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HOMOGENEOUS MONOUNARY ALGEBRAS

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Homogeneous algebras have been investigated by Marczewski [4], Ganter, Płonka and Werner [3] and Csákány [1]. Some related problems for graphs were studied by Droste, Giraudet and Macpherson [2].

In this paper we deal with homogeneous monounary algebras. We show that a connected monounary algebra is homogeneous if and only if either (i) (A, f) is a cycle, or (ii) $\operatorname{card} f^{-1}(x) = \operatorname{card} f^{-1}(y)$ for each $x, y \in A$. Further, we prove that for each cardinal $\alpha > 0$ there is, up to isomorphism, a unique connected monounary algebra (B_α, f) possessing no cycle and having the property that $\operatorname{card} f^{-1}(x) = \alpha$ for each $x \in B_\alpha$; we give a constructive description of (B_α, f).

The case of non-connected monounary algebras can be easily reduced to the connected case.

Next, we find necessary and sufficient conditions under which a monounary algebra can be embedded into a homogeneous monounary algebra.

1. Basic theorem

For the following definition cf. [1].

1.1. Definition. An algebra A is said to be homogeneous if for each $x, y \in A$ there is an automorphism φ of A such that $\varphi(x) = y$.

1.2. Lemma. *Let (A, f) be a monounary algebra. If there are $x, y \in A$ such that x belongs to a cycle of (A, f) and y does not belong to any cycle, then (A, f) is not homogeneous.*

P r o o f. It is obvious that if the assumption is valid, then x can be mapped into y by no homomorphism of (A, f), thus (A, f) is not homogeneous. $\square$

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1.3. Corollary. *Let (A, f) be a homogeneous monounary algebra. Then either*

(i) *(A, f) consists of cycles,*

or

(ii) *(A, f) contains no cycle.*

From the definition of homogeneity we immediately obtain

1.4. Lemma. *Assume that (A, f) is a homogeneous monounary algebra fulfilling* (ii) *of* 1.3. *If $x, y \in A$, then* $\operatorname{card} f^{-1}(x) = \operatorname{card} f^{-1}(y) > 0$.

1.5. Lemma. *Let (A, f), (A', f) be connected monounary algebras and let α be a cardinal. Suppose that neither (A, f) nor (A', f) contains any cycle and that* $\operatorname{card} f^{-1}(a) = \alpha = \operatorname{card} f^{-1}(a')$ *for each $a \in A$, $a' \in A'$. Then $\alpha > 0$ and if $x \in A$, $x' \in A'$, then there is an isomorphism φ of (A, f) onto (A', f) with $\varphi(x) = x'$.*

Proof. Let $x \in A$, $x' \in A'$. We have

$$\alpha = \operatorname{card} f^{-1}(f(x)) \geqslant \operatorname{card}\{x\} = 1,$$

hence $\alpha > 0$.

Let $x \in A$, $x' \in A'$. Put

$$P_0 = \{f^k(x)\colon\ k \in \mathbb{N} \cup \{0\}\}$$

and define by induction, for $n \in \mathbb{N} \cup \{0\}$,

$$P_{n+1} = \{z \in A - \bigcup_{l=0}^{n} P_l\colon\ f(z) \in \bigcup_{l=0}^{n} P_l\}.$$

Then

$$A = \bigcup_{n \in \mathbb{N} \cup \{0\}} P_n.$$

Further,

(1) if $k \in \mathbb{N}$, $t = f^k(x)$, then $f^{-1}(t) - \{f^{k-1}(x)\} \subseteq P_1$, $\operatorname{card} f^{-1}(t) - P_0 = \alpha - 1$;
(2) if $t \in P_m$, $m > 0$ or $m = 0$, $t = x$, then $f^{-1}(t) \subseteq P_{m+1}$, $\operatorname{card} f^{-1}(t) = \alpha$.

Let us define a mapping $\varphi\colon A \to A'$ by induction. Let $z \in P_0$, $z = f^k(x)$ for $k \in \mathbb{N} \cup \{0\}$. We set

$$\varphi(z) = f^k(x').$$

Since there are no cyclic elements in (A, f), we have

(3) $\varphi(y_1) \neq \varphi(y_2)$ for each $y_1, y_2 \in P_0$, $y_1 \neq y_2$.

Let $n \in \mathbb{N}$. Suppose that if $m \in \mathbb{N} \cup \{0\}$, $m < n$, $y \in P_m$, then $\varphi(y)$ is defined and

(4) if $y_1, y_2 \in \bigcup\limits_{m=0}^{n-1} P_m$, $y_1 \neq y_2$, then $\varphi(y_1) \neq \varphi(y_2)$.

