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A NON-COMMUTATIVE GENERALIZATION OF  $MV$ -ALGEBRAS

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## 1. INTRODUCTION

It is well-known that  $MV$ -algebras (as algebras of type  $(\oplus, \odot, \neg, 0, 1)$  of signature  $\langle 2, 2, 1, 0, 0 \rangle$ ) have been introduced and studied by C. C. Chang in [4] and [5] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic. Moreover, Chang proved in [5] that every linearly ordered  $MV$ -algebra is isomorphic to an  $MV$ -algebra in the form  $\Gamma(G, u)$ , where  $G$  is a commutative linearly ordered (additive) group,  $0 \leq u \in G$  is a strong order unit in  $G$ ,  $\Gamma(G, u) = [0, u] = \{x \in G; 0 \leq x \leq u\}$ , and the operations on  $\Gamma(G, u)$  are as follows:  $1 = u$ ,  $x \oplus y = \min(x + y, u)$ ,  $\neg x = u - x$ ,  $x \odot y = \max(x + y - u, 0) = \neg(\neg x \oplus \neg y)$  for any  $x, y \in \Gamma(G, u)$ . D. Mundici generalized this result in [11] to arbitrary  $MV$ -algebras. Namely, let, in general,  $G$  be a commutative lattice ordered group ( $l$ -group) and  $0 \leq u \in G$ . If  $x \oplus y = (x + y) \wedge u$ ,  $\neg x = u - x$ ,  $x \odot y = \neg(\neg x \oplus \neg y)$  for any  $x, y \in [0, u]$ , then  $\Gamma(G, u) = ([0, u], \oplus, \odot, \neg, 0, u)$  is an  $MV$ -algebra. Mundici proved that every  $MV$ -algebra is isomorphic to  $\Gamma(G, u)$  for some commutative  $l$ -group  $G$  and some strong unit  $u \in G$ , and moreover, that the category of  $MV$ -algebras is equivalent to the category of commutative  $l$ -groups with strong units. Mundici further proved in [12] that  $MV$ -algebras are also categorically equivalent to bounded commutative  $BCK$ -algebras. The author in [13] and [14] showed that one can obtain  $MV$ -algebras as special cases of dually residuated  $l$ -monoids ( $DRL$ -monoids) because  $MV$ -algebras are equivalent to  $DRL$ -monoids of a class of bounded  $DRL$ -monoids.

In the paper we define a generalization of  $MV$ -algebras as algebras  $A = (A, \oplus, \odot, \neg, \sim, 0, 1)$  of signature  $\langle 2, 2, 1, 1, 0, 0 \rangle$  in which the binary operations  $\oplus$  and  $\odot$  in general need not be commutative. ( $MV$ -algebras introduced by Chang are then called *commutative*.) In the first part of the paper some properties of non-commutative  $MV$ -algebras are described. In the second part it is proved that (non-commutative)  $MV$ -algebras are in a one-to-one correspondence with

some bounded non-commutative *DRL*-monoids, and in the last section it is shown that every interval  $[0, u]$  of any (non-commutative) *l*-group can be viewed as an *MV*-algebra and that every linearly ordered *MV*-algebra is isomorphic to an analogous interval of some linearly ordered loop.

(Note that the class of commutative *l*-groups is the smallest non-trivial variety of *l*-groups. Therefore, from this point of view, intervals of non-commutative *l*-groups represent a very essential generalization of Chang's *MV*-algebras.)

We use the terminology and results of [2], [7] and [9] concerning the theory of lattice ordered groups and of [1] concerning the theory of loops.

## 2. PROPERTIES OF NON-COMMUTATIVE *MV*-ALGEBRAS

An algebra  $A = (A, \oplus, \odot, \neg, \sim, 0, 1)$  of signature  $\langle 2, 2, 1, 1, 0, 0 \rangle$  is called a (*non-commutative*) *MV*-algebra if for any  $x, y, z, u \in A$ :

- |                                                               |                                                              |
|---------------------------------------------------------------|--------------------------------------------------------------|
| (1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ ;         | (1') $x \odot (y \odot z) = (x \odot y) \odot z$             |
| (2) $x \oplus \sim x = 1 = \neg x \oplus x$ ;                 | (2') $x \odot \sim x = 0 = \neg x \odot x$                   |
| (3) $x \oplus 1 = 1 = 1 \oplus x$ ;                           | (3') $x \odot 0 = 0 = 0 \odot x$                             |
| (4) $x \oplus 0 = x = 0 \oplus x$ ;                           | (4') $x \odot 1 = x = 1 \odot x$                             |
| (5) $\neg(x \oplus y) = \neg x \odot \neg y$ ;                | (5') $\neg(x \odot y) = \neg x \oplus \neg y$                |
| (6) $\sim(x \oplus y) = \sim x \odot \sim y$ ;                | (6') $\sim(x \odot y) = \sim x \oplus \sim y$                |
| (7) $\neg \sim x = x = \sim \neg x$                           |                                                              |
| (8) $\neg 0 = 1 = \sim 0$                                     |                                                              |
| (9) $y \oplus (x \odot \sim y) = (\neg y \odot x) \oplus y$ ; | (9') $y \odot (x \oplus \sim y) = (\neg y \oplus x) \odot y$ |

Denote  $x \vee y = y \oplus (x \odot \sim y)$ ,  $x \wedge y = y \odot (x \oplus \sim y)$ . Then

- |                                                                                             |                                                       |
|---------------------------------------------------------------------------------------------|-------------------------------------------------------|
| (10) $x \vee y = y \vee x$ ;                                                                | (10') $x \wedge y = y \wedge x$                       |
| (11) $x \vee (y \vee z) = (x \vee y) \vee z$ ;                                              | (11') $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ |
| (12a) $x \oplus (y \wedge z) \oplus u = (x \oplus y \oplus u) \wedge (x \oplus z \oplus u)$ |                                                       |
| (12b) $x \oplus (y \vee z) \oplus u = (x \oplus y \oplus u) \vee (x \oplus z \oplus u)$     |                                                       |
| (12'a) $x \odot (y \wedge z) \odot u = (x \odot y \odot u) \wedge (x \odot z \odot u)$      |                                                       |
| (12'b) $x \odot (y \vee z) \odot u = (x \odot y \odot u) \vee (x \odot z \odot u)$          |                                                       |

**Note.** It is obvious that the above system of axioms is not the simplest one, but we use it in this form to have the possibility comparing the introduced notion with that of a commutative *MV*-algebra. We can see that  $A$  can be considered as a commutative (Chang's) *MV*-algebra if and only if both operations  $\oplus$  and  $\odot$  are commutative and the unary operations  $\neg$  and  $\sim$  coincide. (Note that conditions (12b) and (12'a) are not directly required but can be derived for Chang's *MV*-algebras. See e.g. [8], Theorem 3.1.) A simple axiomatic system for commutative *MV*-algebras is used e.g. in [6].

