continuous seminorms on X which induces the topology τ.

The following proposition on regular Borel and σ-Borel extensions of an X-valued Baire measure on T is well known and plays a key role in Section 4. It was first proved in [3], [8] for Banach space valued Baire measures on T and extended to group valued measures in [15]. For a simple and direct proof of the proposition see [4]. Note that a highly technical operator theoretic proof is given in [13] as mentioned in the introduction.

Proposition 6. *Let m be an X-valued Baire measure on T and let X be a quasicomplete lcHs. Then m is Baire regular in T. Moreover, there exists a unique X-valued Borel (σ-Borel) regular σ-additive extension $\hat{m}$ ($\hat{m}_c$) of m on $\mathscr{B}(T)$ ($\mathscr{B}_c(T)$, respectively). Moreover, $\hat{m}\big|_{\mathscr{B}_c(T)} = \hat{m}_c$.*

Throughout this section X denotes a quasicomplete lcHs with the topology τ. Let $u\colon C_0(T) \to X$ be a continuous linear map with the representing measure m. Let $m_0 = m\big|_{\mathscr{B}_0(T)}$ and $m_c = m\big|_{\mathscr{B}_c(T)}$.

Let $\mathscr{E} = \{A \subset X^*\colon A \text{ equicontinuous}\}$, and let $p_A(x) = \sup_{x^*\in A}|x^*(x)|$ and $p_A(x^{**}) = \sup_{x^*\in A}|x^{**}(x^*)|$ for $A \in \mathscr{E}$, $x \in X$ and $x^{**} \in X^{**}$. Then the family of seminorms $\Gamma_{\mathscr{E}} = \{p_A\colon A \in \mathscr{E}\}$ induces the topology τ of X and τ_e of X^{**}.

Let $X_A = X_{p_A}/p_A^{-1}(0)$ and let $Y_A = \widetilde{X_A}$, the completion of the normed space X_A. For $E \in \mathscr{B}(T)$,

$$\|m_{p_A}\|(E) = \sup\{|y^* \circ m|(E)\colon\ y^* \in {Y_A}^*,\ \ \|y^*\| \leqslant 1\}.$$

Lemma 1. *Let $A \in \mathscr{E}$. Then:*

(i) *For $E \in \mathscr{B}(T)$*

$$\|m_{p_A}\|(E) = \|m\|_{p_A}(E) = \sup\{|x^* \circ m|(E)\colon\ x^* \in A\}.$$

(ii) *For $E \in \mathscr{B}_c(T)$*

$$\begin{aligned}\|(m_c)_{p_A}\|(E) = \|m_c\|_{p_A}(E) &= \sup\{|x^* \circ m_c|(E)\colon\ x^* \in A\}\\ &= \sup\{|x^* \circ m|(E)\colon\ x^* \in A\}\end{aligned}$$

where $|x^* \circ m_c|(E) = \mathrm{var}(x^* \circ m_c, \mathscr{B}_c(T))(E)$.

(iii) *For $E \in \mathscr{B}_0(T)$*

$$\begin{aligned}\|(m_0)_{p_A}\|(E) = \|m_0\|_{p_A}(E) &= \sup\{|x^* \circ m_0|(E)\colon\ x^* \in A\}\\ &= \sup\{|x^* \circ m|(E)\colon\ x^* \in A\}\end{aligned}$$

where $|x^* \circ m_0|(E) = var(x^* \circ m_0, \mathscr{B}_0(T))(E)$.

P r o o f. Each element $\tilde{x} \in X_A$ is of the form $\tilde{x} = x + {p_A}^{-1}(0)$ for some $x \in X$ and it is easy to show that the quotient norm $\|\tilde{x}\|_{p_A} = p_A(x)$. For $x^* \in A$, let $\Psi_{x^*}(x+{p_A}^{-1}(0)) = x^*(x)$. Then $\Psi_{x^*}\colon X_A \to \mathbb{C}$ is well defined and linear. Moreover, for $x^* \in A$,

$$|\Psi_{x^*}(x + {p_A}^{-1}(0))| = |x^*(x)| \leqslant p_A(x) = \|x + {p_A}^{-1}(0)\|_{p_A}$$

and hence $\|\Psi_{x^*}\| \leqslant 1$. Then by continuity Ψ_{x^*} has a unique continuous linear extension to the whole of Y_A with the norm less than or equal to one and we denote this extension again by Ψ_{x^*}. Clearly, the mapping $x^* \to \Psi_{x^*}$ of A into ${Y_A}^*$ is injective. For $\tilde{x} = x + {p_A}^{-1}(0) \in X_A$ with $x \in X$ we have

$$\begin{aligned} \|\tilde{x}\|_{p_A} &= \|x + {p_A}^{-1}(0)\|_{p_A} = p_A(x) = \sup_{x^* \in A} |x^*(x)| \\ &= \sup_{x^* \in A} |\Psi_{x^*}(\tilde{x})| \leqslant \sup_{y^* \in {Y_A}^*,\, \|y^*\| \leqslant 1} |y^*(\tilde{x})| = \|\tilde{x}\|_{p_A} \end{aligned}$$

and hence

$$\|\tilde{x}\|_{p_A} = \sup_{x^* \in A} |\Psi_{x^*}(\tilde{x})|. \tag{1}$$

Let us write $\Psi_{x^*}(y) = x^*(y)$ for $x^* \in A$ and $y \in Y_A$. Let $y \in Y_A$ and let $\varepsilon > 0$. Since X_A is dense in Y_A, there exists $\tilde{x} \in X_A$ such that $|y - \tilde{x}|_{p_A} < \varepsilon$. Then by (1) we have