Let $z \in P_n$. Then $t = f(z) \in P_{n-1}$ and $t' = \varphi(t) \in A'$ is defined. If $t = f^k(x)$, $k \in \mathbb{N}$, then we use (1) and (3) and if either $n > 1$ or $n = 1$, $t = x$, then we use (2) and (4); we obtain that there is a bijective mapping ψ of the set $\varphi^{-1}(t) - P_0 = f^{-1}(t) - \bigcup\limits_{m=0}^{n-1} P_m$ onto the set $f^{-1}(t') - \varphi(P_0) = f^{-1}(t) - \bigcup\limits_{m=0}^{n-1} \varphi(P_m)$. We denote

$$\varphi(z) = \psi(z).$$

Let $y_1, y_2 \in \bigcup\limits_{m=0}^{n} P_m$ with $\varphi(y_1) = \varphi(y_2)$, $y_1 \neq y_2$. By the induction hypothesis $\{y_1, y_2\} \nsubseteq \bigcup\limits_{m=0}^{n-1} P_m$. If $\{y_1, y_2\} \subseteq P_n$, then $f(y_1) = f(y_2)$, $\psi(y_1) = \varphi(y_1) = \varphi(y_2) = \psi(y_2)$, thus $y_1 = y_2$, a contradiction. Obviously, if $y_1 \in \bigcup\limits_{m=0}^{n-1}$, $y_2 \in P_n$, then $\varphi(y_1) \neq \varphi(y_2)$.

It can be easily verified that the mapping φ is a homomorphism.

To complete the proof let us show that φ is surjective. Let $v \in A'$. There is the smallest $l \in \mathbb{N} \cup \{0\}$ with

$$f^l(v) \in \{f^k(x') \colon k \in \mathbb{N} \cup \{0\}\}.$$

If $l = 0$, then

$$\varphi(f^k(x)) = f^k(x') = v, \text{ i.e., } \varphi^{-1}(v) = \{f^k(x)\}.$$

Assume that $l > 0$ and that $\varphi^{-1}(f(v)) = \{u\}$, $u \in A$. Since $\varphi(u)$ is defined, the above defintion of φ implies that $\varphi(f^{-1}(u)) = f^{-1}(\varphi(u)) = f^{-1}(f(v))$, i.e.,

$$v \in \varphi(f^{-1}(u)).$$

Hence we have proved that φ is an isomorphism of (A, f) onto (A', f). $\square$

1.6. Lemma. *Let (A, f) be a monounary algebra and let α be a cardinal. Suppose that* (ii) *of* 1.3 *is valid and that* $\operatorname{card} f^{-1}(x) = \alpha$ *for each $x \in A$. Then $\alpha > 0$ and (A, f) is homogeneous.*

Proof. Let $x, x' \in A$. Denote by B and B' the connected components of A containing x or x', respectively. According to 1.5 for (B, f), (B', f) instead of (A, f), (A', f) we obtain that $\alpha > 0$ and that there is an isomorphism φ of (B, f) onto

(B', f) with $\varphi(x) = x'$. Let us define a mapping $\zeta\colon A \to A'$ as follows. If $B = B'$, then we put

$$\zeta(a) = \begin{cases} \varphi(a) & \text{if } a \in B, \\ a & \text{if } a \in A - B. \end{cases}$$

If $B \cap B' = \emptyset$, then we set

$$\zeta(a) = \begin{cases} \varphi(a) & \text{if } a \in B, \\ \varphi^{-1}(a) & \text{if } a \in B', \\ a & \text{otherwise.} \end{cases}$$

Then 1.5 yields that ζ is an automorphism of (A, f) such that $\zeta(x) = x'$, therefore (A, f) is homogeneous. $\square$

1.7. Lemma. *If (A, f) is a homogeneous monounary algebra fulfilling* (i) *of* 1.3, *then (A, f) consists of cycles with the same cardinality.*

P r o o f. Let (A, f) be a homogeneous monounary algebra fulfilling (i) of 1.3 and let $x, y \in A$. There is an automorphism φ of (A, f) with $\varphi(x) = y$. If x belongs to a cycle with n elements and y to a cycle with m elements, then the existence of φ yields that n divides m. Symmetrically we obtain that m divides n, thus $m = n$ and all cycles of (A, f) have the same cardinality. $\square$

1.8. Theorem. *Let (A, f) be a monounary algebra. Then (A, f) is homogeneous if and only if either*

(i) *(A, f) consists of cycles with the same cardinality,*

or

(ii) *(A, f) contains no cycle and there is a cardinal $\alpha > 0$ such that* $\operatorname{card} f^{-1}(x) = \alpha$ *for each $x \in A$.*

P r o o f. Let (A, f) be homogeneous. Then 1.3, 1.7 and 1.4 imply that either (i) or (ii) is valid.

