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ALMOST PERIODIC COMPACTIFICATIONS OF GROUP EXTENSIONS

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Abstract. Let N and K be groups and let G be an extension of N by K. Given a property $\mathcal{P}$ of group compactifications, one can ask whether there exist compactifications N' and K' of N and K such that the universal $\mathcal{P}$-compactification of G is canonically isomorphic to an extension of N' by K'. We prove a theorem which gives necessary and sufficient conditions for this to occur for general properties $\mathcal{P}$ and then apply this result to the almost periodic and weakly almost periodic compactifications of G.

Keywords: group extension, semidirect product, topological group, semitopological semigroup, right topological semigroup, compactification, almost periodic, weakly almost periodic, strongly almost periodic

MSC 2000: 43A60, 22D05

1. Introduction

Let N and K be groups, let $G = N \times K$ be an extension of N by K, and let $\mathcal{P}$ be a property of compactifications (such as that of being a topological group or a semitopological semigroup). In this setting, it is natural to ask whether there exist compactifications N' and K' of N and K such that the universal $\mathcal{P}$-compactification $G^{\mathcal{P}}$ of G is canonically isomorphic to an extension of semigroups $N' \times K'$. Results of this type are known for AP- and LC-compactifications in the special case of a direct or semidirect product (see, for example, [1], [2], [3], [7]), and in the general case for the LC-compactification when N and K are topological groups with N compact and K discrete [6]. In this paper we generalize these results, obtaining compactification theorems of the form $G^{\mathcal{P}} \cong N' \times K^{\mathcal{P}}$ for general and specific properties $\mathcal{P}$.

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The paper is organized as follows. Section 2 presents notation and terminology. In Section 3 we give necessary and sufficient conditions for $G^{\mathcal{P}} \cong N' \times K^{\mathcal{P}}$ to hold for general properties $\mathcal{P}$ when N and K are semitopological semigroups. Section 4 treats the almost periodic compactification, i.e., the universal $\mathcal{P}$-compactification where $\mathcal{P}$ is the property of being a topological group. It is shown that if N, K and G are topological groups then $G^{\mathcal{P}} \cong N' \times K^{\mathcal{P}}$ holds iff the Schreier mapping $[\cdot,\cdot]$ enjoys a certain relative compactness condition. This result is generalized to the semigroup case in the last part of the section. The weakly almost periodic compactification is treated in Section 5. Here $\mathcal{P}$ is the property of being a semitopological semigroup. We show that if G is a topological group and K is compact, then $G^{\mathcal{P}} \cong N' \times K^{\mathcal{P}}$ holds and, moreover, the minimal ideal of $G^{\mathcal{P}}$ is a canonical group extension of the minimal ideal of N' by $K^{\mathcal{P}}$.

Distal and point distal compactifications of group extensions are examined in a separate paper [4] which relies on the basic results established here in §3 and uses their full generality.

2. Preliminaries

Let N and K be groups with identity e and let the *Schreier mappings*

(SM) $$(t,t') \mapsto [t,t']\colon\ K \times K \to N \quad \text{and} \quad t \mapsto t(\cdot)\colon\ K \to \operatorname{Aut}(N)$$

satisfy the Schreier extension formulation conditions ([9] or [10])

(SEF) $$\begin{cases} e(s) = s \ \text{ and } \ [t,e] = [e,t] = e, \\ [t,t'](tt')(s) = t(t'(s))[t,t'], \ \text{ and} \\ [t,t'][tt',t''] = t([t',t''])[t,t't'']. \end{cases}$$

Then $G = N \times K$ with multiplication

(1) $$(s,t)(s',t') = \big(st(s')[t,t'], tt'\big) \qquad (s,s' \in N,\ \ t,t' \in K),$$

is a group; we call G the *extension* of N by K and indicate this situation by

$$G = N \times K \ \ \text{(SEF)}.$$

Conversely, Schreier has shown that if N is a normal subgroup of a group G, then maps satisfying (SEF) can be determined so that G is canonically isomorphic (algebraically) to $N \times K$ (SEF), where $K = G/N$.

The conditions (SEF) still make sense when N and K are semigroups and each $t(\cdot)$ is a homomorphism, and in this case G with multiplication (1) is a semigroup with identity (e, e).

G is said to be a *central extension* if the range of $[\cdot,\cdot]$ is contained in the center of N, and a *semidirect product* of N and K if the range of $[\cdot,\cdot]$ is $\{e\}$. For central extensions and left or right cancellative semigroups N, the middle equation of (SEF) tells us that K acts on N, i.e., the second map of (SM) is a homomorphism. When both maps of (SM) are trivial, then G is the direct product of N and K.

In the sequel we assume that N and K are (at least) semitopological semigroups and that $G = N \times K$ (SEF) has the product topology and is a semitopological semigroup, which places some continuity requirements on the Schreier maps (SM) (e.g., they must be separately continuous). If N, K and $G = N \times K$ (SEF) are topological semigroups, those requirements become stronger; they become somewhat weaker for (right topological semigroup) compactifications $G' = N' \times K'$ (SEF) of G (see the proof of Lemma 3.3).

The remainder of this section is devoted to a brief overview of the theory of semigroup compactifications. For details the reader is referred to [1], especially Chapters 3 and 4.

A (*right topological semigroup*) *compactification* of a semitopological semigroup S is a compact, Hausdorff, right topological semigroup S' together with a continuous homomorphism $\varepsilon_{S'}\colon S \to S'$ (the *compactification map*) such that the image $\varepsilon_{S'}(S)$ is dense in S' and each mapping $s' \mapsto \varepsilon_{S'}(s)s'\colon S' \to S'$ is continuous. The phrase "right topological" means that the map $s' \mapsto s't'\colon S' \to S'$ is continuous for each $t' \in S'$.

Let $C(S)$ denote the C^*-algebra of bounded, continuous, complex-valued functions on S, and let $R(\cdot)$ and $L(\cdot)$ denote the translation operators on $C(S)$, $R(t)f(s) = L(s)f(t) = f(st)$. The C^*-subalgebra $F = \varepsilon^*_{S'}(C(S'))$ of $C(S)$ is called the *function space* of the compactification S'; here $\varepsilon^*_{S'}\colon C(S') \to C(S)$ is the dual map. F is easily seen to be *m-admissible*, i.e., it is a translation invariant C^*-subalgebra of $C(S)$ containing the constant functions and the functions $s \mapsto \mu(L(s)f)$, where $f \in F$ and μ is a member of the spectrum of F. Conversely, if A is an m-admissible C^*-subalgebra of $C(S)$, then the spectrum S^A of A is a compactification of S.

Let S and T be semitopological semigroups with compactifications S' and T' and let $\varphi\colon S \to T$ be continuous. A continuous function $\varphi'\colon S' \to T'$ is an *extension* of φ if $\varphi' \circ \varepsilon_{S'} = \varepsilon_{T'} \circ \varphi$. φ' is unique, and exists iff $\varphi^*(B) \subset A$, where A and B are the function spaces of the compactifications. If φ is a homomorphism then so is φ'.

Let S' and S'' be compactifications of S. Then S'' is a *factor* of S' if the identity map on S has an extension $S' \to S''$. We shall refer to this extension as the associated *compactification homomorphism*. Note that a compactification homomorphism is necessarily surjective and unique. If it is one-to-one then S' and S'' are said to be *isomorphic*.