$$\begin{aligned} \|y\|_{p_A} &< \varepsilon + \|\tilde{x}\|_{p_A} = \varepsilon + \sup_{x^* \in A} |\Psi_{x^*}(\tilde{x})| \\ &\leqslant \varepsilon + \sup_{x^* \in A} |\Psi_{x^*}(\tilde{x} - y)| + \sup_{x^* \in A} |\Psi_{x^*}(y)| \\ &\leqslant \varepsilon + \|\tilde{x} - y\|_{p_A} + \sup_{x^* \in A} |x^*(y)| < 2\varepsilon + \sup_{x^* \in A} |x^*(y)| \end{aligned}$$

and hence

$$\|y\|_{p_A} = \sup_{x^* \in A} |\Psi_{x^*}(y)| = \sup_{x^* \in A} |x^*(y)|$$

for $y \in Y_A$. Thus $\{\Psi_{x^*}\colon\ x^* \in A\}$ is a norm determining subset of $\{y^* \in {Y_A}^*\colon\ \|y^*\| \leqslant 1\}$. Using this result and writing $\Psi_{x^*}(y) = x^*(y)$ for all $x^* \in A$ and $y \in Y_A$ in the proof of the first part of Proposition 11 of [1], one can show that

$$\begin{aligned} \|m\|_{p_A}(E) &= \|m_{p_A}\|(E) = \sup\{|\Psi_{x^*} \circ m|(E)\colon\ x^* \in A\} \\ &= \sup\{|x^* \circ m|(E)\colon\ x^* \in A\} \end{aligned} \tag{2}$$

for $E \in \mathscr{B}(T)$. Thus (i) holds.

Replacing m by m_c (by m_0) and $\mathscr{B}(T)$ by $\mathscr{B}_c(T)$ (by $\mathscr{B}_0(T)$) in the above argument, similarly we have

$$\|(m_c)_{p_A}\|(E) = \|m_c\|_{p_A}(E) = \sup\{|x^* \circ m_c|(E)\colon\ x^* \in A\}$$

for $E \in \mathscr{B}_c(T)$ and

$$\|(m_0)_{p_A}\|(E) = \|m_0\|_{p_A}(E) = \sup\{|x^* \circ m_0|(E)\colon\ x^* \in A\}$$

for $E \in \mathscr{B}_0(T)$. Now a reference to Proposition 1 completes the proofs of (ii) and (iii) of the lemma. $\square$

The following result is the same as Lemma 2 of [13].

Lemma 2. *u^*A is bounded in $M(T)$ for each $A \in \mathscr{E}$.*

Notation 1. $\mathscr{U}_0$ denotes the family of all open Baire sets in T.

Lemma 3. *Suppose $m_0(\mathscr{U}_0) \subset X$. Then:*

(i) *m_0 is σ-additive in $\mathscr{U}_0$ in τ. That is, given a disjoint sequence $(U_n)_1^\infty$ in $\mathscr{U}_0$, then $m_0\bigl(\bigcup_1^\infty U_n\bigr) = \sum_1^\infty m_0(U_n)$ (in the topology τ).*

(ii) *If $(U_n)_1^\infty$ is a disjoint sequence in $\mathscr{U}_0$, then, for each $A \in \mathscr{E}$, $\lim_n \|m_0\|_{p_A}(U_n) = 0$.*

P r o o f. (i) By Proposition 5 (i), $x^* \circ m \in M(T)$ for $x^* \in X^*$ and hence

$$(x^* \circ m_0)\left(\bigcup_1^\infty U_n\right) = \sum_1^\infty (x^* \circ m_0)(U_n)$$

for each $x^* \in X^*$. By hypothesis, $m_0(\mathscr{U}_0) \subset X$ and hence by the Orlicz-Pettis theorem we conclude that $m_0\bigl(\bigcup_1^\infty U_n\bigr) = \sum_1^\infty m_0(U_n)$ in the topology τ. Thus (i) holds.

(ii) If possible, let $\inf_n \|m_0\|_{p_A}(U_n) > 4\delta > 0$ for some $A \in \mathscr{E}$. Then by Lemma 1 we have $\sup_{x^* \in A} |x^* \circ m_0|(U_n) > 4\delta$ for all n. Then there exists an $x_n^* \in A$ such that $|x_n^* \circ m_0|(U_n) > 4\delta$. Consequently, $\sup_{B \in \mathscr{B}_0(T),\, B \subset U_n} |(x_n^* \circ m_0)(B)| > \delta$ and hence there exists $B_n \subset U_n$ in $\mathscr{B}_0(T)$ such that $|(x_n^* \circ m_0)(B_n)| > \delta$. Since $x_n^* \circ m_0$ is a (σ-additive) scalar Baire measure, it is Baire regular and hence there exists an open Baire set G_n with $B_n \subset G_n \subset U_n$ such that $|(x_n^* \circ m_0)(G_n)| > \delta$. Consequently, $\inf_n |(x_n^* \circ m_0)(G_n)| > \delta$. This is absurd, since $|(x_n^* \circ m_0)(G_n)| \leqslant \|m_0(G_n)\|_{p_A} \to 0$ by (i) as (G_n) is a disjoint sequence in $\mathscr{U}_0$. □

Lemma 4. *m_0 is Baire inner regular in $E \in \mathscr{B}_0(T)$ if and only if, for each $A \in \mathscr{E}$ and $\varepsilon > 0$, there exists a compact $K \in \mathscr{K}_0$ with $K \subset E$ such that $\sup_{\mu \in u^*A} |\mu|(E \setminus K) < \varepsilon$; i.e. if and only if, for each $A \in \mathscr{E}$, u^*A is uniformly Baire inner regular in E in the sense of Definition* 1 *of* [12].