If (i) is valid, then it is obvious that (A, f) is homogeneous. If (ii) holds, then (A, f) is homogeneous in view of 1.6. $\square$

1.9. Corollary. *Let (A, f) be a finite connected monounary algebra. Then (A, f) is homogeneous if and only if (A, f) is a cycle.*

2. Construction

Let $\mathscr{H}$ be the class of all homogeneous monounary algebras. In view of 1.8 we have

$$\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2, \qquad \mathscr{H}_1 \cap \mathscr{H}_2 = \emptyset,$$

where

$\mathscr{H}_1$ is the class of all monounary algebras satisfying (i) of 1.8,

$\mathscr{H}_2$ is the class of all monounary algebras satisfying (ii) of 1.8.

The class $\mathscr{H}_1$ is transparent. For a cardinal $\alpha > 0$ we denote by $\mathscr{H}_2^\alpha$ the class of all $(A, f) \in \mathscr{H}_2$ such that $\operatorname{card} f^{-1}(x) = \alpha$ for each $x \in A$.

In this section we show that for each $\alpha > 0$ we have (i) $\mathscr{H}_2^\alpha \neq \emptyset$ and (ii) whenever (A_1, f), (A_2, f) are connected monounary algebras of the class $\mathscr{H}_2^\alpha$, then (A_1, f), (A_2, f) are isomorphic.

For a cardinal α, let $I(\alpha)$ be a set of indices such that $\operatorname{card} I(\alpha) = \alpha$ and $0 \in I(\alpha)$.

2.1. Notation. Let P be the set of all finite words of the form $p_1 p_2 \dots p_n$, where $n \in \mathbb{N}$, $p_1 = 0$ and if $i \in \{1, \dots, n\}$, then $p_i \in I(\alpha)$. For $p_1 p_2 \dots p_n \in P - \{0\}$ we put

$$g(p_1 p_2 \dots p_n) = p_1 p_2 \dots p_{n-1}.$$

Thus g is a partial unary operation on P.

2.2. Notation. Let α be a cardinal, $\alpha > 0$. Denote

$$\begin{aligned} B_\alpha = \mathbb{N} \cup \{(i, p)\colon\ i \in I(\alpha), p \in P\} \cup \\ \cup \{(n, i, p)\colon\ n \in \mathbb{N} - \{1\}, i \in I(\alpha) - \{0\}, p \in P\}. \end{aligned}$$

Let us define a unary operation f on B_α as follows. We set $f(n) = n + 1$ for each $n \in \mathbb{N}$. Let $i \in I(\alpha)$, $p \in P$. Then

$$f((i, p)) = \begin{cases} (i, g(p)) & \text{if } p \neq 0, \\ 1 & \text{if } p = 0. \end{cases}$$

For $n \in \mathbb{N} - \{1\}$, $i \in I(\alpha) - \{0\}$, $p \in P$ we put

$$f((n, i, p)) = \begin{cases} (n, i, g(p)) & \text{if } p \neq 0, \\ n & \text{if } p = 0. \end{cases}$$

2.3. Lemma. *Let α be a cardinal, $\alpha > 0$. Then (B_α, f) is a connected monounary algebra satisfying* (ii) *of* 1.8.

P r o o f. The connectivity of (B_α, f) follows immediately from 2.2. It is also obvious that (B_α, f) contains no cyclic elements. For $x \in B_\alpha$ denote $k(x) = \operatorname{card} f^{-1}(x)$.

We have

$$f^{-1}(1) = \{(i, 0)\colon\ i \in I(\alpha)\},$$

thus

(1) $k(1) = \alpha$.

Further, if $n \in \mathbb{N} - \{1\}$, then

$$f^{-1}(n) = \{n-1\} \cup \{(n, i, 0)\colon\ i \in I(\alpha) - \{0\}\},$$

hence

(2) $k(n) = \alpha$ for each $n \in \mathbb{N}$.

Let $x = (i, p)$, $i \in I(\alpha)$, $p \in P$. Then $p = p_1 p_2 \dots p_n$ and

$$f^{-1}(x) = f^{-1}((i, p)) = \{(i, p_1 p_2 \dots p_n p_{n+1})\colon\ p_{n+1} \in I(\alpha)\},$$

thus

(3) $k(x) = \alpha$.