A compactification with a given property $\mathcal{P}$ is called a *$\mathcal{P}$-compactification.* A *universal $\mathcal{P}$-compactification* of S is a $\mathcal{P}$-compactification of which every $\mathcal{P}$-compactification of S is a factor. Universal $\mathcal{P}$-compactifications, if they exist, are unique (up to isomorphism). We consider only properties $\mathcal{P}$ for which the universal $\mathcal{P}$-compactification exists. (Necessary and sufficient conditions for this are given in [1, 3.3.4].) We denote the universal $\mathcal{P}$-compactification of S by $S^{\mathcal{P}}$ and the function space of $S^{\mathcal{P}}$ by $\mathcal{P}(S)$.

If S and T are semitopological semigroups, $\varphi\colon S \to T$ is a continuous homomorphism, and T' is a compactification of T, then the induced compactification $S' = \overline{\varepsilon_{T'} \circ \varphi(S)}$ (with compactification map $\varepsilon_{S'} = \varepsilon_{T'} \circ \varphi$) is called a *subcompactification* of T'. If $\mathcal{P}$ is a property of compactifications which is inherited by subcompactifications then every continuous homomorphism $S \to T$ has an extension $S^{\mathcal{P}} \to T^{\mathcal{P}}$.

In this paper we shall be concerned with universal $\mathcal{P}$-compactifications $G^{\mathcal{P}}$ of $G = N \times K$ (SEF) for the topological group and the semitopological semigroup properties. The function spaces $\mathcal{P}(G)$ corresponding to these properties are, respectively, SAP(G), the algebra of strongly almost periodic functions on G, and $WAP(G)$, the algebra of weakly almost periodic functions. We shall also need to consider the m-admissible algebra LC(G) of left norm continuous functions, and the algebra RC(G) of right norm continuous functions, which is generally not m-admissible. Note that if G is a topological group then $SAP(G) = AP(G)$, the algebra of almost periodic functions, and $LC(G)$ ($RC(G)$) is the algebra of right (left) uniformly continuous on G. Details concerning these and related spaces may be found in Chapter 4 of [1].

Our goal is to find conditions under which there exists a compactification isomorphism $G^{\mathcal{P}} \cong N' \times K'$ (SEF), where N' and K' are compactifications of N and K uniquely determined by $G^{\mathcal{P}}$. Here, and throughout the paper, it is assumed that the compactification map $\varepsilon_{N' \times K'}$ is the natural one, namely the product map $\varepsilon_{N'} \times \varepsilon_{K'}\colon N \times K \to N' \times K'$. This is equivalent to requiring the Schreier maps $[\cdot,\cdot]\colon K' \times K' \to N'$ and $(x, y) \mapsto y(x)\colon N' \times K' \to N'$ to be extensions of the Schreier maps of $N \times K$.

Let N and K be semitopological semigroups with identity e, and let $G = N \times K$ (SEF) (always with the product topology). We let $q_1\colon N \to G$ and $q_2\colon K \to G$ denote the canonical injections, $p_1\colon G \to N$ and $p_2\colon G \to K$ the projection mappings and set $r_i = q_i \circ p_i$. Note that in the (SEF) case, q_1 and p_2 are homomorphisms, in the semidirect product case q_1, q_2 and p_2 are homomorphisms, and in the direct product case all four mappings are homomorphisms.

The following result was obtained for semidirect products in [3]. The proof there works equally well here, since it relies only on the identity $(s,e)(e,t) = (s,t)$.

Lemma 3.1. *Let F be an m-admissible C^*-subalgebra of $C(G)$ and let A and B denote, respectively, the C^*-subalgebras $q_1^*(F)$ and $q_2^*(F)$. Suppose the following conditions hold:*

(a) *A and B are m-admissible;*

(b) *$p_1^*(A) \subset F$ and $p_2^*(B) \subset F$;*

(c) *for each $f \in F$ either $f(N,\cdot)$ or $f(\cdot,K)$ is relatively compact.*

Then there exists a multiplication on $N^A \times K^B$ relative to which $N^A \times K^B$ is a compactification of G isomorphic to G^F. Conversely, if N' and K' are compactifications of N and K, respectively, and if $G' := N' \times K'$ has a multiplication such that G' and G^F are isomorphic compactifications of G, then conditions (a)–(c) *hold and $A = \varepsilon_{N'}^*(C(N'))$ and $B = \varepsilon_{K'}^*(C(K'))$.*

Remark 3.2. Because of the identity $f(s,t) = L(s,e)f(e,t) = R(e,t)f(s,e)$, condition (c) of Lemma 3.1 is implied by any one of the following:

(i) $F \subset AP(G)$;

(ii) N is compact and $F \subset LC(G)$;

(iii) K is compact and $F \subset RC(G)$.

Lemma 3.3. *Let N' and K' be compactifications of N and K and let $G' := N' \times K'$ have a multiplication relative to which it is a compactification of G. If N' is a topological semigroup or K is compact, then $G' = N' \times K'$* (SEF)*. Moreover, G' is a semidirect product if G is a semidirect product, and it is a central extension if N' is a topological semigroup and G is a central extension.*

P r o o f. Let ε denote the compactification map $\varepsilon_{N'} \times \varepsilon_{K'}$ of G' and let $q_1'\colon N' \to G'$ and $q_2'\colon K' \to G'$ denote the canonical injections and $p_1'\colon G' \to N'$ and $p_2'\colon G' \to K'$ the projection mappings. For $x \in N'$ and $y, y' \in K'$ define $y(x) = p_1'\big(q_2'(y)q_1'(x)\big)$ and $[y,y'] = p_1'\big(q_2'(y)q_2'(y')\big)$. Note that $y(x)$ and $[y,y']$ are continuous in y for fixed

x and y', and $\varepsilon_{K'}(t)(x)$ and $[\varepsilon_{K'}(t), y]$ are continous in x and y, respectively, for fixed t. Moreover, one easily checks that for $s \in N$ and $t, t' \in K$,

$$(2) \qquad \varepsilon_{K'}(t)(\varepsilon_{N'}(s)) = \varepsilon_{N'}(t(s)) \quad \text{and} \quad [\varepsilon_{K'}(t), \varepsilon_{K'}(t')] = \varepsilon_{N'}([t, t']).$$

It follows from (2) that

$$(3) \qquad p_1'(\varepsilon(s,t) \cdot \varepsilon(s',t')) = \varepsilon_{N'}(s) \cdot \varepsilon_{K'}(t)(\varepsilon_{N'}(s')) \cdot [\varepsilon_{K'}(t), \varepsilon_{K'}(t')].$$

Now suppose N' is a topological semigroup. Letting first $\varepsilon(s',t') \to (x',y')$ and then $\varepsilon(s,t) \to (x,y)$ in (3) we get $p_1'((x,y)(x',y')) = xy(x')[y,y']$. Similarly, $p_2'\big((x,y)(x',y')\big) = yy'$. Thus, multiplication in G' is given by $(x,y)(x',y') = (xy(x')[y,y'], yy')$. The (SEF) conditions follow from (2) and continuity or may be deduced directly from the associativity of multiplication in G'. The proof for the case K compact is similar.

If G is a semidirect product of N and K, then the second identity in (2) shows that $[y,y']$ is trivial, hence G' is a semidirect product. The assertion regarding central extensions is clear. □

We may now prove the following general result on $\mathcal{P}$-compactifications of semigroup extensions.