P r o o f. Let m_0 be Baire inner regular in $E \in \mathscr{B}_0(T)$. Given $A \in \mathscr{E}$ and $\varepsilon > 0$, by Definition 4 there exists $K \in \mathscr{K}_0$ with $K \subset E$ such that $\|m_0\|_{p_A}(E \setminus K) < \varepsilon$. Then by Lemma 1 and Proposition 5 (ii) we have

$$\|m_0\|_{p_A}(E \setminus K) = \sup_{x^* \in A} |x^* \circ m|(E \setminus K) = \sup_{\mu \in u^*A} |\mu|(E \setminus K) < \varepsilon.$$

The converse is immediate from Definition 4 and Lemma 1 as $u^*A = \{x^* \circ m\colon\ x^* \in A\}$ by Proposition 5 (ii) and $\Gamma_{\mathscr{E}} = \{p_A\colon\ A \in \mathscr{E}\}$ induces the topology τ_e of X^{**}. □

In the proofs of Lemmas 5 and 6 below we use, respectively, the implications (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) of Theorem 1 of [12].

Lemma 5. *Let $m_0(\mathscr{U}_0) \subset X$. Then:*

(i) *m_0 is Baire inner regular (in τ_e) in each $U \in \mathscr{U}_0$.*

(ii) *For each $\varepsilon > 0$ and for each $A \in \mathscr{E}$, there exists a $K \in \mathscr{K}_0$ such that $\|m\|_{p_A}(T \setminus K) = \sup_{x^* \in A} |x^* \circ m|(T \setminus K) < \varepsilon$.*

P r o o f. Let $A \in \mathscr{E}$. Then by Proposition 5 (ii), $u^*A = \{x^* \circ m \colon x^* \in A\}$ and by Lemma 2, u^*A is bounded in $M(T)$. By Proposition 1, Lemma 1 and Lemma 3 (ii), for each disjoint sequence (U_n) of open Baire sets we have $\lim_n \sup_{x^* \in A} |x^* \circ m|(U_n) = \lim_n \sup_{\mu \in u^*A} |\mu|(U_n) = 0$. Thus by the implication (iii) $\Rightarrow$ (iv) of Theorem 1 of [12] the result holds. $\square$

Lemma 6. *Suppose m_0 is Baire inner regular in each $U \in \mathscr{U}_0$ with respect to the topology τ_e of X^{**} and, for each $\varepsilon > 0$ and for each $A \in \mathscr{E}$, suppose there exists $K \in \mathscr{K}_0$ such that $\|m_0\|_{p_A}(T \setminus K) = \sup_{x^* \in A}\{|x^* \circ m|(B) \colon B \subset T \setminus K,\ B \in \mathscr{B}_0(T)\} < \varepsilon$ (note that the range of m_0 is contained in X^{**}). Then m_0 is Baire inner regular in $\mathscr{B}_0(T)$ with respect to τ_e.*

P r o o f. Let $A \in \mathscr{E}$. Then by Lemma 2, u^*A is bounded in $M(T)$. Since m_0 is Baire inner regular in each open Baire set, Lemma 4 implies that u^*A is uniformly Baire inner regular (in the sense of Definition 1 of [12]) in each open Baire set.

Claim 1.

$$\|m\|_{p_A}(T \setminus K) = \sup_{x^* \in A} |x^* \circ m|(T \setminus K) = \sup_{\mu \in u^*A} |\mu|(T \setminus K) < \varepsilon. \tag{3}$$

In fact, by the second hypothesis, by the Borel regularity of $|x^* \circ m|$, by Theorem 50.D of [7] and by Lemma 1 (i), Proposition 1 and Proposition 5 (ii), we have

$$\begin{aligned}
\|m\|_{p_A}(T \setminus K) &= \sup_{x^* \in A} |x^* \circ m|(T \setminus K) \\
&= \sup_{\mu \in u^*A} \sup_{C \in \mathscr{K},\, C \subset T \setminus K} |\mu|(C) \\
&= \sup_{\mu \in u^*A} \sup_{C \in \mathscr{K}_0,\, C \subset T \setminus K} |\mu|(C) \\
&= \sup_{x^* \in A} \sup_{C \in \mathscr{K}_0,\, C \subset T \setminus K} |x^* \circ m_0|(C) \\
&< \varepsilon.
\end{aligned}$$

Hence the claim holds.

Thus, in virtue of (3), the hypotheses of the lemma show that u^*A satisfies the hypothesis of the statement (iv) of Theorem 1 of [12]. Consequently, by (iv) $\Rightarrow$ (v) of Theorem 1 of [12], u^*A is uniformly Baire inner regular in each $E \in \mathscr{B}_0(T)$. Since this holds for all $A \in \mathscr{E}$ and since $\Gamma_{\mathscr{E}}$ induces the topology τ_e, Lemma 4 yields that m_0 is Baire inner regular in $\mathscr{B}_0(T)$. $\square$

Lemma 7. *Suppose m (m_c, m_0) is Borel (σ-Borel, Baire) inner regular (in τ_e) in $\mathscr{B}(T)$ ($\mathscr{B}_c(T)$, $\mathscr{B}_0(T)$). Then m (m_c, m_0, respectively) is σ-additive in τ_e.*

Proof. Let $A \in \mathscr{E}$ and let $\varepsilon > 0$. Let $\mathscr{S} = \mathscr{B}(T)$ and $\gamma = m$ ($\mathscr{S} = \mathscr{B}_c(T)$ and $\gamma = m_c$; $\mathscr{S} = \mathscr{B}_0(T)$ and $\gamma = m_0$, respectively). Since $\|\gamma(E)\|_{p_A} \leqslant \|\gamma\|_{p_A}(E)$ for $E \in \mathscr{S}$, it suffices to show that $\lim_n \|\gamma\|_{p_A}(E_n) = 0$ whenever (E_n) is a decreasing sequence in $\mathscr{S}$ with $\bigcap_1^\infty E_n = \emptyset$. By hypothesis, for each n there exists a compact $K_n \in \mathscr{S}$ with $K_n \subset E_n$ such that $\|\gamma\|_{p_A}(E_n \setminus K_n) < \varepsilon/2^n$. Then adapting suitably the proof at the end of p. 158 and at the top of p. 159 of [1], we can show that there exists n_0 such that $\|\gamma\|_{p_A}(E_n) < \varepsilon$ for $n \geqslant n_0$. Hence the lemma holds. $\square$

4. Characterizations of weakly compact operators on $C_0(T)$

Let X be a quasicomplete lcHs. Using Propositions 1–6 and Lemmas 1–7 of the preceding sections and Theorem 1 of [11] we will obtain below all the 35 characterizations given in [13] for a continuous linear map $u\colon C_0(T) \to X$ to be weakly compact. As mentioned at the outset, the Borel extension theorem (Proposition 6) for σ-additive X-valued Baire measures on T plays a key role in the present proof in contrast to the proofs of the characterization theorems of [13].