Finally, let $y = (n, i, p)$, $n \in \mathbb{N}$, $i \in I(\alpha)$, $p = p_1 p_2 \dots p_n$. Analogously as above,

$$f^{-1}(y) = f^{-1}((n, i, p)) = \{(n, i, p_1 p_2 \dots p_{n+1})\colon\ p_{n+1} \in I(\alpha)\},$$

which implies

(4) $k(y) = \alpha$.

Therefore (B_α, f) fulfils (ii) of 1.8. $\square$

2.4.1. Notation. Let γ be a cardinal, $\gamma > 0$ and let (A, f) be a monounary algebra. We denote by $\gamma \cdot (A, f)$ a monounary algebra (B, f) such that

$$B = \{(\lambda, a)\colon\ \lambda \in I(\gamma), a \in A\},$$
$$f((\lambda, a)) = (\lambda, f(a)) \text{ for each } \lambda \in I(\gamma), a \in A;$$

i.e., $\gamma \cdot (A, f)$ consists of γ copies of (A, f).

2.4.2. Notation. For $\alpha \in \mathbb{N}$ let $(\mathbb{Z}_\alpha, f)$ be a monounary algebra such that $\mathbb{Z}_\alpha = \{0, 1, \dots, \alpha - 1\}$, $f(i) \equiv i + 1 \pmod{\alpha}$ for each $i \in \mathbb{Z}_\alpha$.

2.5. Theorem. *Let (A, f) be a monounary algebra. Then (A, f) is homogeneous if and only if there are cardinals $\alpha > 0$, $\gamma > 0$ such that either*

(i) $\alpha \in \mathbb{N}$ *and* $(A, f) \cong \gamma \cdot (\mathbb{Z}_\alpha, f)$,

or

(ii) $(A, f) \cong \gamma \cdot (B_\alpha, f)$.

Proof. Let $(A, f) \in \mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2$. If $(A, f) \in \mathscr{H}_1$, then 1.8(i) implies that (i) is valid. Suppose that $(A, f) \in \mathscr{H}_2$. Let γ be the number of connected components of (A, f) and let

$$\{(A_\lambda, f)\}_{\lambda \in I(\gamma)}$$

be the system of all connected components of (A, f). The definition of $\gamma \cdot (B_\alpha, f)$ yields that if

$$B_{\alpha\lambda} = \{(\lambda, b)\colon\ b \in B_\alpha\},$$

then

$$\{(B_{\alpha\lambda}, f)\}_{\lambda \in I(\gamma)}$$

is the system of connected components of $\gamma \cdot (B_\alpha, f)$.

Let $\lambda \in I(\gamma)$. We can apply 1.5 to the monounary algebras (A_λ, f) and $(B_{\alpha\lambda}, f)$. Thus there is an isomorphism φ_λ of (A_λ, f) onto $(B_{\alpha\lambda}, f)$. Put

$$\varphi(x) = \varphi_\lambda(x) \text{ for each } x \in A_\alpha.$$

Then φ is an isomorphism of (A, f) onto $\gamma \cdot (B_\alpha, f)$.

Conversely, let either (i) or (ii) be valid. If (i) holds, then $(A, f) \in \mathscr{H}_1$. Let (ii) hold. Then 2.3 and 1.8 imply that $(A, f) \in \mathscr{H}_2$. □

3. Embedding into a homogeneous algebra

By an embedding of algebras we understand an injective homomorphism. In this section we will describe all monounary algebras (A, f) such that there are a homogeneous algebra (B, f) and an embedding φ of (A, f) into (B, f); in this case we will say that (A, f) is embeddable into (B, f) or that (A, f) is embeddable into a homogeneous algebra.

Let $(\mathbb{Z}_\alpha, f)$ for $\alpha \in \mathbb{N}$ and (B_α, f) for a cardinal $\alpha > 0$ be as in the previous section.

3.1. Lemma. *Let (A, f) be a monounary algebra which is embeddable into a homogeneous algebra. If (A, f) contains a cyclic element, then (A, f) is homogeneous and then $(A, f) \cong \gamma \cdot (\mathbb{Z}_\alpha, f)$ for some cardinal $\gamma > 0$ and $\alpha \in \mathbb{N}$.*

P r o o f. Suppose that there is a cycle C of (A, f). Denote $\alpha = \operatorname{card} C$. By assumption, there are a homogeneous algebra (B, f) and an injective homomorphism φ of (A, f) into (B, f). Then $\varphi(C)$ is an α-element cycle of (B, f). We apply 2.5 to the algebra (B, f); thus

$$(B, f) \cong \beta \cdot (\mathbb{Z}_\alpha, f)$$

for some cardinal $\beta > 0$. Hence there is a cardinal $0 < \gamma \leqslant \beta$ such that $(A, f) \cong \gamma \cdot (\mathbb{Z}_\alpha, f)$. The algebra (A, f) is homogeneous in view of 2.5. □