**Theorem 1.** *If  $A$  is an MV-algebra,  $x, y \in A$ , then*

- (a)  $\neg 1 = 0 = \sim 1$ ;
- (b)  $x \odot y = \neg(\sim x \oplus \sim y) = \sim(\neg x \oplus \neg y)$ ;
- (c)  $x \vee 0 = x = x \wedge 1$ ,  $x \wedge 0 = 0$ ,  $x \vee 1 = 1$ ;
- (d)  $\neg(x \vee y) = \neg x \wedge \neg y$ ,  $\neg(x \wedge y) = \neg x \vee \neg y$ ,  
 $\sim(x \vee y) = \sim x \wedge \sim y$ ,  $\sim(x \wedge y) = \sim x \vee \sim y$ ;
- (e)  $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ .

*Proof.* (a) It follows from (7) and (8).

(b) From (5), (6) and (7).

(c) From (4') and (8).

$$\begin{aligned} \text{(d)} \quad \neg(x \vee y) &= \neg(y \oplus (x \odot \sim y)) = \neg y \odot \neg(x \odot \sim y) \\ &= \neg y \odot (\neg x \oplus \neg \sim y) = \neg y \odot (\neg x \oplus \sim \neg y) = \neg x \wedge \neg y. \end{aligned}$$

The remaining equalities are analogous.

$$\begin{aligned} \text{(e)} \quad x \wedge (x \vee y) &= x \wedge ((\neg x \odot y) \oplus x) = x \odot (((\neg x \odot y) \oplus x) \oplus \sim x) \\ &= x \odot ((\neg x \odot y) \oplus 1) = x \odot 1 = x \end{aligned}$$

Analogously the second equality. □

We obtain the following theorem as a consequence of (10), (10'), (11), (11'), and (c) and (e) from Theorem 1.

**Theorem 2.** *If  $A$  is an MV-algebra then  $(A, \vee, \wedge)$  is a bounded lattice with the least element 0 and the greatest element 1.*

Hence we can consider the order relation  $\leq$  on  $A$  induced by the lattice  $(A, \vee, \wedge)$ , i.e.

$$x \leq y \iff x \vee y = y \iff x \wedge y = x.$$

**Theorem 3.** *For any  $x, y, z, u \in A$  we have*

$$\begin{aligned} \text{(a)} \quad x \leq y &\implies u \oplus x \oplus z \leq u \oplus y \oplus z, \\ &u \odot x \odot z \leq u \odot y \odot z \end{aligned}$$

(i.e.  $(A, \oplus, \leq)$  and  $(A, \odot, \leq)$  are ordered monoids);

$$\text{(b)} \quad x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y;$$

$$\text{(c)} \quad x \leq y \iff \neg y \leq \neg x \iff \sim y \leq \sim x.$$

**P r o o f.** (a) If  $x \leq y$  then  $u \oplus (x \wedge y) \oplus z = u \oplus x \oplus z$ , hence by (12a)  $(u \oplus x \oplus z) \wedge (u \oplus y \oplus z) = u \oplus x \oplus z$ , thus  $u \oplus x \oplus z \leq u \oplus y \oplus z$ .

Using (12) and (12') we also obtain the remaining implications.

(b) Since  $y \leq 1$ , by (a) and (4') we get  $x \odot y \leq x \odot 1 = x$ . Similarly  $x \odot y \leq y$ , and so  $x \odot y \leq x \wedge y$ .

The equality  $x \vee y \leq x \oplus y$  is dual.

(c) Let  $x \leq y$ . Then  $\neg(x \vee y) = \neg y$ , hence by Theorem 1(d) we have  $\neg x \wedge \neg y = \neg y$ , and thus  $\neg y \leq \neg x$ .

Suppose that  $\neg y \leq \neg x$ . Then  $\sim(\neg y \wedge \neg x) = \sim \neg y$ , hence by Theorem 1(d),  $\sim \neg y \vee \sim \neg x = \sim \neg y$ , that means  $y \vee x = y$ , therefore  $x \leq y$ .

The second equivalence is analogous. □

**Theorem 4.** For any  $x, y \in A$  we have

- (a)  $x \oplus y = 0 \implies x = 0 = y$ ;
- (b)  $x \odot y = 1 \implies x = 1 = y$ .

**P r o o f.** (a) Let  $x \oplus y = 0$ . Then by Theorem 1(c) and conditions (4) and (12a),

$$0 = x \wedge 0 = x \wedge (x \oplus y) = (x \oplus 0) \wedge (x \oplus y) = x \oplus 0 = x.$$

Hence  $0 = x \oplus y = 0 \oplus y = y$ .

(b) Dually. □

**Theorem 5.** For any  $x, y \in A$  the following conditions are equivalent:

- (a)  $x \leq y$ ;
- (b)  $y \oplus \sim x = 1$ ;
- (c)  $\neg x \oplus y = 1$ ;
- (d)  $x \odot \sim y = 0$ ;
- (e)  $\neg y \odot x = 0$ .

**P r o o f.** (a) $\Leftrightarrow$ (b): Let  $x \leq y$ . Then by (2) and Theorem 3,  $1 = x \oplus \sim x \leq y \oplus \sim x$ , hence  $y \oplus \sim x = 1$ .

Conversely, if  $y \oplus \sim x = 1$ , then by (4') we get  $x \wedge y = x \odot (y \oplus \sim x) = x \odot 1 = x$ , thus  $x \leq y$ .

(a) $\Leftrightarrow$ (c): Analogously.

(a) $\Leftrightarrow$ (d), (a) $\Leftrightarrow$ (e): Dually. □

**Theorem 6.** a) If  $x \oplus z = y \oplus z$  ( $z \oplus x = z \oplus y$ ) and  $x, y \leq \neg z$  ( $x, y \leq \sim z$ , respectively), then  $x = y$ .

b) If  $x \odot z = y \odot z$  ( $z \odot x = z \odot y$ ) and  $\neg z \leq x, y$  ( $\sim z \leq x, y$ , respectively), then  $x = y$ .

**Proof.** a) Let  $x \oplus z = y \oplus z$  and  $x, y \leq \neg z$ . Then by Theorem 5, (9') and (10'),

$$\begin{aligned} x &= 1 \odot x = (\neg x \oplus \neg z) \odot x = \neg z \odot (x \oplus \sim \neg z) \\ &= \neg z \odot (x \oplus z) = \neg z \odot (y \oplus z) = y. \end{aligned}$$

Analogously the second implication.

b) Dually. □

**Theorem 7.** For any  $x, y \in A$  the following conditions are equivalent:

- (a)  $x \oplus y = y$ ;
- (b)  $x \odot y = x$ ;
- (c)  $\sim x \vee y = 1$ ;
- (d)  $x \wedge \neg y = 0$ .

**Proof.** (a) $\Leftrightarrow$ (d): Let  $x \oplus y = y$ . Then

$$x \wedge \neg y = \neg y \odot (x \oplus \sim \neg y) = \neg y \odot y = 0.$$

Conversely, let  $x \wedge \neg y = 0$ . Then

$$\begin{aligned} y &= 0 \oplus y = (x \wedge \neg y) \oplus y = (x \oplus y) \wedge (\neg y \oplus y) \\ &= (x \oplus y) \wedge 1 = x \oplus y. \end{aligned}$$

(b) $\Leftrightarrow$ (c): If  $x \odot y = x$  then

$$\sim x \vee y = (\neg \sim x \odot y) \oplus \sim x = x \oplus \sim x = 1.$$

Conversely, if  $\sim x \vee y = 1$  then

$$\begin{aligned} x &= x \odot 1 = x \odot (\sim x \vee y) = (x \odot \sim x) \vee (x \odot y) \\ &= 0 \vee (x \odot y) = x \odot y. \end{aligned}$$

(c) $\Leftrightarrow$ (d): If  $\sim x \vee y = 1$  then  $\neg(\sim x \vee y) = \neg 1$ , hence  $x \wedge \neg y = 0$ .