Theorem 1. *Let $u\colon C_0(T) \to X$ be a continuous linear map, where X is a quasicomplete lcHs. Let m be the representing measure of u and let $m_c = m|_{\mathscr{B}_c(T)}$ and $m_0 = m|_{\mathscr{B}_0(T)}$. Then the following assertions are equivalent.*

(i) *u is weakly compact.*
(ii) *The range of m is contained in X.*
(iii) *The range of m_c is contained in X.*
(iv) *The range of m_0 is contained in X.*
(v) *$m(U) \in X$ for all open sets U in T.*
(vi) *$m(F) \in X$ for all closed sets F in T.*
(vii) *$m(U) \in X$ for all σ-Borel open sets U in T.*
(viii) *$m(U) \in X$ for all open Baire sets U in T.*
(ix) *$m(U) \in X$ for all open sets U in T which are σ-compact.*
(x) *$m(F) \in X$ for all closed sets F in T which are G_δ.*

(xi) $m(U) \in X$ *for all open sets* U *in* T *which are* F_σ.

(xii) *For each increasing sequence* $(f_n)_1^\infty \subset C_0(T)$ *with* $0 \leqslant f_n \leqslant 1$, (uf_n) *converges weakly in* X.

(xiii) m *is* σ*-additive in the topology* τ_e *of* X^{**}.

(xiv) m_c *is* σ*-additive in the topology* τ_e *of* X^{**}.

(xv) m_0 *is* σ*-additive in the topology* τ_e *of* X^{**}.

(xvi) m *is strongly additive in the topology* τ_e *of* X^{**}.

(xvii) m_c *is strongly additive in the topology* τ_e *of* X^{**}.

(xviii) m_0 *is strongly additive in the topology* τ_e *of* X^{**}.

(xix) m *is Borel regular in* τ_e *of* X^{**}.

(xx) m *is Borel inner regular in* τ_e *of* X^{**}.

(xxi) m *is Borel inner regular (in* τ_e*) in each open set* U *in* T.

(xxii) m *is Borel outer regular (in* τ_e*) in each compact set* K *in* T *and Borel inner regular (in* τ_e*) in the set* T.

(xxiii) m_c *is* σ*-Borel regular in* τ_e *of* X^{**}.

(xxiv) m_c *is* σ*-Borel inner regular in* τ_e *of* X^{**}.

(xxv) m_c *is* σ*-Borel inner regular (in* τ_e*) in each* σ*-Borel open set* U *in* T *and in the set* T.

(xxvi) m_c *is* σ*-Borel outer regular (in* τ_e*) in each compact set* K *in* T *and* σ*-Borel inner regular (in* τ_e*) in the set* T.

(xxvii) m_0 *is Baire regular in* τ_e *of* X^{**}.

(xxviii) m_0 *is Baire inner regular in* τ_e *of* X^{**}.

(xxix) m_0 *is Baire inner regular (in* τ_e*) in each open Baire set* U *in* T *and in the set* T.

(xxx) m_0 *is Baire outer regular (in* τ_e*) in each compact* G_δ *in* T *and Baire inner regular (in* τ_e*) in the set* T.

(xxxi) *All bounded Borel measurable scalar functions* f *on* T *are* m*-integrable and* $\int_T f\,\mathrm{d}m \in X$.

(xxxii) *All bounded* $\mathscr{B}_c(T)$*-measurable scalar functions* f *on* T *are* m_c*-integrable and* $\int_T f\,\mathrm{d}m_c \in X$.

(xxxiii) *All bounded Baire measurable scalar functions* f *on* T *are* m_0*-integrable and* $\int_T f\,\mathrm{d}m_0 \in X$.

(xxxiv) *All bounded scalar functions* f *belonging to the first Baire class in* T *are* m_0*-integrable and* $\int_T f\,\mathrm{d}m_0 \in X$.

(xxxv) $u^{**}f \in X$ *for all bounded scalar functions* f *belonging to the first Baire class in* T.

Proof. In the sequel we will prove only those implications which are not obvious.

(i) $\Rightarrow$ (ii): By (i) and Proposition 2, $u^{**}C_0^{**}(T) \subset X$ and by Proposition 5 (v), $m(E) = u^{**}(\chi_E)$ for $E \in \mathscr{B}(T)$. As $\mathscr{B}(T) \subset C_0^{**}(T)$, (ii) holds.

(viii) $\Rightarrow$ (iv): In fact, by hypothesis (viii) and by Lemmas 5 and 6, m_0 is Baire inner regular in τ_e of X^{**}. Given $K \in \mathscr{K}_0$, by Theorem 50.D of Halmos [7] there exists $U \in \mathscr{U}_0$ such that $K \subset U$ and hence $m_0(K) = m_0(U) - m_0(U \setminus K) \in X$. Thus $m_0(\mathscr{K}_0) \subset X$. Let $E \in \mathscr{B}_0(T)$. Let $D(E) = \{K \in \mathscr{K}_0 \colon K \subset E\}$ and let $K_1 \geqslant K_2$ for $K_1, K_2 \in D(E)$ if $K_1 \supset K_2$. Then by the Baire inner regularity of m_0 in E, $\lim_{D(E)} m_0(K) = m_0(E)$ so that the net $\{m_0(K) \colon K \in D(E)\}$ is τ_e-Cauchy with the limit $m_0(E)$. Since by Proposition 5 (iv), m has τ_e-bounded range in X^{**}, $m_0(\mathscr{K}_0)$ is τ-bounded in X. Thus there exists a τ-bounded closed set H in X such that $m_0(\mathscr{K}_0(T)) \subset H$. Since X is quasicomplete, we conclude that $m_0(E) \in H \subset X$. Thus m_0 has the range in X.

