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Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 2, 319–327

Persistent URL: http://dml.cz/dmlcz/127719

Czechoslovak Mathematical Journal, 52 (127) (2002), 319–327

A CLASS OF TORSION-FREE ABELIAN GROUPS CHARACTERIZED BY THE RANKS OF THEIR SOCLES

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(Received March 17, 1999)

Abstract. Butler groups formed by factoring a completely decomposable group by a rank one group have been studied extensively. We call such groups, bracket groups. We study bracket modules over integral domains. In particular, we are interested in when any bracket R-module is R tensor a bracket group.

Keywords: Dedekind domain

MSC 2000: 13A15, 13F05, 13B22

1. Introduction

In 1983, F. Richman studied a certain class of Butler groups to great advantage [12]. His work was later generalized by Hill and Megibben [8] and separately, by Arnold and Vinsonhaler (see [2] for a summary and a list of related papers). The class of groups *dual* the class studied in [2] was examined by Wu Yen Lee in [9], and will be studied more generally in our next section. A matrix-oriented approach to the study of the groups of Lee was proffered in [4], and later, in the same vein, in [6].

We will adopt the colloquial terminology of calling the groups from [9], *bracket* groups.

Specifically, given subgroups $C_1, \ldots, C_n$ of the group $\mathbb{Q}$ of rational numbers, a bracket group is $C_1 \oplus \ldots \oplus C_n / C_0$, where C_0 is any rank 1 pure subgroup of $C_1 \oplus \ldots \oplus C_n$ generated by some $(c_1, \ldots, c_n)$ with each $c_i \neq 0$. As mentioned above, these groups, and their duals, have been studied extensively over the past few years. Of special interest here is the result from [7] that two backet groups G and H are quasi-isomorphic if and only if rank $G(\tau)$ = rank $H(\tau)$ for all types τ, where $G(\tau)$ represents the τ-*socle* of G, and is defined to be $G(\tau) = \{x \in G \mid \text{type } x \geqslant \tau\}$.

In this article we discuss a class of torsion-free abelian groups of finite rank which are distinguished (up to quasi-isomorphism) by the ranks of their socles. Specifically, we study the class of groups of the form $R \otimes G$ where G is a bracket group and R is a certain subring of an algebraic number field. Below, the unadorned *Hom* and $\otimes$ symbols are meant to be with respect to the group $\mathbb{Z}$ of integers.

2. A class of Butler modules

Throughout this text, R denotes a subring of an algebraic number field, and $Q = \mathbb{Q}R$ represents the quotient field of R. The integral closure of R in its Q is denoted by $\overline{R}$, and is a Dedekind overring of R with $\overline{R}/R$ finite [1].

Definitions.

- An R-module A is called *completely decomposable*, if $A = A_1 \oplus \ldots \oplus A_n$ where $A_1, \ldots, A_n$ are rank one R-modules.
- A torsion-free R-module M of finite rank is called a *bracket R-module*, if there is a completely decomposable module $A = A_1 \oplus \ldots \oplus A_n$, such that $M \cong A/A_0$, where A_0 is some rank one pure submodule of A whose projection onto each coordinate is nonzero.

In the definition, we are distinguishing between the quotient of a completely decomposable R-module $A = A_1 \oplus \ldots \oplus A_n$ modulo *any* rank one pure submodule, and the quotient A/A_0 where A_0 is a pure submodule of A generated by some $(a_1, \ldots, a_n)$ with $a_j \neq 0$ for all j. The latter module is a bracket module, and the former is the direct sum of a completely decomposable module and a bracket module.

Definition. By $\pi(R)$ we mean the set of integral primes p such that $pR \neq R$. We will say that R is a $\pi(R)$-primary ring, if for each $p \in \pi(R)$, pR is a primary ideal of R. We will call R an integrally primary ring, *IP ring* for short, if for each $p \in \pi(R)$, $p\overline{R}$ is a primary ideal of $\overline{R}$; equivalently, R is an IP ring if $\overline{R}$ is a $\pi(\overline{R})$-primary ring.

The following conditions are equivalent for a subring S of an algebraic number field: pS is primary; pS contains a power of a prime ideal; and there is a unique prime ideal of S containing p. It follows that R is an IP ring if and only if R is a $\pi(R)$-primary ring, and for each non zero prime ideal P of R, there is a unique prime ideal of $\overline{R}$ lying over P.

The following appears in [3] in essence, however there is an error in the statement of Proposition 2.2 in [3], which is corrected here.