Conversely, let  $x \wedge \neg y = 0$ . Then  $\sim(x \wedge \neg y) = \sim 0 = 1$ , thus  $\sim x \vee y = 1$ . □

The proof of the following theorem is analogous.

**Theorem 8.** For any  $x, y \in A$  the following conditions are equivalent:

- (a)  $x \oplus y = x$ ;
- (b)  $x \odot y = y$ ;

- (c)  $x \vee \neg y = 1$ ;
- (d)  $\sim x \wedge y = 0$ .

As a consequence we get

**Theorem 9.** *For any  $x \in A$  the following conditions are equivalent:*

- (a)  $x \oplus x = x$ ;
- (b)  $x \odot x = x$ ;
- (c)  $\sim x \vee x = 1$ ;
- (d)  $x \vee \neg x = 1$ ;
- (e)  $x \wedge \neg x = 0$ ;
- (f)  $\sim x \wedge x = 0$ ;
- (g)  $\neg x \oplus \neg x = \neg x$ ;
- (h)  $\neg x \odot \neg x = \neg x$ ;
- (i)  $\sim x \oplus \sim x = \sim x$ ;
- (j)  $\sim x \odot \sim x = \sim x$ .

Let us now consider the set of all elements in  $A$  having the properties from the preceding theorem.

**Theorem 10.** *The set  $B = \{x \in A; x \oplus x = x\}$  is a sublattice of the lattice  $(A, \vee, \wedge)$  containing 0 and 1 such that if  $x \in B$  then also  $\neg x$  and  $\sim x$  belong to  $B$  and both are complements of  $x$  in  $B$ .*

*P r o o f.* Let  $x, y \in B$ . Then by (12'b) and Theorem 3(b),

$$\begin{aligned} (x \vee y) \odot (x \vee y) &= (x \odot x) \vee (x \odot y) \vee (y \odot x) \vee (y \odot y) \\ &= x \vee y \vee (x \odot y) \vee (y \odot x) = x \vee y, \end{aligned}$$

hence  $x \vee y \in B$ . Similarly we get  $x \wedge y \in B$ .

By Theorem 9,  $\neg x, \sim x \in B$ , by (4) and (4'),  $0, 1 \in B$ , and thus by Theorem 9,  $\neg x$  and  $\sim x$  are complements of  $x$  in  $B$ . □

Now let us denote by  $\mathcal{MV}$  the class of all (non-commutative)  $MV$ -algebras. From the definition it is obvious that  $\mathcal{MV}$  is a variety of algebras of type  $(\oplus, \odot, \neg, \sim, 0, 1)$ . Recall that a variety of algebras is called *arithmetical* if it is congruence distributive and permutable. It is well known (e.g. [3], Theorem II.12.5) that by Pixley's theorem a variety  $\mathcal{V}$  is arithmetical if and only if there is a term  $w(x, y, z)$  of given type such that the identities

$$w(y, y, x) = w(x, y, x) = w(x, y, y) = x$$

are satisfied in  $\mathcal{V}$ .

**Theorem 11.** *The variety  $\mathcal{MV}$  of  $MV$ -algebras is arithmetical.*

**Proof.** Set

$$w(x, y, z) = ((\neg y \oplus z) \odot x) \vee ((\neg y \oplus x) \odot z) \vee (x \wedge z).$$

It is obvious that  $w(x, y, z)$  is indeed a term in the language  $(\oplus, \odot, \neg, \sim, 0, 1)$  because the operations  $\vee$  and  $\wedge$  only express abbreviations of the corresponding terms of the type of  $MV$ -algebras.

Let  $A$  be an  $MV$ -algebra and  $x, y, z \in A$ . Then

$$w(y, y, x) = ((\neg y \oplus x) \odot y) \vee ((\neg y \oplus y) \odot x) \vee (y \wedge x),$$

and so (9'), (2) and (4') of the definition of an  $MV$ -algebra yield

$$w(y, y, x) = (x \wedge y) \vee x \vee (x \wedge y) = x.$$

Further,

$$w(x, y, x) = ((\neg y \oplus x) \odot x) \vee ((\neg y \oplus x) \odot x) \vee x,$$

and because  $(\neg y \oplus x) \odot x \leq x$  by Theorem 3, we get

$$w(x, y, x) = x.$$

Finally,

$$\begin{aligned} w(x, y, y) &= ((\neg y \oplus y) \odot x) \vee ((\neg y \oplus x) \odot y) \vee (x \wedge y) \\ &= x \vee (x \wedge y) \vee (x \wedge y) = x. \end{aligned}$$

□

### 3. CONNECTIONS BETWEEN $MV$ -ALGEBRAS AND DUALY RESIDUATED LATTICE ORDERED SEMIGROUPS

**Definition.** An algebra  $A = (A, +, 0, \vee, \wedge, \dashv, \lhd)$  of signature  $\langle 2, 0, 2, 2, 2, 2 \rangle$  is called a *dually residuated (non-commutative) lattice ordered monoid* (a *DRL-monoid*) if:

- (1)  $(A, +, 0)$  is a (non-commutative) monoid;
- (2)  $(A, \vee, \wedge)$  is a lattice;



(3)  $(A, +, 0, \vee, \wedge)$  is a lattice ordered monoid, i.e. for any  $x, y, z, v \in A$ ,

$$\begin{aligned}x + (y \vee z) + v &= (x + y + v) \vee (x + z + v), \\x + (y \wedge z) + v &= (x + y + v) \wedge (x + z + v);\end{aligned}$$

(4) if  $\leq$  denotes the order on  $A$  induced by the lattice  $(A, \vee, \wedge)$  then for any  $x, y \in A$ ,  
 $x \rightarrow y$  is the least element  $s \in A$  such that  $s + y \geq x$ ,  
 $x \leftarrow y$  is the least element  $r \in A$  such that  $y + r \geq x$ ;

(5)  $A$  satisfies the identities

- (a)  $((x \rightarrow y) \vee 0) + y \leq x \vee y, y + ((x \leftarrow y) \vee 0) \leq x \vee y,$
- (b)  $x \rightarrow x \geq 0, x \leftarrow x \geq 0.$

**Remark.** Condition (4) is equivalent to the following identities:

- (4a)  $(x \rightarrow y) + y \geq x, y + (x \leftarrow y) \geq x;$
- (4b)  $(x \rightarrow y) \leq ((x \vee z) \rightarrow y), (x \leftarrow y) \leq ((x \vee z) \leftarrow y);$
- (4c)  $((x + y) \rightarrow y) \leq x, ((y + x) \leftarrow y) \leq x.$

Therefore *DRL*-monoids form a variety of algebras of signature  $\langle 2, 0, 2, 2, 2, 2 \rangle$ .