(iv) $\Rightarrow$ (i): In fact, by hypothesis, Proposition 5 (i) and the Orlicz-Pettis theorem, m_0 is σ-additive in τ. Then by Proposition 6 there exists a unique X-valued Borel regular σ-additive extension $\hat{m}$ of m_0 on $\mathscr{B}(T)$. As each $f \in C_0(T)$ is a bounded Baire measurable function by Theorem 51.B of Halmos [7], by Proposition 5 (iii) we have

$$x^*uf = \int_T f \, \mathrm{d}(x^* \circ m) = \int_T f \, \mathrm{d}(x^* \circ m_0) = \int_T f \, \mathrm{d}(x^* \circ \hat{m}), \quad f \in C_0(T).$$

Since $x^* \circ m \in M(T)$ by Proposition 5 (i) and since $x^* \circ \hat{m} \in M(T)$ as $\hat{m}$ is Borel regular and σ-additive, it follows by the uniqueness part of the Riesz representation theorem that $x^* \circ m = x^* \circ \hat{m}$ for each $x^* \in X^*$. Since m has the range in X^{**} and $\hat{m}$ has the range in X we conclude that $m = \hat{m}$ and hence m not only has the range in X but also is σ-additive in $\mathscr{B}(T)$ in τ. Thus, given a disjoint sequence (U_n) of open sets in T, $m(\bigcup_1^\infty U_n) = \sum_1^\infty m(U_n)$ and in particular, $\lim_n m(U_n) = 0$. Thus, for each equicontinuous subset A of X^*, Proposition 5 (ii) yields $\lim_n \|m(U_n)\|_{p_A} = \lim_n \sup_{x^* \in A} |(x^* \circ m)(U_n)| = \lim_n \sup_{\mu \in u^*A} |\mu(U_n)| = 0$. Moreover, by Lemma 2, u^*A is bounded in $M(T)$. Therefore, by Proposition 3, u^*A is relatively weakly compact in $M(T)$. Consequently, by Proposition 2, u is weakly compact. Thus (i) holds.

(x) $\Rightarrow$ (xi): Let U be an open set in T such that it is a countable union of closed sets. Then $T \setminus U$ is a closed set which is G_δ and hence by hypothesis (x) we have $m(U) = m(T) - m(T \setminus U) \in X$. Hence (xi) holds.

(ix) $\Rightarrow$ (viii): by § 14, Chapter III of Dinculeanu [2].

(ii) $\Rightarrow$ (xii): Let (f_n) be as in (xii). Then $\lim_n f_n(t) = f(t)$ exists in [0,1] for each $t \in T$ and f is Borel measurable. Then the hypothesis (ii) combined with Proposition 5 (i) and the Orlicz-Pettis theorem implies that m is σ-additive in $\mathscr{B}(T)$.

Consequently, by Proposition 4 we obtain

$$\lim_n \int_T f_n \,\mathrm{d}m = \int_T f \,\mathrm{d}m \in X.$$

Then by Propositions 4 (i) and 5 (iii) we have

$$\lim_n x^* u f_n = \lim_n \int_T f_n \,\mathrm{d}(x^* \circ m) = x^* \left(\lim_n \int_T f_n \,\mathrm{d}m \right) = x^* \left(\int_T f \,\mathrm{d}m \right)$$

for all $x^* \in X^*$. Thus (xii) holds.

(xii) $\Rightarrow$ (viii): Let $U \in \mathscr{U}_0$. Then by § 14, Chapter III of Dinculeanu [2], there exists a sequence $(K_n) \subset \mathscr{K}_0$ such that $K_n \nearrow U$. By Urysohn's lemma we can choose an increasing sequence g_n of non negative continuous functions with compact supports such that $g_n \nearrow \chi_U$. Then by hypothesis there exists a vector $x_0 \in X$ such that $\lim_n x^* u g_n = x^* x_0$ for all $x^* \in X^*$. Therefore, by the Lebesgue bounded convergence theorem and by Proposition 5 we have $x^* x_0 = \lim_n \int_T g_n \,\mathrm{d}(x^* \circ m) = x^* m(U)$ for all $x^* \in X^*$. Since $m(U) \in X^{**}$, it follows that $m(U) = x_0 \in X$. Hence (viii) holds.

(ii) $\Rightarrow$ (xiii): By (ii) m has the range in X and hence by Proposition 5 (i) and the Orlicz-Pettis theorem m is σ-additive in τ. Since $\tau_e|_X = \tau$, (xiii) holds.

(xv) $\Rightarrow$ (i): Let Y be the completion of (X^{**}, τ_e). Then by hypothesis $m_0 \colon \mathscr{B}_0(T) \to Y$ is σ-additive in τ_e and hence by Proposition 6 there exists a unique Y-valued Borel regular σ-additive (in τ_e) extension $\tilde{m}$ of m_0 on $\mathscr{B}(T)$. Each $f \in C_0(T)$ is a bounded Baire measurable function by Theorem 51.B of Halmos [7] and consequently, by Proposition 5 (iii) we have

$$x^* u f = \int_T f \,\mathrm{d}(x^* \circ m) = \int_T f \,\mathrm{d}(x^* \circ m_0) = \int_T f \,\mathrm{d}(x^* \circ \tilde{m})$$