Lemma 1. *Let R be a subring of an algebraic number field. Then, every rank one module is of the form $I \otimes X$ for some ideal I of R and some subgroup X of $\mathbb{Q}$, if and only if R is an IP ring.*

P r o o f. We will first show that R is an IP ring under the stated condition. If there are two nonzero prime ideals P_1 and P_2 of $\overline{R}$ lying over a single integral prime $p \in \pi(R)$, then consider the localizations $\overline{R}_j = \overline{R}_{P_j}$ of $\overline{R}$ at the prime ideals P_j, $j = 1, 2$. Since each of $\overline{R}_1$ and $\overline{R}_2$ is p-local as an abelian group, and since any two ideals of R are quasi-equal, from the hypothesis we must conclude that both $\overline{R}_1, \overline{R}_2$ are quasi-equal to $R \otimes \mathbb{Z}_p \cong R\mathbb{Z}_p \subseteq Q$. Therefore $\overline{R}_1 \cdot \overline{R}_2$ is quasi-equal to $\overline{R}_1$. But this contradicts the fact that $\overline{R}_P \cdot \overline{R}_{P'} = Q$ for any two distinct nonzero prime ideals of $\overline{R}$ since $\overline{R}$ is Dedekind. Thus, R is an IP ring.

Given a rank one R-module A, set $\pi = \{p \mid pA \neq A\}$. We will use the notation $R_\pi = \bigcap_{p\in\pi} R_p$ and $\mathbb{Z}_\pi = \bigcap_{p\in\pi} \mathbb{Z}_p$. If $A \cong J \otimes X$ for some rank one subgroup X of $\mathbb{Q}$ and some ideal J of $S = R_\pi$, then $J \cong I \otimes S$ for some ideal I of R, and $A \cong I \otimes X'$ for $X' = \mathbb{Z}_\pi \otimes X$. Since S is an IP ring, we may assume, without loss of generality, that $\pi = \pi(R)$.

Let X and Y be rank one pure subgroups of the rank one R-module A. With $0 \neq a \in X$ and $0 \neq b \in Y$, $q = b/a \in Q = \mathbb{Q}R$, so there is an integer $k \neq 0$ for which $kq \in R$. Then, left multiplication on A by kq is a group monomorphism sending X into Y. From this we conclude that type X = type Y, and A is homogeneous as a group. As Warfield has shown in [14], the natural map $\mathrm{Hom}(X, A) \otimes X \to A$ is an isomorphism. It remains to show that $I = \mathrm{Hom}(X, A)$ is isomorphic to an ideal of R. Identify the rank one R-module I with $\{t \in Q \mid tX \subseteq A\}$.

Consider $\bar{I} = \overline{R}I \subseteq Q$. Once we have shown that $\bar{I}$ is a fractional ideal of $\overline{R}$, we are assured of finding an integer $m \neq 0$ such that $m\bar{I} \subseteq \overline{R}$. But since $\overline{R}/R$ is finite, $m'\overline{R} \subseteq R$ for some integer $m' \neq 0$, and then $mm'I \subseteq mm'\bar{I} \subseteq R$, so I is isomorphic to an ideal of R.

The rank one $\overline{R}$-module $\bar{I}$ is homogeneous as a group, of type equal to the type of $\mathbb{Z}_\pi$, where $\pi = \pi(R)$. It is easy to see that $\bar{I}$ is a fractional ideal of $\overline{R}$ if and only if $\bar{I}_P = \overline{R}_P$ for almost all maximal ideals P of $\overline{R}$, and for any maximal ideal P of $\overline{R}$, $\bar{I}_P$ is a fractional ideal of $\overline{R}_P$. Let P be a maximal ideal of $\overline{R}$.

Since $\overline{R}_P$ is a *dvr*, $\bar{I}_P$ is either a fractional ideal of $\overline{R}_P$, or it is Q. If $P\bar{I}_P = \bar{I}_P$, then because R is an IP ring, we know that $p\overline{R} = P^n$ for some integer n and some integral prime p. But then $p\bar{I}_P = \bar{I}_P$ implying that I is p-divisible. But $p \in \pi = \pi(R)$ and so $pA \neq A$. Thus, $\bar{I}_P$ is a fractional ideal of $\overline{R}_P$ for all P.

Suppose $\bar{I}_P \neq \overline{R}_P$ for infinitely many primes P. The theory of heights for $\mathbb{Z}$ carries over to modules over $\overline{R}$ in the following manner. Define the height sequence $h^{\bar{I}}(x)\colon \mathrm{spec}(\overline{R}) \to \mathbb{N}up\{\infty\}$, for $x \in \bar{I}$, by $h^{\bar{I}}_P(x) = n$ provided $x \in P^n\bar{I} \setminus P^{n+1}\bar{I}$, and $h^{\bar{I}}_P(x) = \infty$ otherwise. Then any two height sequences of elements in the rank one module $\bar{I}$ have equivalent height sequences (in that, $\sum_P |h^{\bar{I}}_P(x) - h^{\bar{I}}_P(x')| < \infty$), and $\bar{I}$ is a fractional ideal of R precisely when $\sum_P h^{\bar{I}}_P(x) < \infty$ for each $0 \neq x \in \bar{I}$. We have seen that $h^{\bar{I}}_P(x) < \infty$ for each P, so we must now show that $h^{\bar{I}}_P(x) = 0$ for almost all prime ideals P.