3.2. Notation. Let (A, f) be a monounary algebra. We denote by $\gamma_0(A, f)$ the number of connected components of (A, f). If (A, f) contains no cyclic elements, then we put

$$\alpha_0(A, f) = \sup\{\operatorname{card} f^{-1}(x)\colon\ x \in A\}.$$

If (A, f) contains cyclic elements, then let $\alpha_0(A, f)$ be the minimum of cardinalities of the cycles of (A, f).

3.3. Lemma. *Let (A, f) be a monounary algebra containing no cyclic element and let α, γ be cardinals. If $0 < \alpha < \alpha_0(A, f)$ or $0 < \gamma < \gamma_0(A, f)$, then (A, f) is not embeddable into $\gamma \cdot (B_\alpha, f)$.*

P r o o f. Suppose that (A, f) is embeddable into $\gamma \cdot (B_\alpha, f)$ and that φ is the corresponding embedding. Then the number of connected components of (A, f) is not greater than the number of connected componets of $\gamma \cdot (B_\alpha, f)$, i.e.,

$$\gamma_0(A, f) \leqslant \gamma.$$

Assume that $0 < \alpha < \alpha_0(A, f)$. Then there is $x \in A$ with

$$\operatorname{card} f^{-1}(x) > \alpha.$$

Then

$$\alpha = \operatorname{card} f^{-1}(\varphi(x)) \geqslant \operatorname{card} f^{-1}(x) > \alpha,$$

which is a contradiction. □

3.4. Lemma. *Let (A, f) be a connected monounary algebra containing no cyclic element. If $\alpha \geqslant \alpha_0(A, f)$, then (A, f) is embeddable into (B_α, f).*

P r o o f. The required embedding can be defined by induction analogously as the bijection φ in 1.5. □

3.5. Corollary. *Let (A, f) be a monounary algebra containing no cyclic element. If $\alpha \geqslant \alpha_0(A, f)$ and $\gamma \geqslant \gamma_0(A, f)$, then (A, f) is embeddable into $\gamma \cdot (B_\alpha, f)$.*

P r o o f. Let $\alpha \geqslant \alpha_0(A, f)$, $\gamma \geqslant \gamma_0(A, f)$. Let $\{A_i\}_{i \in I(\gamma_0)}$ be the system of connected components of (A, f). By 3.4, (A_i, f) (for each $i \in I(\gamma_0)$) is embeddable into (B_α, f) and the relation $\gamma \geqslant \gamma_0(A, f)$ implies that then (A, f) is embeddable into $\gamma \cdot (B_\alpha, f)$. □

3.6. Proposition. *Let (A, f) be embeddable into a homogeneous algebra. Put $\alpha_0 = \alpha_0(A, f)$, $\gamma_0 = \gamma_0(A, f)$. Then either*

(i) a) *(A, f) is embeddable into $\gamma \cdot (B_\alpha, f)$ for all cardinals $\gamma \geqslant \gamma_0$, $\alpha \geqslant \alpha_0$,*

b) *if γ, α are cardinals, $0 < \gamma < \gamma_0$ or $0 < \alpha < \alpha_0$, then (A, f) is not embeddable into $\gamma \cdot (B_\alpha, f)$,*

or

(ii) a) *$\alpha_0 \in \mathbb{N}$, (A, f) is embeddable into $\gamma \cdot (\mathbb{Z}_{\alpha_0}, f)$ for each cardinal $\gamma \geqslant \gamma_0$,*

b) *if γ is a cardinal, $0 < \gamma < \gamma_0$, then (A, f) is not embeddable into $\gamma \cdot (\mathbb{Z}_{\alpha_0}, f)$.*

P r o o f. First, let (A, f) contain no cyclic element. Then 3.3 and 3.5 imply that (i) is valid. Now assume that (A, f) contains a cyclic element. By 3.1, (ii) holds. □

3.7. Theorem. *A monounary algebra (A, f) is embeddable into a homogeneous algebra if and only if either*

(i) *(A, f) is homogeneous, $(A, f) \cong \gamma \cdot (\mathbb{Z}_\alpha, f)$ for some cardinal $\gamma > 0$ and $\alpha \in \mathbb{N}$,*

or

(ii) *(A, f) contains no cyclic element.*

P r o o f. It is a consequence of 3.1 and 3.5. □

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