for each $f \in C_0(T)$. By Proposition 5 (i), $x^* \circ m \in M(T)$. Since each $x^* \in X^*$ is τ_e-continuous in X^{**}, it follows that $x^* \circ \tilde{m}$ is a σ-additive regular Borel complex measure on T and hence $x^* \circ \tilde{m} \in M(T)$. Thus the continuous linear functional $x^* u$ on $C_0(T)$ is represented by both $x^* \circ m$ and $x^* \circ \tilde{m}$ belonging to $M(T)$ and hence $x^* \circ m = x^* \circ \tilde{m}$ for all $x^* \in X^*$. Since m takes values in X^{**} and $\tilde{m}$ takes values in Y, it follows that $m = \tilde{m}$ so that $\tilde{m}$ has values in X^{**}. Moreover, m $(= \tilde{m})$ is σ-additive in τ_e. Consequently, given a disjoint sequence (U_n) of open sets in T, by Proposition 5 (ii) we have $\lim_n \|m(U_n)\|_{p_A} = \lim_n \sup_{x^* \in A} |(x^* \circ m)(U_n)| = \lim_n \sup_{\mu \in u^* A} |\mu(U_n)| = 0$ for each $A \in \mathscr{E}$. Moreover, for such A, by Lemma 2, $u^* A$ is bounded in $M(T)$. Then by an argument similar to that in the end of the proof of (iv) $\Rightarrow$ (i) we conclude that u is weakly compact. Hence (i) holds.

(xviii) $\Rightarrow$ (i): Let $\Sigma(\mathscr{B}_0(T))$ be the Banach space of all bounded complex functions which are uniform limits of sequences of $\mathscr{B}_0(T)$-simple functions, with pointwise addition and scalar multiplication and with the supremum norm $\|\cdot\|_T$. Let

$$Vf = \int_T f \,\mathrm{d}m_0, \quad f \in \Sigma(\mathscr{B}_0(T)).$$

By Proposition 5 (iv), m_0 is a τ_e-bounded vector measure and hence, by Lemma 6 of [11], V is a well defined X^{**}-valued continuous linear map. Then as the representing measure m_0 of V (see Definition 2 of [11]) is strongly additive by hypothesis (xviii), by Theorem 1 of [11] V is a weakly compact operator. By Theorem 51.B of Halmos [7] each $f \in C_0(T)$ is Baire measurable and bounded and hence is the uniform limit of a sequence of Baire simple functions. Hence $C_0(T) \subset \Sigma(\mathscr{B}_0(T))$. In particular, $V|_{C_0(T)}$ is weakly compact. Moreover, by Propositions 4 (i) and 5 (iii), we have

$$x^*Vf = \int_T f \,\mathrm{d}(x^* \circ m_0) = \int_T f \,\mathrm{d}(x^* \circ m) = x^*uf, \quad f \in C_0(T)$$

for each $x^* \in X^*$. Since $Vf \in X^{**}$ and $uf \in X$, we conclude that $Vf = uf$ for each $f \in C_0(T)$. Consequently, $u = V|_{C_0(T)}$ and hence $\{uf\colon \|f\|_T \leqslant 1\}$ is relatively $\sigma(X^{**}, X^{***})$-compact. Since $u(C_0(T)) \subset X$, it follows that $\{uf\colon \|f\|_T \leqslant 1\}$ is relatively weakly compact in X. Thus u is weakly compact. Hence (i) holds.

(ii) $\Rightarrow$ (xix): By (ii), Proposition 5 (i) and the Orlicz-Pettis theorem, m is σ-additive in $\mathscr{B}(T)$ in the topology τ of X. Then m_0 is σ-additive in $\mathscr{B}_0(T)$ and has the range in X. Therefore, by Proposition 6 there exists a unique Borel regular X-valued σ-additive (in τ) extension $\hat{m}$ of m_0 on $\mathscr{B}(T)$. Then by Proposition 5 (iii) and by the fact that each $f \in C_0(T)$ is bounded and Baire measurable (by Theorem 51.B of [7]), we have

$$x^*uf = \int_T f \,\mathrm{d}(x^* \circ m) = \int_T f \,\mathrm{d}(x^* \circ m_0) = \int_T f \,\mathrm{d}(x^* \circ \hat{m})$$

for each $x^* \in X^*$ and $f \in C_0(T)$. Since $x^* \circ m \in M(T)$ by Proposition 5 (i) and since $x^* \circ \hat{m} \in M(T)$ as $\hat{m}$ is Borel regular and σ-additive in τ with values in X, we conclude that $x^* \circ m = x^* \circ \hat{m}$ for each $x^* \in X^*$. Since by hypothesis m has the range in X and $\hat{m}$ in X, it follows that $m = \hat{m}$. Thus m is Borel regular in τ and hence m is Borel regular in τ_e as $\tau_e\big|_X = \tau$. Thus (xix) holds.

(xxi) (or (xxv), (xxix)) $\Rightarrow$ (xxviii): Let $U \in \mathscr{U}_0$ or let $U = T$. Let $A \in \mathscr{E}$ and $\varepsilon > 0$. Then by hypothesis and by Theorem 50.D of Halmos [7] there exists a compact G_δ K such that $K \subset U$ and $\|m\|_{p_A}(U \setminus K) < \varepsilon$ ($\|m_c\|_{p_A}(U \setminus K) < \varepsilon$,

$\|m_0\|_{p_A}(U \setminus K) < \varepsilon$, respectively). Thus, in particular, $\|m_0\|_{p_A}(E) < \varepsilon$ for all $E \in \mathscr{B}_0(T)$ with $E \subset U \setminus K$. Since this holds for all $U \in \mathscr{U}_0$ and for $U = T$, the conditions of Lemma 6 are satisfied by m_0. Therefore, m_0 is Baire inner regular in $\mathscr{B}_0(T)$. Hence (xxviii) holds.

(xxviii) $\Rightarrow$ (xv): by Lemma 7.