Commutative *DRL*-monoids have been introduced and studied by K. L. N. Swamy in [15], [16] and [17] as common generalizations of commutative lattice ordered groups and Brouwerian (and so also Boolean) algebras. (For commutative *DRL*-monoids, the operations  $\rightarrow$  and  $\leftarrow$  coincide and the common result  $x \rightarrow y = x \leftarrow y$  is denoted by  $x - y$  for any elements  $x$  and  $y$ . Then commutative *DRL*-monoids are regarded as algebras in the language  $(+, 0, \vee, \wedge, -)$  of signature  $\langle 2, 0, 2, 2, 2 \rangle$ .) Connections between commutative *MV*-algebras and *DRL*-monoids were described by the author in [13] and [14]. The properties of non-commutative *DRL*-monoids were studied by T. Kovář in [10].

Now we will deal with connections between *MV*-algebras and *DRL*-monoids in non-commutative cases.

**Theorem 12.** *Let  $A = (A, +, 0, \vee, \wedge, \rightarrow, \leftarrow)$  be a *DRL*-monoid with a greatest element 1 which satisfies the conditions*

- (i)  $\forall x \in A; 1 \leftarrow (1 \rightarrow x) = x = 1 \rightarrow (1 \leftarrow x);$
- (ii)  $\forall x, y \in A; 1 \rightarrow ((1 \leftarrow x) + (1 \leftarrow y)) = 1 \leftarrow ((1 \rightarrow x) + (1 \rightarrow y)).$

Set

$$\neg x = 1 \rightarrow x, \quad \sim x = 1 \leftarrow x, \quad x \cdot y = \neg(\sim x + \sim y)$$

for any  $x, y \in A$ .

Then  $(A, +, \cdot, \neg, \sim, 0, 1)$  is an *MV*-algebra.

Proof. By [10], Theorem 1.2.3, if a *DRL*-monoid  $A$  is bounded above, then it is bounded below too, and moreover,  $0$  is the least element in  $A$ .

We will verify the axioms of an *MV*-algebra for  $A$ . Since  $(A, +, 0)$  is a monoid, (1) and (4) are satisfied.

(2) By [10], Lemma 1.7, in any *DRL*-monoid we have

$$((x \rightarrow y) \vee 0) + y = x \vee y = y + ((x \leftarrow y) \vee 0).$$

Hence in our case we get for any  $x \in A$

$$x + \sim x = x + (1 \leftarrow x) = 1 + (x \leftarrow 1) \geq 1 + 0 = 1.$$

Analogously,

$$\neg x + x = (1 \rightarrow x) + x = (x \rightarrow 1) + 1 \geq 0 + 1 = 1.$$

$$(3) \quad x + 1 \geq 0 + 1 = 1, \quad 1 + x \geq 1 + 0 = 1.$$

(7) By condition (i) we get

$$\neg \sim x = 1 \rightarrow (1 \leftarrow x) = x, \quad \sim \neg x = 1 \leftarrow (1 \rightarrow x) = x.$$

(5) By (7),

$$\neg x \cdot \neg y = \neg(\sim \neg x + \sim \neg y) = \neg(x + y).$$

(6) By condition (ii) and property (7) we have

$$\sim x \cdot \sim y = \sim(\neg \sim x + \neg \sim y) = \sim(x + y).$$

(8)  $\neg 0 = 1 \rightarrow 0$  is the smallest of the elements  $r \in A$  such that  $0 + r \geq 1$ , hence  $\neg 0 = 1$ . Similarly  $\sim 0 = 1$ .

$$(1') \quad \begin{aligned} x \cdot (y \cdot z) &= x \cdot (\neg(\sim y + \sim z)) \\ &= \neg(\sim x + \sim \neg(\sim y + \sim z)) = \neg(\sim x + (\sim y + \sim z)), \\ (x \cdot y) \cdot z &= \neg(\sim \neg(\sim x + \sim y) + \sim z) = \neg((\sim x + \sim y) + \sim z), \end{aligned}$$

and thus by (1), condition (1') is satisfied.

(2') By (7) and (2) we have

$$x \cdot \sim x = \sim(\neg x + \neg \sim x) = \sim(\neg x + x) = \sim 1 = 0.$$

Similarly  $\neg x \cdot x = 0$ .

(3') By (3),

$$x \cdot 0 = \neg(\sim x + \sim 0) = \neg(\sim x + 1) = \neg 1 = 0,$$

and similarly also  $0 \cdot x = 0$ .

(4')

$$x \cdot 1 = \neg(\sim x + \sim 1) = \neg(\sim x + 0) = \neg \sim x = x.$$

Similarly  $1 \cdot x = x$ .

(5') By condition (ii),

$$\neg(x \cdot y) = \neg \sim(\neg x + \neg y) = \neg x + \neg y.$$

(6')

$$\sim(x \cdot y) = \sim \neg(\sim x + \sim y) = \sim x + \sim y.$$

Put now  $x \preceq_1 y \iff y + (x \cdot \sim y) = y$  for any  $x, y \in A$ . Let  $x \preceq_1 y$ . Then by the definition of the operation  $\leftarrow$  we have  $x \cdot \sim y \geq x \leftarrow y$ , and hence  $y + (x \leftarrow y) = y$ , that means, by [10], Lemma 1.1.7,  $x \vee y = y$ , and therefore  $x \leq y$ .

Conversely, let  $x \leq y$ . Then the equality  $(1 \rightarrow x) + x = 1$  implies  $(1 \rightarrow x) + y = 1$ . Hence

$$\begin{aligned} y + (x \cdot \sim y) &= y + \sim(\neg x + y) = y + (1 \leftarrow ((1 \rightarrow x) + y)) \\ &= y + (1 \leftarrow 1) = y + 0 = y, \end{aligned}$$

that means  $x \preceq_1 y$ .

Therefore the relation  $\preceq_1$  coincides with the relation  $\leq$  of the *DRI*-monoid  $A$ .

Similarly, put  $x \preceq_2 y \iff (\neg y \cdot x) + y = y$  for any  $x, y \in A$ . Analogously we can prove (using the definition of the operation  $\rightarrow$  and [10], Lemma 1.1.7) that the relation  $\preceq_2$  also coincides with  $\leq$ .

Hence  $x \vee y = y + (x \cdot \sim y) = (\neg y \cdot x) + y$  is the lattice join and  $x \wedge y = y \cdot (x + \sim y) = (\neg y + x) \cdot y$  is the lattice meet also in the *MV*-algebra constructed. (So at the same time we have verified the validity of conditions (9) and (9').) Therefore (10), (10'), (11) and (11') are satisfied.

(12) It follows from the fact that  $(A, +, 0, \leq)$  is an *l*-monoid.