Theorem 3.4. *Let $\mathcal{P}$ be a property of semigroup compactifications which is inherited by subcompactifications, and let A denote the m-admissible algebra $q_1^*(\mathcal{P}(G))$. Suppose that the following conditions hold:*
(a) *$q_2^*(\mathcal{P}(G)) \subset \mathcal{P}(K)$;*
(b) *$p_1^*(A) \subset \mathcal{P}(G)$;*
(c) *for each $f \in \mathcal{P}(G)$ either $f(N, \cdot)$ or $f(\cdot, K)$ is relatively compact;*
(d) *N^A is a topological semigroup or K is compact.*
Then N^A is a $\mathcal{P}$-compactification of N and $G^{\mathcal{P}} \cong N^A \times K^{\mathcal{P}}$ (SEF). *Moreover, $G^{\mathcal{P}}$ is a semidirect product if G is a semidirect product, and $G^{\mathcal{P}}$ is a central extension if G is a central extension and N^A is a topological semigroup.*

Conversely, if $G^{\mathcal{P}} \cong N' \times K'$ (SEF) *for some compactifications N' and K' and if $\mathcal{P}$ is inherited by factors, then* (a)–(c) *hold, $N' \cong N^A$ and $K' \cong K^{\mathcal{P}}$.*

P r o o f. Set $B := q_2^*(\mathcal{P}(G))$. Since $p_2 \colon G \to K$ is a continuous homomorphism and $\mathcal{P}$ is inherited by subcompactifications, p_2 has an extension $\overline{p}_2 \colon G^{\mathcal{P}} \to K^{\mathcal{P}}$. Then $p_2^*(\mathcal{P}(K)) = \varepsilon_{G^{\mathcal{P}}}^*\big(\overline{p}_2^*(C(K^{\mathcal{P}}))\big) \subset \mathcal{P}(G)$, hence $\mathcal{P}(K) = q_2^* p_2^*(\mathcal{P}(K)) \subset B$ by (a). Therefore $B = \mathcal{P}(K)$, and Lemmas 3.1 and 3.3 imply that $G^{\mathcal{P}} \cong N^A \times K^{\mathcal{P}}$ (SEF). Since N^A is a subcompactification of $G^{\mathcal{P}}$ (via q_1), N^A is a $\mathcal{P}$-compactification of N.

The converse follows from Lemma 3.1 and the observation that K', as a factor of $G^{\mathcal{P}}$, is a $\mathcal{P}$-compactification of K so (a) holds and $K' \cong K^{\mathcal{P}}$. □

Remark 3.5. If $\mathcal{P}$-compactifications of semitopological semigroups are topological groups, then condition (b) of Theorem 3.4 is implied by condition (a). Indeed, if (a) holds then $r_2^*(\mathcal{P}(G)) \subset \mathcal{P}(G)$, so r_2 has an extension $\overline{r}_2\colon G^{\mathcal{P}} \to G^{\mathcal{P}}$. It follows that $x \mapsto x\overline{r}_2(x)^{-1}$ is an extension of r_1, which implies (b).

4. Almost periodic compactifications

For the purposes of the next theorem we define, for $(s,t) \in G = N \times K$, operators $S_\ell(s,t), S_r(s,t)\colon C(N) \to C(G)$ by

$$[S_\ell(s,t)g](s',t') = g(st(s')[t,t']) \quad \text{and} \quad [S_r(s,t)g](s',t') = g(s't'(s)[t',t]).$$

Note that for $f \in C(G)$,

$$S_\ell(s,t)q_1^* f = L(s,t)r_1^* f \quad \text{and} \quad S_r(s,t)q_1^* f = R(s,t)r_1^* f. \tag{4}$$

Theorem 4.1. *Let N, K and $G = N \times K$ (SEF) be topological groups and let $A = q_1^*\big(AP(G)\big)$. The following conditions are equivalent:*

(a) $G^{AP} \cong N' \times K'$ (SEF) *for some compactifications N' and K' of N and K.*

(b) $q_2^*(AP(G)) \subset AP(K)$.

(c) $\varepsilon_{G^{AP}}([K,\cdot] \times e) \subset (G^{AP})^K$ *is relatively compact in the topology of uniform convergence.*

(d) $\varepsilon_{G^{AP}}([\cdot,K] \times e) \subset (G^{AP})^K$ *is relatively compact in the topology of uniform convergence.*

If (a) *holds then $K' \cong K^{AP}$ and $N' \cong N^A$; moreover,*

$$A = \{g \in C(N)\colon\ S_\ell(G)g \text{ is norm relatively compact in } C(G)\} \tag{5}$$

$$\quad = \{g \in C(N)\colon\ S_r(G)g \text{ is norm relatively compact in } C(G)\}, \tag{6}$$

and A is generated by the coefficients of the finite dimensional irreducible unitary representations $\sigma\colon N \to \mathcal{B}(\mathcal{H}_\sigma)$ with the property that the sets

$$\sigma([K,\cdot]) = \{\sigma([t,\cdot])\colon\ t \in K\} \subset \mathcal{B}(\mathcal{H}_\sigma)^K \text{ and } \sigma(K(\cdot)) = \{\sigma(t(\cdot))\colon\ t \in K\} \subset \mathcal{B}(\mathcal{H}_\sigma)^N$$

are relatively compact in the topologies of uniform convergence.

Proof. By Theorem 3.4, (a) implies (b) as well as the isomorphisms $K' \cong K^{AP}$ and $N' \cong N^A$. Also, by Theorem 3.4 and Remarks 3.1 and 3.5, (b) implies (a).

To show that (b) implies (c), let $\{t_\alpha\}$ be a net in K and choose a subnet $\{t_\beta\}$ such that $\varepsilon_{K^{AP}}(t_\beta)$ and $\varepsilon_{G^{AP}}(e, t_\beta)$ converge. If (b) holds then q_2 has an extension $\overline{q}_2\colon K^{AP} \to G^{AP}$, and the identity

$$\begin{aligned}\varepsilon_{G^{AP}}([t_\beta, t], e) &= \varepsilon_{G^{AP}}(e, t_\beta)\varepsilon_{G^{AP}}(e, t)\varepsilon_{G^{AP}}(e, t_\beta t)^{-1} \\ &= \varepsilon_{G^{AP}}(e, t_\beta)\varepsilon_{G^{AP}}(e, t)\overline{q}_2(\varepsilon_{K^{AP}}(t_\beta)\varepsilon_{K^{AP}}(t))^{-1}\end{aligned}$$

implies that $\varepsilon_{G^{AP}}([t_\beta, t], e)$ converges uniformly in t. Therefore (b) implies (c). The converse follows from a rearrangement of the preceding identity. Using right instead of left translations in these arguments shows that (b) and (d) are equivalent.

Now assume that the equivalent conditions (a)–(d) hold and let A_1 denote the algebra defined in (5). It is easy to see that $p_1^*(A_1) \subset AP(G)$, hence $A_1 \subset A$. The reverse inclusion follows from the first identity in (4) and from the inclusion $r_1^*(AP(G)) \subset AP(G)$, which is (b) of Theorem 3.4. Thus, $A_1 = A$. A similar argument shows that A also coincides with the algebra defined in (6).