(xxii) $\Rightarrow$ (i): Let $K \in \mathscr{K}$ and let $A \in \mathscr{E}$. Given $\varepsilon > 0$, by hypothesis there exists an open set U in T such that $\|m\|_{p_A}(U \setminus K) < \varepsilon$. Then by Lemma 1 (i) and Proposition 5 (ii) we have $\sup\{|x^* \circ m|(U \setminus K)\colon\ x^* \in A\} = \sup_{\mu \in u^*A} |\mu|(U \setminus K) < \varepsilon$ and by Lemma 2, u^*A is bounded in $M(T)$. Thus condition (iv) (a) of Proposition 3 is satisfied by u^*A. Since m is inner regular in T, there exists a compact set C such that $\|m\|_{p_A}(T \setminus C) < \varepsilon$ so that by an argument similar to that above we have $\sup_{\mu \in u^*A} |\mu|(T \setminus C) < \varepsilon$. Therefore, condition (iv) (b) of Proposition 3 is also satisfied by u^*A. Hence by Proposition 3, u^*A is relatively weakly compact in $M(T)$ and consequently, by Proposition 2, u is weakly compact. Thus (i) holds.

(ii) $\Rightarrow$ (xxiii): Proceeding as in the proof of (ii) $\Rightarrow$ (xix), we have $m = \hat{m}$ on $\mathscr{B}(T)$. Since $\hat{m}|_{\mathscr{B}_c(T)}$ is σ-Borel regular by Proposition 6, we conclude that m_c is σ-Borel regular in τ and hence in τ_e. Thus (xxiii) holds.

(xxiv) $\Rightarrow$ (xiv): by Lemma 7.

(xxiii) implies the first part of (xxv) and (xix) implies the second part of (xxv). As (xxv) $\Rightarrow$ (xxviii), it follows that (i) $\Leftrightarrow$ (xxv).

(xix) $\Rightarrow$ (xxvi): Given $K \in \mathscr{K}$, $A \in \mathscr{E}$ and $\varepsilon > 0$, then by hypothesis there exists an open set U with $U \supset K$ such that $\|m\|_{p_A}(U \setminus K) < \varepsilon$. By Theorem 50.D of Halmos [7] we can choose a $V \in \mathscr{U}_0$ such that $K \subset V \subset U$ so that $\|m_c\|_{p_A}(V \setminus K) < \varepsilon$. Thus m_c is σ-Borel outer regular in K. Clearly, m_c is σ-Borel inner regular in T as m is, by hypothesis, Borel inner regular in T. Hence (xxvi) holds.

(xxvi) $\Rightarrow$ (i): Let $K \in \mathscr{K}$. Proceeding as in the proof of (xxii) $\Rightarrow$ (i), we have $\|m_c\|_{p_A}(U \setminus K) < \varepsilon$, where U is a σ-Borel open set containing K. Thus by Lemma 1 (ii) and Proposition 1 we have $\|m_c\|_{p_A}(U \setminus K) = \sup_{x^* \in A} |x^* \circ m_c|(U \setminus K) = \sup_{x^* \in A} |x^* \circ m|(U \setminus K) < \varepsilon$. Hence, by Proposition 5 (ii), $\sup_{\mu \in u^*A} |\mu|(U \setminus K) < \varepsilon$. Since u^*A is bounded in $M(T)$ by Lemma 2, condition (iv) (a) of Proposition 3 is satisfied by u^*A. Again by hypothesis, there exists a compact C such that $\|m_c\|_{p_A}(T \setminus C) < \varepsilon$. Thus for each compact $K \subset T \setminus C$, by Lemma 1 (ii) we have $\sup_{x^* \in A} |x^* \circ m|(K) < \varepsilon$. As $|x^* \circ m|$ is Borel regular by Proposition 5 (i) for each $x^* \in A$, and $x^* \circ m = u^*x^*$ by Proposition 5 (ii), it follows that $\sup_{x^* \in A} |x^* \circ m|(T \setminus C) = \sup_{\mu \in u^*A} |\mu|(T \setminus C) \leqslant \varepsilon$. Thus condition (iv) (b) of Proposition 3 is also satisfied by u^*A. Therefore, u^*A is relatively weakly compact in $M(T)$ for each $A \in \mathscr{E}$. Now by Proposition 2 we conclude that u is weakly compact. Hence (i) holds.

(xv) $\Rightarrow$ (xxvii): Since m_0 is σ-additive in τ_e, by the first part of Proposition 6, m_0 is Baire regular in τ_e. Thus (xxvii) holds.

(xxix) $\Rightarrow$ (xxviii): by Lemma 6.

(xix) $\Rightarrow$ (xxix): Let $U \in \mathscr{U}_0$, $A \in \mathscr{E}$ and $\varepsilon > 0$. By hypothesis, there exists a compact $K \subset U$ such that $\|m\|_{p_A}(U \setminus K) < \varepsilon$. By Theorem 50.D of Halmos [7] there exists a compact $C \in \mathscr{K}_0$ such that $K \subset C \subset U$. Then $\|m_0\|_{p_A}(U \setminus C) < \varepsilon$. Hence m_0 is Baire inner regular in U. As m is Borel inner regular in T, there exists $K \in \mathscr{K}$ such that $\|m\|_{p_A}(T \setminus K) < \varepsilon$. By Theorem 50.D of Halmos [7] there exists $C \in \mathscr{K}_0$ such that $K \subset C$ and hence $\|m_0\|_{p_A}(B) < \varepsilon$ for all $B \in \mathscr{B}_0(T)$ with $B \subset T \setminus C$. Thus m_0 is Baire inner regular in T. Hence (xxix) holds.

(xix) $\Rightarrow$ (xxx): Let $K \in \mathscr{K}_0$, $A \in \mathscr{E}$ and $\varepsilon > 0$. By hypothesis and by Theorem 50.D of Halmos [7] there exists $U \in \mathscr{U}_0$ with $K \subset U$ such that $\|m\|_{p_A}(U \setminus K) < \varepsilon$ so that by (i) and (iii) of Lemma 1 we have $\|m_0\|_p(U \setminus K) < \varepsilon$. Similarly, we can show that m_0 is Baire inner regular in T. Hence (xxx) holds.