(12') First verify that  $\sim(x \vee y) = \sim x \wedge \sim y$  and  $\neg(x \wedge y) = \neg x \vee \neg y$ .

$$\begin{aligned} \sim(x \vee y) &= \sim((\neg y \cdot x) + y) = \sim(\neg y \cdot x) \cdot \sim y \\ &= (y + \sim x) \cdot \sim y = (\neg \sim y + \sim x) \cdot \sim y = \sim x \wedge \sim y, \\ \neg(x \wedge y) &= \neg(y \cdot (x + \sim y)) = \neg y + \neg(x + \sim y) \\ &= \neg y + (\neg x \cdot y) = \neg y + (\neg x \cdot \sim \neg y) = \neg x \vee \neg y. \end{aligned}$$

Now we get

$$\begin{aligned}
 x \cdot (y \vee z) &= \neg(\sim x + \sim(y \vee z)) = \neg(\sim x + (\sim y \wedge \sim z)) \\
 &= \neg((\sim x + \sim y) \wedge (\sim x + \sim z)) = \neg(\sim(x \cdot y) \wedge \sim(x \cdot z)) \\
 &= \neg\sim(x \cdot y) \vee \neg\sim(x \cdot z) = (x \cdot y) \vee (x \cdot z).
 \end{aligned}$$

The equality  $(y \vee z) \cdot x = (y \cdot x) \vee (z \cdot x)$  can be proved analogously, hence (12'b) is proved. Dually (12'a).  $\square$

In the next theorem we will show that also conversely, any *MV*-algebra induces some *DRL*-monoid.

**Theorem 13.** *Let  $A = (A, \oplus, \odot, \neg, \sim, 0, 1)$  be an *MV*-algebra. For any  $x, y \in A$  set*

$$x \rightarrow y = \neg y \odot x, \quad x \leftarrow y = x \odot \sim y.$$

If  $x \vee y$  ( $x \wedge y$ ) denotes the supremum (the infimum) of elements  $x, y \in A$  in the order induced on  $A$ , then  $(A, \oplus, 0, \vee, \wedge, \rightarrow, \leftarrow)$  is a bounded *DRL*-monoid with the greatest element 1 satisfying conditions (i) and (ii) of Theorem 12.

**P r o o f.** By the definition of an *MV*-algebra and by Theorem 2 we get that  $y \oplus (x \odot \sim y) = x \vee y \geq x$ . Let  $v \in A$  be such that  $y \oplus v \geq x$ . Then by Theorem 5,  $x \odot \sim y \leq v$ , and hence  $x \odot \sim y$  is the least of the elements  $v \in A$  with the property  $y \oplus v \geq x$ , thus  $x \odot \sim y = x \leftarrow y$ .

Similarly  $(\neg y \odot x) \oplus y = x \vee y \geq x$ , and if  $w \oplus y \geq x$  for  $w \in A$ , then by Theorem 5 we get  $\neg y \odot x \leq w$ , that means  $\neg y \odot x = x \rightarrow y$ .

We will show the validity of conditions (5) of the definition of a *DRL*-monoid.

(a) Let  $x, y \in A$ . Then (9) of the definition of an *MV*-algebra yields

$$\begin{aligned}
 (x \rightarrow y) \oplus y &= (\neg y \odot x) \oplus y = x \vee y, \\
 y \oplus (x \leftarrow y) &= y \oplus (x \odot \sim y) = x \vee y.
 \end{aligned}$$

(b) For any  $x \in A$  we get by (2) and (2') of the definition of an *MV*-algebra:

$$\begin{aligned}
 x \rightarrow x &= \neg x \odot x = 0, \\
 x \leftarrow x &= x \odot \sim x = 0.
 \end{aligned}$$

Hence  $(A, \oplus, 0, \vee, \wedge, \rightarrow, \leftarrow)$  is a *DRL*-monoid and 1 is its greatest element.

We will verify conditions (i) and (ii).

$$\begin{aligned}
 \text{(i)} \quad 1 \leftarrow (1 \rightarrow x) &= 1 \leftarrow (\neg x \odot 1) = 1 \leftarrow \neg x = 1 \odot \sim \neg x = x, \\
 1 \rightarrow (1 \leftarrow x) &= 1 \rightarrow (1 \odot \sim x) = 1 \rightarrow \sim x = \neg \sim x \odot 1 = x.
 \end{aligned}$$

$$\begin{aligned}
(ii) \quad 1 \rightarrow ((1 \leftarrow x) \oplus (1 \leftarrow y)) &= 1 \rightarrow ((1 \odot \sim x) \oplus (1 \odot \sim y)) \\
&= 1 \rightarrow (\sim x \oplus \sim y) = \neg(\sim x \oplus \sim y) \odot 1 = x \odot y, \\
1 \leftarrow ((1 \rightarrow x) \oplus (1 \rightarrow y)) &= 1 \leftarrow ((\neg x \odot 1) \oplus (\neg y \odot 1)) \\
&= 1 \leftarrow (\neg x \oplus \neg y) = 1 \odot \sim(\neg x \oplus \neg y) = x \odot y.
\end{aligned}$$

□

**Remark.** Let  $A = (A, \oplus, \odot, \neg, \sim, 0, 1)$  be an  $MV$ -algebra and let  $A_1 = (A, \oplus, 0, \vee, \wedge, \rightarrow, \leftarrow)$  be the  $DRL$ -monoid generated by  $A$  by the method of Theorem 13, that means, in  $A_1$ ,  $x \rightarrow y = \neg y \odot x$ ,  $x \leftarrow y = x \odot \sim y$  for any  $x, y \in A$ . Then in the  $MV$ -algebra  $A_2 = (A, \oplus, \cdot, \neg, \approx, 0, 1)$  induced by  $A_1$  by the method of Theorem 12, we have in  $A_2$  for any  $x \in A$ ,

$$\begin{aligned}
\neg x &= 1 \rightarrow x = \neg x \odot 1 = \neg x, \\
\approx x &= 1 \leftarrow x = 1 \cdot \sim x = \sim x,
\end{aligned}$$

and hence by Theorem 1(b) also

$$x \cdot y = x \odot y$$

for any  $x, y \in A$ . Therefore the  $MV$ -algebras  $A$  and  $A_2$  coincide and thus every  $MV$ -algebra is induced by a  $DRL$ -monoid.

Let  $\mathcal{C}_\infty$  and  $\mathcal{C}_\infty$  be classes of algebras of given types. Recall that two classes  $\mathcal{C}_\infty$  and  $\mathcal{C}_\infty$  of algebras are *equivalent* if there exists a one-to-one correspondence  $\mathcal{F}$  between  $\mathcal{C}_\infty$  and  $\mathcal{C}_\infty$  such that for any  $A \in \mathcal{C}_\infty$ ,  $A$  and  $\mathcal{F}(A)$  have the same underlying set, and for each  $A, B \in \mathcal{C}_\infty$  and each mapping  $f$  of  $A$  into  $B$ ,  $f$  is a  $\mathcal{C}_\infty$ -homomorphism of the algebra  $A$  into the algebra  $B$  if and only if  $f$  is a  $\mathcal{C}_\infty$ -homomorphism of  $\mathcal{F}(A)$  into  $\mathcal{F}(B)$ . It is obvious that the algebraic categories corresponding to the equivalent classes  $\mathcal{C}_\infty$  and  $\mathcal{C}_\infty$  are isomorphic.

Let us denote by  $\mathcal{MV}$  the class of  $MV$ -algebras (which is a variety of algebras of signature  $\langle 2, 2, 1, 1, 0, 0 \rangle$ ). For the class of bounded  $DRL$ -monoids we will consider the greatest element 1 as a nullary operation and so we will extend the type of such  $DRL$ -monoids to  $(+, 0, \vee, \wedge, \rightarrow, \leftarrow, 1)$  of signature  $\langle 2, 0, 2, 2, 2, 2, 0 \rangle$ . Now, let us denote by  $\mathcal{DRl}_{1(i)(ii)}$  the class of  $DRL$ -monoids with 1 satisfying conditions (i) and (ii) viewed as a variety of the above type. Then we get the following theorem.