To prove the last assertion of the theorem, first observe that A is generated by functions $q_1^*(f)$, where f is a coefficient of a finite dimensional irreducible unitary representation $\pi\colon G \to \mathcal{B}(\mathcal{H}_\pi)$, say $f(s,t) = f_{\xi\zeta}(s,t) = \langle \pi(s,t)\xi, \zeta\rangle$. Let $g_{\xi\zeta} = q_1^*(f_{\xi\zeta})$ and define $\sigma(s) = \pi(s,e)$, $s \in N$. Since $S_\ell(G)g_{\xi\zeta}$ is norm relatively compact for all $\xi, \zeta \in \mathcal{H}_\pi$, $\tilde{S}_\ell(G)\sigma \subset \mathcal{B}(\mathcal{H}_\pi)^G$ is relatively compact in the topology of uniform convergence, where

$$\tilde{S}_\ell(s,t)\sigma(s', t') = \sigma(st(s')[t, t']).$$

It follows that $\sigma([K, \cdot])$ and $\sigma(K(\cdot))$ are relatively compact in the topologies of uniform convergence of $\mathcal{B}(\mathcal{H}_\pi)^K$ and $\mathcal{B}(\mathcal{H}_\pi)^N$, respectively. Since σ is a finite dimensional unitary representation of N it is a direct sum of irreducible unitary representations, hence $g_{\xi\zeta}$ is a linear combination of coefficients of finite dimensional irreducible unitary representations of N, each with the required compactness property.

Conversely, let $\sigma\colon N \to \mathcal{B}(\mathcal{H}_\sigma)$ be a finite dimensional unitary representation with the stated compactness property. Since $\tilde{S}_\ell(s,t)\sigma(s', t') = \sigma(s)\sigma(t(s'))\sigma([t, t'])$, $\tilde{S}_\ell(G)\sigma$ is relatively compact in the topology of uniform convergence of $\mathcal{B}(\mathcal{H}_\sigma)^G$. Hence, if $g_{\xi\zeta}$ is a coefficient of σ, then $S_\ell(G)g_{\xi\zeta}(\cdot) = \langle \tilde{S}_\ell(G)\sigma(\cdot)\xi, \zeta\rangle$ is norm relatively compact and therefore $g_{\xi\zeta} \in A$. □

Corollary 4.2. *Let $[K, \cdot]$ or $[\cdot, K]$ be relatively compact in the topology of uniform convergence of N^K (with respect to either the left or right uniformity on N). For $t \in K$ define an operator $T(t)$ on $C(N)$ by $T(t)g(s) = g(t(s))$. Then $G^{AP} \cong N^A \times K^{AP}$* (SEF), *where*

(7) $A = q_1^*(AP(G)) = \{g \in AP(N)\colon\ T(K)g$ *is norm relatively compact in* $C(N)\}$.

Moreover, A is generated by the coefficients of the finite dimensional irreducible unitary representations $\sigma\colon N \to \mathcal{B}(\mathcal{H}_\sigma)$ with the property that $\sigma(K(\cdot))$ is relatively compact in the topology of uniform convergence of $\mathcal{B}(\mathcal{H}_\sigma)^N$. Hence, if $K(\cdot)$ is relatively compact in the topology of uniform convergence of N^N, then $G^{AP} \cong N^{AP} \times K^{AP}$ (SEF).

Proof. By Theorem 4.1, $G^{AP} \cong N^A \times K^{AP}$ (SEF), where $A = q_1^*(AP(G))$. It remains to show that A equals the algebra A' defined by the operators $T(t)$ in (7).

It is clear from (8) that $A \subset A'$, since $L(s')g(s) = S_\ell(s', e)g(s,t)$ and $T(t)g(s) = S_\ell(e,t)g(s,e)$. For the reverse inclusion, note first that $T(K)AP(N) \subset AP(N)$. Indeed, if $g \in AP(N)$ then $R(s)T(t)g = T(t)R(t(s))g$, hence $R(N)T(t)g \subset T(t)R(N)g$. Next, note that A' is translation invariant. To see this let $g \in A'$, $s \in N$, $t \in K$ and let a_t denote the inverse of the automomorphism $t(\cdot)$ (see (3)). Then $T(t)R(s)g = R(a_t(s))T(t)g$, hence $T(K)R(s)g \subset R(N)T(K)g$. Since $T(K)g$ is a norm relatively compact subset of $AP(N)$, $R(s)g \in A'$. Similarly, $L(s)g \in A'$.

Now let $g \in A'$. Given a net $\{(s_\alpha, t_\alpha)\}$ in G, choose a subnet $\{(s_\beta, t_\beta)\}$ such that $T(t_\beta) \to u$ and $L(s_\beta) \to v$ in the strong operator topology of $\mathcal{B}(A')$ and $[t_\beta, t]$ converges uniformly in t to some function $r(t)$. Since $S_\ell(s_\beta, t_\beta)g(s,t) = T(t_\beta)L(s_\beta)R([t_\beta, t])g(s)$, we have

$$\begin{aligned}
|S_\ell(s_\beta, t_\beta)g(s,t) - uvR(r(t))g(s)| &\leqslant \|T(t_\beta)L(s_\beta)R([t_\beta,t])g - uvR(r(t))g\| \\
&\leqslant \|T(t_\beta)L(s_\beta)R([t_\beta,t])g - uL(s_\beta)R([t_\beta,t])g\| \\
&\quad + \|L(s_\beta)R([t_\beta,t])g - vR([t_\beta,t])g\| \\
&\quad + \|R([t_\beta,t])g - R(r(t))g\| \\
&= a_\beta(t) + b_\beta(t) + c_\beta(t).
\end{aligned}$$

The mixed translates $L(s_\beta)R([t_\beta,t])g$ belong to a totally bounded set in A', hence $\|a_\beta\| = \sup_{t\in K} a_\beta(t) \to 0$. Similarly, $\|b_\beta\| \to 0$, and clearly $\|c_\beta\| \to 0$. Therefore $S_\ell(s_\beta, t_\beta)g$ converges in norm and $g \in A$. □

The first part of the following corollary is due to M. Landstad [5]. (See also [1], [3] and [7].)

Corollary 4.3. *If G is a semidirect product of N and K, then G^{AP} is a semidirect product of N^A and K^{AP}. If $K(\cdot) \subset N^N$ is relatively compact in the topology of uniform convergence, then G^{AP} is a semidirect product of N^{AP} and K^{AP}.*

Corollary 4.4. *If K is compact then $G^{AP} \cong N^A \times K$* (SEF).

The analog of Corollary 4.4 for the case N compact is generally false (see Example 4.15, below).

Example 4.5. Let $G = N \times K = \mathbb{C}^* \times \mathbb{R}^2$ have multiplication

$$(z, x, y)(z', x', y') = (zz' r^{yx'} e^{\mathrm{i}\theta yx'}, x + x', y + y'),$$

where $r > 0$ and $\theta \in \mathbb{R}$. Here $[(x, y), (x', y')] = r^{yx'} e^{\mathrm{i}\theta yx'}$, and the automorphism determined by (x, y) is the identity map, so G is an extension of N by K. G is also a semidirect product of the subgroups $N_1 = \mathbb{C}^* \times \mathbb{R} \times 0$ and $K_1 = 1 \times 0 \times \mathbb{R}$ ($G = N_1 K_1$ and $N_1 \lhd G$); K_1 acts on N_1 by

$$y\colon\ (z, x) \mapsto y(z, x) = (1, 0, y)(z, x, 0)(1, 0, y)^{-1} = (zr^{xy} e^{\mathrm{i}\theta xy}, x).$$