Suppose $0 \neq x \in \bar{I}$ is such that $h^{\bar{I}}_P(x) > 0$ for infinitely many prime ideals P. It is well known that $\overline{R} = \bigcap_{P \in \mathcal{S}} T_P$ where T is the ring of integers in the algebraic number field $\mathbb{Q}R$ and $\mathcal{S}$ is a certain subset of spec(T). Recall that almost all integral primes p are unramified in T, meaning that pT is a product of distinct prime ideals. The condition that R is an IP ring can be interpreted as saying that $\mathcal{S}$ cannot contain two prime ideals of T lying over the same integral prime.

For almost all integral primes $p \in \pi(R)$, $p\overline{R}$ is a prime ideal. Since every prime ideal of $\overline{R}$ lies over some integral prime, we conclude that almost all prime ideals of $\overline{R}$ are of the form $p\overline{R}$ for some $p \in \pi(R)$. Therefore, $x \in p\bar{I}$ for infinitely many integral primes, contradicting the fact that, as a group, $\bar{I}$ is homogeneous of type equal to the type of $\mathbb{Z}_\pi$. This contradiction shows that $\bar{I}$ is a fractional ideal of $\overline{R}$, and the proof is complete. □

Recall that a subring R of an algebraic number field, is called *strongly homogeneous*, if every element of R is an integral multiple of a unit of R. Arnold's development in [1] advances the following equivalence: R is strongly homogeneous; pR is a maximal ideal of R for every $p \in \pi(R)$; and, every ideal of R is generated by an integer. So strongly homogeneous rings are pid's that are IP rings.

Lemma 2. *Let R be a subring of an algebraic number field that is an IP ring. Then, for any module $R \subseteq A \subseteq Q$, there is a subgroup C of $\mathbb{Q}$ and fractional ideal J of R, containing R, such that $A = JC$. If in addition, R is a pid, then there is an element $r \in R$ such that $A = r^{-1}RC$.*

Proof. From the proof of Lemma 1, $A \cong_{\text{nat}} \text{Hom}(C, A) \otimes C$ for any rank one pure subgroup C of A. In particular, we shall take C equal to the $\mathbb{Z}$-purification of 1 in A, and identify $\text{Hom}(C, A)$ with the submodule $J = \{t \in Q \mid tC \subseteq A\}$. As verified in the proof of Lemma 1, J is a fractional ideal of R which clearly contains R. Evaluating the natural isomorphism results in $A = JC$. When R is also a pid, then $J = tR$ for some $t \in Q$, but since $1 \in J$, $t^{-1} \in R$. Thus, $A = tRC$ as desired. □

Theorem 3. *Let R be a strongly homogeneous ring and M a torsion-free R-module of finite rank. The following are equivalent:*

(a) *M is a bracket R-module.*

(b) *$M \cong R \otimes G$ for some bracket group G.*

P r o o f. (b) $\to$ (a) holds generally, and is clear. For the reverse implication, in accordance with the definition of M as a bracket R-module, let

$$0 \longrightarrow A_0 \longrightarrow A_1 \oplus \ldots \oplus A_n \xrightarrow{\varphi} M \longrightarrow 0$$

be an exact sequence of R-modules with $A_0, \ldots, A_n$ having rank one. Let

$$(a_1, \ldots, a_n) \in \operatorname{Ker} \varphi,$$

and recall that each $a_j \neq 0$. Multiplication by a_i sends $a_i^{-1} A_i$ isomorphically onto A_i and $1 \mapsto a_i$. Therefore, replacing A_i with $a_i^{-1} A_i$, we may assume that each A_i contains 1 and that $(1, \ldots, 1) \in \operatorname{Ker} \varphi \equiv A_0$.

By virtue of Lemma 2, there are rational subgroups $C_i \subseteq \mathbb{Q}$, and elements $r_i \in R$ such that $A_i = r_i^{-1} R C_i$. Since r_i is an integral multiple of a unit of R, there are integers n_i for $i = 1, \ldots, n$ such that $A_i = n_i^{-1} R C_i$. Set $D = D_1 \oplus \ldots \oplus D_n$ where $D_i = n_i^{-1} C_i$ for each i. We claim that $G = \varphi(D)$ is a bracket group and that $M \cong R \otimes G$. Note that $(1, \ldots, 1) \in K = \operatorname{Ker} \varphi \cap D$.