**Theorem 14.** *The classes  $\mathcal{MV}$  and  $\mathcal{DRl}_{1(i)(ii)}$  are equivalent.*

**Proof.** If  $A = (A, \oplus, \odot, \neg, \sim, 0, 1)$  is an  $MV$ -algebra, denote by  $\mathcal{F}(A) = (A, \oplus, 0, \vee, \wedge, \rightarrow, \leftarrow, 1)$  the induced  $DRL$ -monoid. By the preceding remark it is clear

that  $\mathcal{F}$  is a one-to-one correspondence between  $\mathcal{MV}$  and  $\mathcal{DR}l_{1(i)(ii)}$ . Moreover, if  $f: A \rightarrow B$  is an  $MV$ -homomorphism then (since the operations  $\vee, \wedge, \rightarrow$  and  $\leftarrow$  are defined by  $MV$ -terms)  $f$  is also an  $\mathcal{DR}l_1$ -homomorphism, and vice versa.  $\square$

**Corollary 15.** *The categories  $\mathcal{MV}$  and  $\mathcal{DR}l_{1(i)(ii)}$  are isomorphic.*

Let  $\mathcal{CMV}$  denote the class of commutative  $MV$ -algebras and  $\mathcal{CDR}l_{1(i)}$  the class of commutative  $\mathcal{DR}l$ -monoids with the greatest element 1 satisfying condition (i) which is now in the form  $1 - (1 - x) = x$ . (Note that condition (ii) is trivially satisfied for commutative  $\mathcal{DR}l$ -monoids.)

Thus as a consequence we have (see also [14]):

**Theorem 16.** *The classes  $\mathcal{CMV}$  and  $\mathcal{CDR}l_{1(i)}$  are equivalent (and hence the corresponding equational categories are isomorphic).*

Moreover, by [10], Theorem 1.1.23, the lattice  $(A, \vee, \wedge)$  of any  $\mathcal{DR}l$ -monoid  $A$  is distributive. Hence we have:

**Theorem 17.** *If  $A = (A, \oplus, \odot, \neg, \sim, 0, 1)$  is any  $MV$ -algebra, then the lattice  $(A, \vee, \wedge)$  is distributive.*

At the same time we obtain as an immediate consequence of Theorem 10:

**Corollary 18.** *For every  $MV$ -algebra  $A$  the set  $B = \{x \in A; x \oplus x = x\}$  of its additive idempotents is a Boolean algebra in which the complement  $x'$  of arbitrary element  $x \in B$  fulfils  $x' = \neg x = \sim x$ .*

#### 4. INTERVALS OF LATTICE ORDERED GROUPS AND LOOPS

Let  $G$  be a commutative  $l$ -group,  $0 \leq u \in G$  and  $A = [0, u] = \{x \in G; 0 \leq x \leq u\}$ . Set  $x \oplus y = (x + y) \wedge u$ ,  $\neg x = u - x$  and  $x \odot y = \neg(\neg x \oplus \neg y)$  for any  $x, y \in A$ . Then  $\Gamma(G, u) = (A, \oplus, \odot, \neg, 0, u)$  is a commutative  $MV$ -algebra. C. C. Chang in [5] proved that any linearly ordered commutative  $MV$ -algebra is isomorphic to  $\Gamma(G, u)$  for a commutative linearly ordered group  $G$  and a strong order unit  $u$  in  $G$ . D. Mundici in [11] generalized this result to arbitrary commutative  $MV$ -algebras. Namely, he showed that if  $A$  is any commutative  $MV$ -algebra then there are a commutative  $l$ -group  $G$  and a strong unit  $u$  in  $G$  such that  $A$  is isomorphic to  $\Gamma(G, u)$ .

In this section we will examine analogous intervals of non-commutative  $l$ -groups.

**Theorem 19.** Let  $G = (G, +, 0, -(\cdot), \vee, \wedge)$  be a (non-commutative) lattice ordered group,  $0 \leq u \in G$  and  $A = [0, u]$ . For any  $x, y \in A$  set

$$x \oplus y = (x + y) \wedge u, \quad x \rightarrow y = (x - y) \vee 0, \quad x \leftarrow y = (-y + x) \vee 0.$$

Then  $(A, \oplus, 0, \vee, \wedge, \rightarrow, \leftarrow)$  is a bounded *DRL*-monoid (with the greatest element  $u$ ) in which any  $x, y \in A$  satisfy

- (i)  $u \leftarrow (u \rightarrow x) = x = u \rightarrow (u \leftarrow x)$ ;
- (ii)  $u \rightarrow ((u \leftarrow x) \oplus (u \leftarrow y)) = u \leftarrow ((u \rightarrow x) \oplus (u \rightarrow y))$ .

**P r o o f.** We will verify conditions (1)–(5) of a *DRL*-monoid.

Condition (1):

$$x \oplus (y \oplus z) = (x + y + z) \wedge u = (x \oplus y) \oplus z,$$

hence  $(A, \oplus, 0)$  is a monoid.

Condition (2), (3):  $(A, \vee, \wedge)$  is a sublattice of the lattice  $(G, \vee, \wedge)$  and e.g.

$$\begin{aligned} x \oplus (y \wedge z) &= (x + y) \wedge (x + z) \wedge u = ((x + y) \wedge u) \wedge ((x + z) \wedge u) \\ &= (x \oplus y) \wedge (x \oplus z), \end{aligned}$$

hence  $(A, \oplus, 0, \vee, \wedge)$  is an *l*-monoid.

Condition (4):

$$\begin{aligned} y \oplus (x \leftarrow y) &= y \oplus ((-y + x) \vee 0) = (y \oplus (-y + x)) \vee y \\ &= (x \wedge u) \vee y = x \vee y, \end{aligned}$$

thus  $y \oplus (x \leftarrow y) \geq x$ .

Let  $r \in A$ ,  $y \oplus r \geq x$ . Then  $(y + r) \wedge u \geq x$ , hence  $y + r \geq x$ , i.e.  $r \geq -y + x$ . Moreover,  $r \geq 0$ , therefore  $r \geq (-y + x) \vee 0 = x \leftarrow y$ . Thus  $x \leftarrow y$  satisfies condition (4). Similarly for  $x \rightarrow y$ . (Moreover,  $(x \rightarrow y) \oplus y = x \vee y$ , as well.)

Condition (5a): Since  $0$  is the least element in  $A$ , we have

$$\begin{aligned} ((x \rightarrow y) \vee 0) \oplus y &= (x \rightarrow y) \oplus y = x \vee y, \\ y \oplus ((x \leftarrow y) \vee 0) &= y \oplus (x \leftarrow y) = x \vee y. \end{aligned}$$

Condition (5b): Evidently  $x \rightarrow x = 0 = x \leftarrow x$ .

It remains to show the validity of conditions (i) and (ii).