By Corollary 4.3, G^{AP} is a semidirect product $N_1^{A_1} \times K_1^{AP} = (\mathbb{C}^* \times \mathbb{R})^{A_1} \times \mathbb{R}^{AP}$, where A_1 is the algebra generated by the continuous characters

$$\chi(z, x) = \chi_{a,k}(z)\chi_b(x) = \left(\frac{z}{|z|}\right)^k e^{\mathrm{i}a \ln |z|} e^{\mathrm{i}bx} \quad (k \in \mathbb{Z},\ a, b \in \mathbb{R})$$

with the property that for each net $\{y_\alpha\}$ in $\mathbb{R}$ there exists a subnet $\{y_\beta\}$ such that

$$\begin{aligned} \chi(y_\beta(z, x)) &= \chi(zr^{xy_\beta} e^{\mathrm{i}\theta xy_\beta}, x) \\ &= \left(\frac{z}{|z|}\right)^k e^{\mathrm{i}xy_\beta(k\theta + a \ln r)} e^{\mathrm{i}(a \ln |z| + bx)} \end{aligned}$$

converges uniformly in (z, x). This clearly forces $k\theta + a \ln r = 0$. Thus, if $r = 1$, then G^{AP} is a semidirect product $((0, \infty)^{AP} \times \mathbb{R}^{AP}) \times \mathbb{R}^{AP}$ or $(\mathbb{C}^{*AP} \times \mathbb{R}^{AP}) \times \mathbb{R}^{AP}$, according as $\theta \neq 0$ or $\theta = 0$. If $r \neq 1$ then G^{AP} is a semidirect product $(\mathbb{T} \times (0, \infty)^B \times \mathbb{R}^{AP}) \times \mathbb{R}^{AP}$, where B is the algebra generated by the characters $t \mapsto e^{\mathrm{i}a \ln t}$, $a = k\theta / \ln r$, $k \in \mathbb{Z}$. If $\theta = 0$ this reduces to the semidirect product $(\mathbb{T} \times \mathbb{R}^{AP}) \times \mathbb{R}^{AP}$.

In each case we may express G^{AP} as an extension of N^A by K^{AP}. For example, if $r = 1$ and $\theta \neq 0$ then

$$G^{AP} = ((0, \infty)^{AP} \times \mathbb{R}^{AP} \times 0) \cdot (1 \times 0 \times \mathbb{R}^{AP}) = (0, \infty)^{AP} \times \mathbb{R}^{2AP} = \mathbb{C}^{*A} \times \mathbb{R}^{2AP} \text{ (SEF)},$$

where A is the algebra generated by the characters $\chi_{a,0}$.

Example 4.6. Let $G = N \times K = \mathbb{C}^* \times \mathbb{Z}^2$ with multiplication

$$(z, m, n)(z', m', n') = (zz' \lambda^{nm'}, m + m', n + n'),$$

where $\lambda = re^{2\pi \mathrm{i}\theta} \in \mathbb{C}^*$. Here, $[(m, n), (m', n')] = \lambda^{nm'}$, and the automorphism determined by (m, n) is the identity. This is a subgroup of the group in the preceding example and may be analysed in the same way. Instead, we use Theorem 4.1 to show directly that

$$(\mathbb{C}^* \times \mathbb{Z}^2)^{AP} \cong \mathbb{C}^{*A} \times \mathbb{Z}^{2AP} \cong \mathbb{C}^{*A} \times (\mathbb{Z}^{AP})^2 \text{ (SEF)}, \tag{8}$$

where A is the algebra generated by the continuous characters

$$\chi_{k,a}(z) = \Big(\frac{z}{|z|}\Big)^k e^{2\pi i a \ln |z|} \quad (k \in \mathbb{Z},\ a \in \mathbb{R},\ a \ln r + k\theta \in \mathbb{Q}). \tag{9}$$

To establish (8) we show that $q_2^*(AP(G)) \subset AP(K)$. Let $f \in AP(G)$ and $g = q_2^*(f)$. We may assume that f is a coefficient of an irreducible finite dimensional unitary representation π of G. Following [6] we note that if $U = \pi(1,0,1)$ and $V = \pi(1,1,0)$ then $UVU^{-1}V^{-1} = \pi(\lambda, 0, 0)$ commutes with π, hence Schur's Lemma implies that $UVU^{-1}V^{-1} = e^{i2\pi t} I$ for some $t \in \mathbb{R}$. If t were irrational, then U and V would generate the irrational rotation algebra, which is infinite dimensional [8]. Thus, t is rational and there exists a positive integer p such that $\pi(\lambda^p, 0, 0) = I$. For $n \in \mathbb{Z}$ let r_n denote the remainder on division of n by p. Given a net $\{(m_\alpha, n_\alpha)\}$ in K choose a subnet $\{(m_\beta, n_\beta)\}$ such that r_{m_β} is constant, say $r_{m_\beta} = r_0$, and such that $R(1, m_\beta, n_\beta) f \to h \in AP(G)$. Since

$$\begin{aligned}\pi(1, m + m_\beta, n + n_\beta) &= \pi(\lambda^{-n m_\beta}, m, n)\pi(1, m_\beta, n_\beta)\\ &= \pi(\lambda^{jp + r_{-n} r_{m_\beta}}, m, n)\pi(1, m_\beta, n_\beta)\\ &= \pi\big((\lambda^{r_{-n} r_0}, m, n)(1, m_\beta, n_\beta)\big),\end{aligned}$$

we have

$$R(m_\beta, n_\beta) g(m, n) = f(1, m + m_\beta, n + n_\beta) = R(1, m_\beta, n_\beta) f(\lambda^{r_{-n} r_0}, m, n),$$

which converges uniformly to $h(\lambda^{r_{-n} r_0}, m, n)$. Therefore $g \in AP(K)$, as required.

From Theorem 4.1 we conclude that (8) holds, where $A = q_1^*(AP(G))$ is generated by the characters $\chi_{k,a}$ of $\mathbb{C}^*$ with the property that for each net $\{(m_\alpha, n_\alpha)\}$ in $\mathbb{Z}^2$ there exists a subnet $\{(m_\beta, n_\beta)\}$ such that $\chi_{k,a}([(m_\beta, n_\beta), (m, n)]) = e^{2\pi i m n_\beta(\theta k + a \ln r)}$ converges uniformly in $m \in \mathbb{Z}$. This is possible only if $\theta k + a \ln r \in \mathbb{Q}$.

We consider some special cases:

(a) $r = 1$ and θ is rational. Then $A = AP(\mathbb{C}^*)$ and

$$(\mathbb{C}^* \times \mathbb{Z}^2)^{AP} \cong \mathbb{C}^{*AP} \times (\mathbb{Z}^{AP})^2 \cong (\mathbb{T} \times (0, +\infty)^{AP}) \times (\mathbb{Z}^{AP})^2 \text{ (SEF)}.$$

(b) $r = 1$ and θ is irrational. Then $A = AP((0, \infty))$ and

$$(\mathbb{C}^* \times \mathbb{Z}^2)^{AP} \cong (0, +\infty)^{AP} \times (\mathbb{Z}^{AP})^2 \text{ (SEF)}.$$

(c) $r \neq 1$ and $\ln r$ and θ are rational. Then

$$(\mathbb{C}^* \times \mathbb{Z}^2)^{AP} \cong (\mathbb{T} \times (0,\infty)^B) \times (\mathbb{Z}^{AP})^2 \text{ (SEF)},$$

where B is the algebra generated by the characters $s \mapsto e^{2\pi a \ln s}$ of $(0,\infty)$ with a rational.