(xxx) $\Rightarrow$ (xxix): Clearly, it suffices to show that m_0 is Baire inner regular in each open Baire set. Given $A \in \mathscr{E}$ and $\varepsilon > 0$, by the hypothesis of Baire inner regularity of m_0 in T and by Theorem 50.D of Halmos [7] there exists a compact $\Omega \in \mathscr{K}_0$ such that $\|m_0\|_{p_A}(T \setminus \Omega) < \varepsilon/2$. Let $U \in \mathscr{U}_0$ such that U is relatively compact.

Claim 1. m_0 is Baire inner regular in U.

In fact, by Theorem 50.D of Halmos [7] we can choose a compact $C \in \mathscr{K}_0$ such that $\overline{U} \subset C$. Then $U = C \setminus (C \setminus U)$ and $C \setminus U \in \mathscr{K}_0$ by Theorem 51.D of Halmos [7]. Therefore, by hypothesis there exists $W \in \mathscr{U}_0$ with $W \supset C \setminus U$ such that $\|m_0\|_{p_A}(W \setminus (C \setminus U)) < \varepsilon$. Now $U = C \setminus (C \setminus U) \supset C \setminus W$ and $C \setminus W \in \mathscr{K}_0$ again by Theorem 51.D of Halmos [7]. Moreover, $U \setminus (C \setminus W) = U \cap ((T \setminus C) \cup W) = U \cap W$. On the other hand, $W \setminus (C \setminus U) \supset W \cap U$. Therefore, $\|m_0\|_{p_A}(U \setminus (C \setminus W)) < \varepsilon$. Thus the claim holds.

Now let $U \in \mathscr{U}_0$. Choose by Theorem 50.D of Halmos [7] a relatively compact open Baire set V such that $\Omega \subset V$. Then $U \cap V$ is relatively compact and belongs to $\mathscr{U}_0$. Therefore, by Claim 1, m_0 is Baire inner regular in $U \cap V$ and hence there exists a compact $K \in \mathscr{K}_0$ with $K \subset U \cap V$ such that $\|m_0\|_{p_A}((U \cap V) \setminus K) < \varepsilon/2$. Then $K \subset U$ and $\|m_0\|_{p_A}(U \setminus K) \leqslant \|m_0\|_{p_A}((U \cap V) \setminus K) + \|m_0\|_{p_A}(U \setminus \Omega) < \varepsilon$. Therefore, m_0 is Baire inner regular in each open Baire set and hence (xxix) holds.

(ii) $\Rightarrow$ (xxxi), (xxxii) and (xxxiii): By (ii), Proposition 5 (i) and the Orlicz-Pettis theorem m is X-valued and σ-additive in τ. Since every bounded Borel (σ-Borel, Baire) measurable scalar function is the uniform limit of a sequence of Borel (σ-Borel, Baire) simple functions and m is a τ-bounded X-valued vector measure, f is m-integrable (see Definition 1 of [11]) and $\int_T f\,\mathrm{d}m \in X$ (f is m_c-integrable and $\int_T f\,\mathrm{d}m_c \in X$, f is m_0-integrable and $\int_T f\,\mathrm{d}m_0 \in X$, respectively).

(xxxi) (or (xxxii), (xxxiii)) $\Rightarrow$ (ii) ((iii), (iv)): Let $E \in \mathscr{B}(T)$ $(E \in \mathscr{B}_c(T)$, $E \in \mathscr{B}_0(T))$. Then by hypothesis, $m(E)$ $(m_c(E), m_0(E))$ belongs to X. Thus (ii) ((iii), (iv), respectively) holds.

(xxxiv) $\Rightarrow$ (viii): Let U be an open Baire set. Then by § 14, Chapter III of Dinculeanu [2], there exists an increasing sequence K_n of compact G_δ sets such that $U = \bigcup_1^\infty K_n$. Then by Urysohn's lemma we can choose non negative continuous functions g_n with compact supports such that $g_n \nearrow \chi_U$. Thus χ_U belongs to the first Baire class and is bounded. Then by hypothesis, $m_0(U) \in X$. Thus (viii) holds.

(i) $\Rightarrow$ (xxxv): If u is weakly compact, then by Proposition 2, u^{**} has the range in X. Since the bounded scalar functions of the first Baire class belong to $C_0^{**}(T)$, (xxxv) holds.

(xxxv) $\Rightarrow$ (viii): By Proposition 5 (v), $u^{**}(\chi_U) = m(U)$ for $U \in \mathscr{U}_0$. As observed in the proof of (xxxiv) $\Rightarrow$ (viii), χ_U is bounded and belongs to the first Baire class. Hence, by hypothesis, $m(U) \in X$. Thus (viii) holds.

This completes the proof of the theorem. □

Remark 2. As in [13], the strict Dunford-Pettis property of $C_0(T)$ is an immediate consequence of the above theorem and the proof of the latter is not based on this property unlike the proof of Theorem 6 of Grothendieck [6]. Theorem 5.3 of Thomas [16] is also deducible from the above theorem by the same argument as that used in the proof of Theorem 13 in [13].

Remark 3. All these 35 characterizations are given in [13] in Theorems 2–9. Some of the proofs given here are the same as those in [13] (for example, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) of Theorem 2 of [13], (i) $\Leftrightarrow$ (xi) of Theorem 3 of [13] and Theorem 9 of [13]) but, for the sake of completeness, we have given the proofs of all non obvious equivalences of these 35 characterizations. In the present proof the use of Theorems 1 and 2 of [12] has been dispensed with unlike the proof in [13] and instead, the Borel extension theorem has been used along with the first part of Theorem 1 of [13], Lemma 1 and Theorem 2 of [6], Theorem 1 of [11] and Lemmas 1–7.

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