Condition (i):

$$u \leftarrow (u \rightarrow x) = -(u - x) + u = x, u \rightarrow (u \leftarrow x) = u - (-x + u) = x.$$

Condition (ii):

$$\begin{aligned} u \rightarrow ((u \leftarrow x) \oplus (u \leftarrow y)) &= u \rightarrow ((-x + u) \oplus (-y + u)) \\ &= u - ((-x + u - y + u) \wedge u) = (y - u + x) \vee 0, \\ u \leftarrow ((u \rightarrow x) \oplus (u \rightarrow y)) &= u \leftarrow ((u - x) \oplus (u - y)) \\ &= -((u - x + u - y) \wedge u) + u = (y - u + x) \vee 0. \end{aligned}$$

□

As an immediate consequence of Theorems 12 and 19 we obtain

**Theorem 20.** *Let  $G = (G, +, 0, -(\cdot), \vee, \wedge)$  be a (non-commutative)  $l$ -group,  $0 \leq u \in G$  and  $A = [0, u]$ . If we set*

$$\begin{aligned} x \oplus y &= (x + y) \wedge u, & x \odot y &= (y - u + x) \vee 0, \\ \neg x &= u - x, & \sim x &= -x + u \end{aligned}$$

for any  $x, y \in A$  then  $\Gamma(G, u) = (A, \oplus, \odot, \neg, \sim, 0, u)$  is an  $MV$ -algebra.

**Remark.** a) By Remark after the proof of Theorem 12, for arbitrary  $0 \leq u \in G$ , the set  $B_u$  of additive idempotents of  $\Gamma(G, u)$  is a Boolean algebra. Evidently  $x \in B_u$  if and only if  $x \wedge (u - x) = x \wedge (-x + u) = 0$ .

b) If for  $0 < u \in G$  the interval  $[0, u]$  is a chain (i.e.  $u$  is a basic element in the  $l$ -group  $G$ ), then  $B_u = \{0, u\}$ . Hence, if  $G$  is a linearly ordered group, then for each  $0 < u \in G$ ,  $B_u$  is a two-element Boolean algebra.

c) Let an  $l$ -group  $G$  be the direct sum of linearly ordered groups  $G_1, \dots, G_n$  and let  $0 < u = (u_1, \dots, u_n) \in G$ . If  $u_i \neq 0$  for each  $i = 1, \dots, n$ , then  $B_u$  is a finite Boolean algebra having  $2^n$  elements. Hence for any finite Boolean algebra  $B$  there are an  $l$ -group  $G$  and  $0 \leq u \in G$  such that  $B \cong B_u$ .

d) Let  $G$  be an  $l$ -group and  $0 < u \in G$ . Let us suppose that  $u$  is a singular element in  $G$ , i.e. for any  $s, t \in [0, u]$ ,  $s + t = u$  implies  $s \wedge t = 0$ . Since for every  $x \in [0, u]$ ,  $x + (-x + u) = u = (u - x) + x$ , we have  $x \wedge \sim x = 0 = \neg x \wedge x$ , and so  $x \in B_u$ . Hence in this case  $B_u = A = [0, u]$ , therefore  $A$  is a Boolean algebra and  $x' = \neg x = \sim x$  for any  $x \in A$ .

Recall that any commutative  $MV$ -algebra is isomorphic to  $\Gamma(G, u)$  for an appropriate commutative lattice ordered group  $G$  and  $0 \leq u \in G$  (where moreover  $u$  can



be a strong unit of  $G$ ). This fact has been proved for linearly ordered  $MV$ -algebras and linearly ordered groups by C. C. Chang in [5], and it has been generalized to arbitrary  $MV$ -algebras and commutative  $l$ -groups by D. Mundici in [11] using the possibility of expressing any commutative  $MV$ -algebra as a subdirect sum of linearly ordered  $MV$ -algebras.

We will show that for non-commutative  $MV$ -algebras the similar construction does not lead in general to a group, but, nevertheless, we will show that any linearly ordered  $MV$ -algebra can be represented as an interval of a linearly ordered loop.

**Definition.** a) Let  $(G, +, 0, /, \backslash)$  be a loop and let  $\leq$  be an order on  $G$ . Then  $G = (G, +, 0, /, \backslash, \leq)$  is called an *ordered loop* if

$$\forall x, y, z, v \in G; x \leq y \Rightarrow v + x + z \leq v + y + z.$$

b) If  $G$  is an ordered loop and  $(G, \leq)$  is a lattice with the lattice operations  $\vee$  and  $\wedge$ , then  $G = (G, +, 0, /, \backslash, \vee, \wedge)$  is called a *lattice ordered loop* if for any  $x, y, z, v \in G$ ,

$$\begin{aligned} v + (x \vee y) + z &= (v + x + z) \vee (v + y + z), \\ v + (x \wedge y) + z &= (v + x + z) \wedge (v + y + z). \end{aligned}$$

(Recall that for any  $x, y \in G$ ,  $y/x$  ( $y \backslash x$ ) denotes the unique solution  $v$  ( $w$ ) of the equation  $x + v = y$  ( $w + x = y$ , respectively).)

**Lemma 21.** *If  $G$  is an ordered loop and  $x \in G$ , then*

$$0 \leq x \Leftrightarrow 0/x \leq 0 \Leftrightarrow 0 \backslash x \leq 0.$$

**Proof.** If  $0 \leq x$  then  $0/x \leq x + (0/x) = 0$ , and if  $0/x \leq 0$  then  $0 = x + (0/x) \leq x$ . Similarly  $0 \leq x$  if and only if  $0 \backslash x \leq 0$ .  $\square$

**Lemma 22.** *Let  $G$  be a lattice ordered loop,  $0 \leq u \in G$ , and  $A = [0, u]$ . For any  $x, y \in A$  set*

$$\begin{aligned} x \oplus y &= (x + y) \wedge u, \neg x = (0 \backslash x) + u, \sim x = u + (0/x), \\ x \odot y &= \sim(\neg x \oplus \neg y). \end{aligned}$$

*Then  $\oplus$  and  $\odot$  are binary and  $\neg$  and  $\sim$  are unary operations on  $A$ . (Denote  $\Gamma(G, u) = (A, \oplus, \odot, \neg, \sim, 0, u)$ .)*

**Proof.** Let  $x, y \in A$ . Obviously  $x \oplus y \in A$ . By Lemma 21,  $0 \setminus x \leq 0$ , hence  $(0 \setminus x) + u \leq u$ . Further,  $0 = (0 \setminus x) + x \leq (0 \setminus x) + u$ , thus  $(0 \setminus x) + u \in A$ . Analogously  $u + (0/x) \in A$ .