Examples 4.1 and 4.2 have higher dimensional analogs, which may be handled in a similar manner. For instance, one could take $N = \mathbb{C}^{*2}$, $K = \mathbb{Z}^4$ and multiplication

$$(z,w,j,k,m,n)(z',w',j',k',m',n') = (zz'\lambda^{jm'}, ww'\mu^{kn'}, j+j', k+k', m+m', n+n').$$

Example 4.7. Let $G = N \times K = \mathbb{C}^{*2} \times \mathbb{Z}^2$ with multiplication

$$(z_1, z_2, m, n)(z_1', z_2', m', n') = (z_1 z_1' z_2'^n \lambda^{m'n(n-1)/2}, z_2 z_2' \lambda^{m'n}, m+m', n+n').$$

Here $[(m,n),(m',n')] = (\lambda^{m'n(n-1)/2}, \lambda^{nm'})$, $\lambda \in \mathbb{C}^*$, and the automorphism determined by (m,n) is the map $(z_1,z_2) \mapsto (z_1 z_2^n, z_2)$. If $u = (1,1,0,1)$ and $v = (1,1,1,0)$, then $uvu^{-1}v^{-1} = (1,\lambda,0,0) = w$ and $uwu^{-1}w^{-1} = (\lambda,1,0,0)$, and using this one can argue as in Example 4.6 to show that

$$(\mathbb{C}^{*2} \times \mathbb{Z}^2)^{AP} \cong \mathbb{C}^{*2A} \times (\mathbb{Z}^{AP})^2 \text{ (SEF)},$$

where $A = q_2^*(AP(G))$. One can also use the method of Example 4.5, since G is a semidirect product $(\mathbb{C}^{*2} \times \mathbb{Z}) \times \mathbb{Z}$.

Example 4.8 (Heisenberg groups)**.** Let $G = N \times K = \mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}^n)$ with multiplication

$$(s,x,y)(s',x',y') = (s+s'+y\cdot x', x+x', y+y').$$

Here, $[(x,y),(x',y')] = y \cdot x'$, and the automorphism determined by (x,y) is trivial. G is a semidirect product of the subgroups $N_1 = \mathbb{R} \times \mathbb{R}^n \times 0$ and $K_1 = 0 \times 0 \times \mathbb{R}^n$, so K_1 acts on N_1 by automorphisms

$$y\colon\ (s,x,0) \mapsto y(s,x,0) = (s + y\cdot x, x, 0).$$

G^{AP} is therefore a semidirect product $N_1^{A_1} \times K_1^{AP} = (\mathbb{R} \times \mathbb{R}^n)^{A_1} \times (\mathbb{R}^{AP})^n$, where A_1 is generated by the characters $\chi_{a,y}(s,x) = e^{\mathrm{i}(as+y\cdot x)}$ with the property that for each net $\{y_\alpha\}$ in $\mathbb{R}^n$ there exists a subnet $\{y_\beta\}$ such that $\chi_{a,y}(s+y_\beta\cdot x, x)$ converges uniformly in (s,x). This is possible only if $a=0$, so $N_1^{A_1} = (\mathbb{R}^{AP})^n$ and G^{AP} reduces to the direct product $(\mathbb{R}^{AP})^{2n}$.

A similar analysis shows that the AP-compactification of the subgroup $H = \mathbb{Z} \times (\mathbb{Z}^n \times \mathbb{Z}^n)$ is a semidirect product $(\mathbb{Z}^B \times (\mathbb{Z}^{AP})^n) \times (\mathbb{Z}^{AP})^n$, where B is the algebra generated by the characters $m \mapsto \lambda^m$, λ a root of unity. Thus, while the factor $\mathbb{R}$ of G is annihilated by taking the AP-compactification, the factor $\mathbb{Z}$ of H is not. One consequence of this is that not every member of $AP(H)$ extends to a member of $AP(G)$. Note that H^{AP} may also be viewed as an extension of $\mathbb{Z}^B$ by $(\mathbb{Z}^{AP})^{2n}$.

Next, we consider generalizations of the above results to the semigroup case.

Theorem 4.9. *Let N, K and $G = N \times K$ (SEF) be semitopological semigroups and let $A = q_1^*(SAP(G))$. The following conditions are equivalent:*

(a) $G^{SAP} \cong N' \times K'$ (SEF) *for some compactifications N' and K' of N and K.*

(b) $q_2^*(SAP(G)) \subset SAP(K)$.

(c) $\varepsilon_{G^{SAP}}([K,\cdot] \times e) \subset (G^{SAP})^K$ *is relatively compact in the topology of uniform convergence, and the mapping $[\cdot,\cdot]\colon (K \times K, \mathcal{T}_{SAP} \times \mathcal{T}_{SAP}) \to (N, \mathcal{T}_A)$ is continuous, where $\mathcal{T}_{SAP}$ is the initial topology on K for the family $SAP(K)$, and $\mathcal{T}_A$ is the initial topology on N for the family A.*

If (a) *holds then $K' \cong K^{SAP}$, $N' \cong N^A$, and A is generated by the coefficients of the finite dimensional irreducible unitary representations of N having the compactness property described in Theorem* 4.1.

P r o o f. As in the proof of Theorem 4.1, (a) implies (b) and the isomorphisms $K' \cong K^{SAP}$ and $N' \cong N^A$, and (b) implies the first part of (c). To show that (b) implies the second part of (c), let $\{(t_\alpha, t'_\alpha)\}$ be a net in $K \times K$ which $\mathcal{T}_{SAP} \times \mathcal{T}_{SAP}$-converges to (t, t'). If (b) holds then, $\varepsilon_{G^{SAP}}(e, t_\alpha) \to \varepsilon_{G^{SAP}}(e, t)$, $\varepsilon_{G^{SAP}}(e, t'_\alpha) \to \varepsilon_{G^{SAP}}(e, t')$, and $\varepsilon_{G^{SAP}}(e, t_\alpha t'_\alpha) \to \varepsilon_{G^{SAP}}(e, tt')$, so

$$\begin{aligned}\varepsilon_{G^{SAP}}([t_\alpha, t'_\alpha], e) &= \varepsilon_{G^{SAP}}(e, t_\alpha)\varepsilon_{G^{SAP}}(e, t'_\alpha)\varepsilon_{G^{SAP}}(e, t_\alpha t'_\alpha)^{-1} \\ &\to \varepsilon_{G^{SAP}}(e, t)\varepsilon_{G^{SAP}}(e, t')\varepsilon_{G^{SAP}}(e, tt')^{-1} \\ &= \varepsilon_{G^{SAP}}([t, t'], e).\end{aligned}$$

Therefore $[t_\alpha, t'_\alpha] \to [t, t']$ in $(N, \mathcal{T}_A)$.

Now assume that (c) holds. As in the proof of Theorem 4.1, $q_2^*(SAP(G)) \subset AP(K)$. Let $f \in SAP(G)$, $g = q_2^*(f)$ and $\hat{g} = (\varepsilon_{K^{AP}}^*)^{-1}(g)$. To show that $g \in SAP(K)$ it suffices to show that g is distal, i.e., $\hat{g}(\varepsilon_{K^{AP}}(t) y \varepsilon_{K^{AP}}(t')) = g(tt')$ for t, $t' \in K$ and $y = y^2 \in K^{AP}$ [1, 4.6.6]. Let $\varepsilon_{K^{AP}}(t_\alpha) \to y$. Since $q_2^*(SAP(G)) \subset AP(K)$ and y is an idempotent, $\varepsilon_{G^{SAP}}(e, t_\alpha)$ and $\varepsilon_{G^{SAP}}(e, t_\alpha^2)$ converge to the same member of G^{SAP}. Moreover, $t_\alpha \to e$ and $tt_\alpha \to t$ in $(K, \mathcal{T}_{SAP})$, hence, by the continuity hypothesis, $[t, t_\alpha] \to e$, $[t_\alpha, t_\alpha] \to e$ and $[tt_\alpha, t'] \to [t, t']$ in $(N, \mathcal{T}_A)$. Thus, setting $\varepsilon = \varepsilon_{G^{SAP}}$, we have

$$\varepsilon(e, t_\alpha) = \varepsilon([t_\alpha, t_\alpha], e)\varepsilon(e, t_\alpha^2)\varepsilon(e, t_\alpha)^{-1} \to \varepsilon(e, e),$$
$$\varepsilon(e, tt_\alpha) = \varepsilon([t, t_\alpha], e)^{-1}\varepsilon(e, t)\varepsilon(e, t_\alpha) \to \varepsilon(e, t)$$

and

$$\begin{aligned}\varepsilon(e, tt_\alpha t') &= \varepsilon([tt_\alpha, t'], e)^{-1}\varepsilon(e, tt_\alpha)\varepsilon(e, t') \\ &\to \varepsilon([t, t'], e)^{-1}\varepsilon(e, t)\varepsilon(e, t') = \varepsilon(e, tt').\end{aligned}$$

In particular, $g(tt_\alpha t') \to g(tt')$. Therefore, (c) implies (b).

The remaining assertions of the theorem are proved as in Theorem 4.1. □

Corollary 4.10. *Let K be a compact topological group. Then $G^{SAP} \cong N^A \times K$ (SEF).*

Corollary 4.11. *Suppose that $[t, t']$ is a homomorphism in one of the variables for each fixed value of the other. If $A := q_2^*(SAP(G)) \subset AP(K)$, then $G^{SAP} \cong N^A \times K^{SAP}$ (SEF).*

P r o o f. Let π be a finite dimensional unitary representation of G. Since $A \subset AP(K)$, $\pi_2 := \pi \circ q_2\colon K \to \mathcal{U}$ has an extension $\overline{\pi}_2\colon K^{AP} \to \mathcal{U}$, where $\mathcal{U}$ is the closure of $\pi(G)$. Set $\theta(t, t') = \pi([t, t'], e)^{-1}$. The identity

$$\theta(t, t') = \pi_2(tt')\pi_2(t')^{-1}\pi_2(t)^{-1}$$

implies that $\theta(t, t')$ is almost periodic and hence has an extension $\overline{\theta}\colon (K \times K)^{AP} = K^{AP} \times K^{AP} \to \mathcal{U}$ which is a homomorphism in one of its variables and satisfies

$$\overline{\pi}_2(uv) = \overline{\theta}(u, v)\overline{\pi}_2(u)\overline{\pi}_2(v) \quad (u, v \in K^{AP}).$$

It follows from the fact that $\mathcal{U}$ is a group that $\overline{\pi}_2(uvw) = \overline{\pi}_2(uw)$ if $v^2 = v$. Therefore (b) of Theorem 4.9 holds: If f is a coefficient of π then $q_2^*(f)$ is distal and almost periodic and hence strongly almost periodic. □

From 4.11 we have immediately

Corollary 4.12 [3]**.** *If G is a semidirect product of N and K, then G^{SAP} is a semidirect product of N^A and K^{SAP}. If $K(\cdot) \subset N^N$ is relatively compact in the topology of uniform convergence, then G^{SAP} is a semidirect product of N^{SAP} and K^{SAP}.*

Example 4.13. Let $\mathbb{Z}_p$ denote the p-adic integers and let N be a subsemigroup of $(\mathbb{C},\cdot)$ which is invariant under rotations. Give $G = N \times \mathbb{Z}_p^2$ multiplication

$$(z,x,y)(z',x',y') = (zz'e^{2\pi i x_0 y x'}, x+x', y+y'),$$

where x_0 is a fixed p-adic number. By Corollary 4.10, $(N \times \mathbb{Z}_p^2)^{SAP} = N^A \times \mathbb{Z}_p^2$ (SEF).

Example 4.14. Consider the subsemigroup $G_1 = N \times K_1 = \mathbb{C}^* \times {\mathbb{Z}_+}^2$ of the group G of Example 4.6. As in that example, $q_2^*(SAP(G_1)) \subset AP(K_1)$, so by Corollary 4.11,

$$(\mathbb{C}^* \times {\mathbb{Z}_+}^2)^{SAP} \cong \mathbb{C}^{*A} \times ({\mathbb{Z}_+}^{SAP})^2 \text{ (SEF)},$$

where $A = q_1^*(SAP(G_1))$.

Similarly, for the subsemigroup $\mathbb{C}^{*2} \times {\mathbb{Z}_+}^2$ of the group G in Example 4.7,

$$(\mathbb{C}^{*2} \times {\mathbb{Z}_+}^2)^{SAP} \cong \mathbb{C}^{*2A} \times ({\mathbb{Z}_+}^{SAP})^2 \text{ (SEF)}.$$

Example 4.15. Let N be an abelian topological group, K a semitopological semigroup and let $\psi\colon K \to N$ be a continuous function such that $\psi(e) = e$. Then $[t,t'] = \psi(t)\psi(t')\psi(tt')^{-1}$ satisfies the cocycle identity of (SEF) (in fact it is a coboundary), hence $G = N \times K$ (SEF), where $t \mapsto t(\cdot)$ is trivial. If $\psi^*(AP(N)) \subset SAP(K)$ then $[\cdot,\cdot]$ satisfies condition (c) of Theorem 4.9, hence $G^{SAP} \cong N^{AP} \times K^{SAP}$ (SEF).

Now take $N = \mathbb{T}$ and $K = \mathbb{Z}$. Then $\chi(z,t) = z\psi(t)$ defines a continuous character of G such that $q_2^*(\chi) = \psi$, so if $\psi \notin AP(G)$ then $q_2^*(AP(G)) \not\subset AP(K)$ and hence G^{AP} cannot be isomorphic to $N' \times K'$ (SEF) for any compactifications N' and K'. For instance if $\psi(n) = e^{in/(1+n^2)}$ or $\psi(n) = e^{in^2}$ then $(\mathbb{T} \times \mathbb{Z})^{AP} \not\cong \mathbb{T}' \times \mathbb{Z}'$.

Similarly, if $\psi(t) = e^{it^2}$ then $(\mathbb{T} \times \mathbb{R})^F \not\cong \mathbb{T}' \times \mathbb{R}'$ (SEF) for $F = AP$ or WAP, and $(\mathbb{T} \times \mathbb{R}_+)^{SAP} \not\cong \mathbb{T}' \times {\mathbb{R}_+}'$ (SEF).

5. Weakly almost periodic compactifications

Theorem 5.1. *If K is compact then $G^{WAP} \cong N^W \times K$ (SEF), where $W = q_1^*(WAP(G))$. Moreover, the homomorphism $\tilde{t}(\cdot)\colon N^W \to N^W$ corresponding to $t \in K$ is an automorphism.*

P r o o f. We show first that each map $t(\cdot)\colon N \to N$ extends to an automorphism $\tilde{t}(\cdot)\colon N^W \to N^W$ such that $(t,x) \to \tilde{t}(x)\colon K \times N^W \to N^W$ is continuous. For $t \in K$ let $T(t)\colon C(N) \to C(N)$ denote the dual of the map $t(\cdot)$. The identity $T(t)q_1^*(f) = q_1^*(L(e,t)R\big((e,t)^{-1}\big)f)$ shows that $T(t)W \subset W$, hence $t(\cdot)$ has an extension $\tilde{t}(\cdot)$:

$N^W \to N^W$. To show that $(t,x) \to \tilde{t}(x)$ is continuous, let $t_\alpha \to t \in K$, $f \in WAP(G)$ and set $g = q_1^*(f) = \varepsilon_{N^A}^*(\hat{g})$, where $\hat{g} \in C(N^W)$. Since $WAP(G) \subset LC(G) \cap RC(G)$, $\hat{g}\big(\tilde{t}_\alpha(\varepsilon_{N^W}(s))\big) = g(t_\alpha(s)) = f\big((e,t_\alpha)(s,e)(e,t_\alpha)^{-1}\big)$ converges to $f\big((e,t)(s,e)(e,t)^{-1}\big) = \hat{g}\big(\tilde{t}(\varepsilon_{N^W}(s))\big)$ uniformly in s. It follows that $\hat{g}(\tilde{t}_\alpha(x)) \to \hat{g}(\tilde{t}(x))$ uniformly in $x \in N^W$, which is equivalent to the continuity of $(t,x) \to \tilde{t}(x)$.