Finally,  $x \odot y = \sim(-x \oplus \neg y) = (0 \setminus ((u + (0 \setminus x)) \oplus (u + (0 \setminus y)))) + u$ , and because  $u + (0 \setminus x), u + (0 \setminus y) \in A$ , we have  $x \odot y \in A$ .  $\square$

**Theorem 23.** *Let  $A$  be a bounded linearly ordered DRI-monoid satisfying (i) and (ii). Then there exist a linearly ordered loop  $G$  and  $0 \leq u \in G$  such that  $\Gamma(G, u)$  is an MV-algebra and  $A$  is isomorphic to  $\Gamma(G, u)$ .*

**Proof.** Let  $A$  be a linearly ordered non-commutative DRI-monoid with a greatest element 1 satisfying conditions (i) and (ii) and let  $\mathbb{Z}$  be the additive group of integers linearly ordered by the natural order. Denote by  $B = \mathbb{Z} \vec{\times} A$  the cartesian product of  $\mathbb{Z}$  and  $A$  ordered by the lexicographic order. (That means,  $(m, x) < (n, y)$  if and only if  $m < n$  or  $m \leq n$  and  $x < y$ .) We will define a binary operation  $\oplus$  on  $B$  as follows: If  $(m, x), (n, y) \in B$  then

$$(m, x) \oplus (n, y) = (m + n, x + y), \quad \text{if } x + y < 1,$$

$$(m, x) \oplus (n, y) = (m + n + 1, 1 \rightarrow ((1 \leftarrow x) + (1 \leftarrow y))), \quad \text{if } x + y = 1.$$

Moreover, for any  $m \in \mathbb{Z}$  put

$$(m, 1) = (m + 1, 0).$$

a) Let us show that for any  $(m, x), (n, y) \in B$  there exists a unique  $(s, w) \in B$  such that  $(m, x) \oplus (s, w) = (n, y)$ .

αα) Suppose that  $x \leq y$ .

ααI) Let  $y \neq 1$ . Then

$$(m, x) \oplus (n - m, y \leftarrow x) = (n, y).$$

Show that  $(s, w) = (n - m, y \leftarrow x)$  is a unique solution of the above equation. Let  $(p, z) \in B$  be such that also  $(m, x) \oplus (p, z) = (n, y)$ .

ααI1) Let  $x + z < 1$ . Then  $m + p = n$ ,  $x + z = y$ , hence  $p = n - m$ ,  $z \geq y \leftarrow x$ , and  $x + (y \leftarrow x) = x + z$ . We have  $y \leftarrow x \leq 1 \leftarrow x$ , and moreover,  $x + z = y < 1$  and  $x + (1 \leftarrow x) = 1$  imply  $z < 1 \leftarrow x$ . Therefore in the induced MV-algebra we get  $x + (y \leftarrow x) = x + z$  and  $z, y \leftarrow x \leq \sim x$ , hence, by Theorem 6,  $z = y \leftarrow x$ .

ααI2) Let  $x + z = 1$ . Then  $m + p + 1 = n$ , thus  $p = n - m - 1$  and  $1 \rightarrow ((1 \leftarrow x) + (1 \leftarrow z)) = y$ . At the same time,  $(1 \leftarrow x) + (1 \leftarrow z) \geq 1 \leftarrow x$ , hence  $1 \rightarrow ((1 \leftarrow x) + (1 \leftarrow z)) \leq 1 \rightarrow (1 \leftarrow x) = x$ , and so  $y \leq x$ . That means  $x = y$ .

Hence by [10], Lemma 1.1.12,  $(1 \rightarrow (1 \leftarrow z)) \rightarrow (1 \leftarrow x) = x$ , and therefore by (i),  $z \rightarrow (1 \leftarrow x) = x$ .

Moreover, from  $x + 1 = 1$  we get  $z \geq 1 \leftarrow x$ , hence by [10], Lemma 1.1.12,  $(z \rightarrow (1 \leftarrow x)) + (1 \leftarrow x) = z$ . Thus  $x + (1 \leftarrow x) = z$ , that means  $z = 1$ .

Therefore  $(p, z) = (n - m - 1, 1) = (n - m, 0)$ . (At the same time, in this case,  $(n - m, y \leftarrow x) = (n - m, x \leftarrow x) = (n - m, 0)$ .)

a $\alpha$ II) Let  $y = 1$ . Then

$$(m, x) \oplus (n - m, 1 \leftarrow x) = (n, 1).$$

Let  $(m, x) \oplus (p, z) = (n, 1)$ . If  $x + z < 1$ , then by the definition of  $\oplus$ ,  $x + z = 1$ , a contradiction. Hence  $x + z = 1$ , i.e.  $z \geq 1 \leftarrow x$ . It is obvious that  $1 \rightarrow ((1 \leftarrow x) + (1 \leftarrow z))$  is equal either to 1 or to 0.

If  $1 \rightarrow ((1 \leftarrow x) + (1 \leftarrow z)) = 1$ , then by Theorem 4 we have  $x = z = 1$ . Then  $(m, 1) \oplus (p, 1) = (n, 1)$ , thus  $m + p + 1 = n$ , i.e.  $p = n - m - 1$ . Hence  $(p, 1) = (n - m - 1, 1) = (n - m, 0) = (n - m, 1 \leftarrow 1)$ .

Let  $1 \rightarrow ((1 \leftarrow x) + (1 \leftarrow z)) = 0$ . Then  $x \odot z = 0$ , hence  $\neg \sim x \odot z = 0$ , and therefore by Theorem 5,  $z \leq \sim x = 1 \leftarrow x$ , i.e.  $z = 1 \leftarrow x$ .

a $\beta$ ) Suppose  $x > y$ .

Let  $(m, x) \oplus (p, z) = (n, y)$ . We have  $x + z > y$ , thus  $x + z = y$  cannot hold. Hence  $x + 1 = 1$ , so  $z \geq 1 \leftarrow x$  and  $1 \rightarrow ((1 \leftarrow x) + (1 \leftarrow z)) = y$ . Therefore  $x \odot z = x \odot ((1 \leftarrow x) + y)$ , and since  $z \geq 1 \leftarrow x$  and  $(1 \leftarrow x) + y \geq 1 \leftarrow x$ , we have, by Theorem 6,  $z = (1 \leftarrow x) + y$ .

b) Analogously, for any  $(m, x), (n, y) \in B$  there is in  $B$  a unique solution  $(r, v)$  of the equation  $(r, v) \oplus (m, x) = (n, y)$ .

Therefore  $B$  is a quasigroup. Obviously,  $(0, 0)$  is a zero element in  $B$ , hence  $B$  is a loop.

Now it is evident that the  $MV$ -algebra corresponding to  $A$  is isomorphic to  $\Gamma(B, (0, 1))$ .  $\square$

**Remark.** If  $A$  is a commutative  $MV$ -algebra, then the loop  $B$  from the proof of Theorem 23 is a (linearly ordered) group. But in general, for a non-commutative case,  $B$  is not a group. Namely, for any  $(m, x) \in B$  we have

$$\begin{aligned} (m, x) \oplus (-m - 1, \sim x) &= (m - m - 1 + 1, x \odot \sim x) = (0, 0), \\ (-m - 1, \neg x) \oplus (m, x) &= (-m - 1 + m + 1, \neg x \odot x) = (0, 0), \end{aligned}$$

that means  $(-m - 1, \sim x)$  is a right and  $(-m - 1, \neg x)$  is a left opposite element of  $(m, x)$  in the loop  $B$ . But in general,  $\sim x$  is not equal to  $\neg x$ , therefore  $(B, \oplus)$  need not be a semigroup, and so  $B$  need not be a group.

**Problem.** The question whether any linearly ordered (non-commutative)  $MV$ -algebra is isomorphic to  $\Gamma(G, u)$  for a linearly ordered group  $G$  and an element  $0 \leq u \in G$  remains open.

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