For $t \in K$ the inverse a_t of the automomorphism $t(\cdot)\colon N \to N$ is given by

$$a_t(s) = [t^{-1},t]^{-1}t^{-1}(s)[t^{-1},t];$$

define $\tilde{a}_t\colon N^W \to N^W$ by $\tilde{a}_t(x) = \varepsilon_{N^W}\big([t^{-1},t]^{-1}\big)\widetilde{t^{-1}}(x)\varepsilon_{N^W}([t^{-1},t])$. Then $\tilde{a}_t$ is an extension of a_t, and the relations $\varepsilon_{N^W}(t(s)) = \tilde{t}(\varepsilon_{N^W}(s))$ and $\varepsilon_{N^W}(a_t(s)) = \tilde{a}_t(\varepsilon_{N^W}(s))$ imply that $\tilde{a}_t \circ \tilde{t}$ and $\tilde{t} \circ \tilde{a}_t$ are the identity mappings on $\varepsilon_{N^W}(N)$ and hence, by continuity, on N^W. Thus, $\tilde{a}_t$ is the inverse of $\tilde{t}$ and $\tilde{t}$ is an automorphism. Moreover, since $(t,x) \to \tilde{t}(x)$ is continuous so is $(t,x) \to \tilde{a}_t(x)$.

Next, we use Grothendieck's double limit criterion to show that $p_1^*(W) \subset WAP(G)$. Let $g \in W$, $\hat{g} = \varepsilon_{N^W}^{*-1}(g)$, $f = p_1^*(g)$, and let $\{(s_n,t_n)\}$ and $\{(s_m,t_m)\}$ be sequences in G such that the following limits exist:

$$a := \lim_m \lim_n f\big((s_n,t_n)(s_m,t_m)\big) \quad \text{and} \quad b := \lim_n \lim_m f\big((s_n,t_n)(s_m,t_m)\big).$$

We need to show that $a = b$. Choose subnets $\{(s_\alpha,t_\alpha)\}$ of $\{(s_n,t_n)\}$ and $\{(s_\beta,t_\beta)\}$ of $\{(s_m,t_m)\}$ such that $\varepsilon_{N^W}(s_\alpha) \to x \in N^W$, $\varepsilon_{N^W}(s_\beta) \to y \in N^W$, $t_\alpha \to t \in K$, and $t_\beta \to u \in K$. Then, for fixed β,

$$t_\alpha(s_\beta)[t_\alpha,t_\beta] = p_1\big((e,t_\alpha)(s_\beta,t_\beta)\big) \to p_1\big((e,t)(s_\beta,t_\beta)\big) = t(s_\beta)[t,t_\beta],$$

and since $g \in WAP(N) \subset LC(N) \cap RC(N)$ we have

$$\begin{aligned} a &= \lim_\beta \lim_\alpha \hat{g}\big(\varepsilon_{N^W}(s_\alpha)\varepsilon_{N^W}(t_\alpha(s_\beta)[t_\alpha,t_\beta])\big) \\ &= \lim_\beta \hat{g}\big(x\varepsilon_{N^W}(t(s_\beta)[t,t_\beta])\big) \\ &= \lim_\beta \hat{g}\big(x\tilde{t}(\varepsilon_{N^W}(s_\beta))\varepsilon_{N^W}([t,t_\beta])\big) \\ &= \hat{g}\big(x\tilde{t}(y)\varepsilon_{N^W}([t,u])\big) = \hat{g}\big(\tilde{t}(\tilde{a}_t(x)y)\varepsilon_{N^W}([t,u])\big) \\ &= \lim_\alpha \hat{g}\big(\tilde{t}_\alpha(\tilde{a}_{t_\alpha}(\varepsilon_{N^W}(s_\alpha))y)\varepsilon_{N^W}([t_\alpha,u])\big) \\ &= \lim_\alpha \hat{g}\big(\varepsilon_{N^W}(s_\alpha)\tilde{t}_\alpha(y)\varepsilon_{N^W}([t_\alpha,u])\big) \\ &= \lim_\alpha \lim_\beta \hat{g}\big(\varepsilon_{N^W}(s_\alpha)\tilde{t}_\alpha(\varepsilon_{N^W}(s_\beta))\varepsilon_{N^W}([t_\alpha,t_\beta])\big) = b. \end{aligned}$$

The theorem now follows from Theorem 3.4 and Remark 3.2. □

The next corollary shows that the minimal ideal $M(G^{WAP})$ of G^{WAP} is a canonical group extension of $M(N^W)$ by K.

Corollary 5.2. $M(G^{WAP}) \cong M(N^W) \times K$ (SEF), *and* $M(N^W)$ *is canonically isomorphic to* N^A, *where* $A = q_1^*(AP(G))$.

P r o o f. Since $\tilde{t}(\cdot)$ is surjective, $\tilde{t}(M(N^W)) = M(N^W)$, hence the first coordinate of $\big(x\tilde{t}(x')[t,t'],tt'\big) = (x,t)(x',t')$ is in $M(N^W)$ whenever x or x' is in $M(N^W)$. Thus $J := M(N^W) \times K$ is a closed ideal of G^{WAP}. Since J is also a group it must coincide with $M(G^{WAP})$. Since G^{AP} is canonically isomorphic to $M(G^{WAP})$ and $G^{AP} \cong N^A \times K$ (SEF) (Corollary 4.4), $M(N^W) \cong N^A$. $\square$

Example 5.3. Let $G = N \times K = \mathbb{C}^* \times \mathbb{Z}_p^2$, as in Example 4.13. By Theorem 5.1, $G^{WAP} \cong \mathbb{C}^{*W} \times \mathbb{Z}_p^2$ (SEF), where $W = q_1^*(WAP(G))$.

The analog of Theorem 5.1 for compact N is false, as the following example shows.

Example 5.4. Consider the subgroup $G_1 = \mathbb{T} \times \mathbb{Z}^2$ of the group G of Example 4.6 (when $|\lambda| = 1$). As observed in [6], if $\lambda = e^{\mathrm{i}}$ then $G_1^{WAP} \ncong \mathbb{T} \times (\mathbb{Z}^2)^{WAP}$. In fact, since $r_1^*(WAP(G_1)) \not\subset WAP(G_1)$ (the function $f(z,m,n) = z(m^2+n^2+1)^{-1}$ will serve), it follows from Theorem 3.4 that G_1^{WAP} cannot be isomorphic to $\mathbb{T}' \times (\mathbb{Z}^2)^{WAP}$ (SEF) for *any* compactification $\mathbb{T}'$ of $\mathbb{T}$.

On the other hand, if λ is a root of unity then $G_1^{WAP} \cong \mathbb{T} \times (\mathbb{Z}^2)^{WAP}$, as can be seen by an application of Grothendieck's double limit criterion.

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