Consider the commutative rectangle

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R \otimes K & \longrightarrow & R \otimes D & \longrightarrow & R \oplus G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A_0 & \longrightarrow & A_1 \oplus \ldots \oplus A_n & \xrightarrow{\varphi} & M & \longrightarrow & 0. \end{array}$$

As $K \neq 0$, the image of α is nonzero. The module A_0 is rank one, implying that α has torsion cokernel (as an abelian group). Since β is an isomorphism, the Snake Lemma implies that $\operatorname{Ker} \gamma \cong \operatorname{Coker} \alpha$, and because $R \otimes G$ is torsion-free, $\operatorname{Ker} \gamma = 0$. Furthermore, γ is an epimorphism. By counting ranks we find that K must have $\mathbb{Z}$-rank one, and consequently G is a bracket group with $R \otimes G \cong M$. $\square$

The assumption on R in Theorem 3 is tight as the next two examples show.

Example 1. Let R be a subring of an algebraic number field. If every bracket R-module is isomorphic to $R \otimes G$ for some bracket group G, then R is a pid that is an IP ring.

P r o o f. Every rank one module is isomorphic to $R \otimes C$ for some subgroup C of $\mathbb{Q}$, so by Lemma 1, R is an IP ring. Furthermore, any ideal I of R is isomorphic to $R \otimes C$ for some $\mathbb{Z} \subseteq C \subseteq \mathbb{Q}$. Since it is clear that we may find C so that $pC = C$ when $p \in \pi(R)$, C must be finitely generated over $\mathbb{Z}_{\pi(R)}$. Therefore, I is principal. $\square$

To facilitate the proof of Theorem 3, more is required of R, other than R is a IP ring that is a pid. We needed to factor a rank one R-module $R \subseteq A \subseteq Q$ into $A = RC$ where $\mathbb{Z} \subseteq C \subseteq \mathbb{Q}$. In light of this and in view of the next example, we feel that the strongly homogeneous assumption on R in Theorem 3 is necessary.

Example 2. Let R be a subring of an algebraic number field. Then, any submodule $R \subseteq A \subseteq Q$ is equal to RC for some $\mathbb{Z} \subseteq C \subseteq \mathbb{Q}$ if and only if R is strongly homogeneous.

Proof. In the notation above, if every A can be expressed as RC, then for any maximal ideal P of R, $P^{-1} = \{t \in Q \mid tP \subseteq R\}$ must be of this form. But P^{-1} is finitely generated over R so we must be able to obtain C so that it is a finitely generated $\mathbb{Z}_{\pi(R)}$-module. It follows that P^{-1} is $n^{-1}R$ for some integer n, but since $P = (P^{-1})^{-1}$ ([10], Theorem 37), $P = nR$. Clearly n must be prime, so every maximal ideal is pR for some integral prime p. On the other hand, any given pR is contained in a maximal ideal of R which implies that the maximal ideals of R are precisely the ideals pR for $p \in \pi(R)$. Therefore R is strongly homogeneous. □

Theorem 4. *The following are equivalent:*

(1) *R is an IP ring.*

(2) *Every bracket R-module is quasi-isomorphic to $R \otimes G$ for some bracket group G.*

Proof. (2) → (1). As in the proof of Lemma 1, suppose P_1 and P_2 are two maximal ideals of $\overline{R}$ over $p\overline{R}$ for some $p \in \pi(R) = \pi(\overline{R})$ and let $R_j = \overline{R}_{P_j}$. Since each R_j is a bracket (rank one) R-module, R_j is quasi-isomorphic to $R \otimes C_j$ for some subgroup C_j of $\mathbb{Q}$. Clearly, we must have $C_1 \cong C_2 \cong \mathbb{Z}_p$. This results in $R_1 \cdot R_2$ being quasi-equal to R_1, although we know that $R_1 \cdot R_2 = Q$. Therefore, $p\overline{R}$ must be primary.

(1) → (2). Let M be a bracket R-module. As argued at the beginning of the proof of Theorem 3, M is the epimorphic image of a completely decomposable R-module $A_1 \oplus \ldots \oplus A_n$ where each A_i is a submodule of Q containing R, and, $(1, \ldots, 1)$ belongs to the kernel. Let $\varphi\colon A_1 \oplus \ldots \oplus A_n \to M$ represent such a scheme.

By Lemma 2 we can factor each $A_j = J_j C_j$ with $R \subseteq J_j \subseteq Q$ a fractional ideal, and $\mathbb{Z} \subseteq C_j \subseteq \mathbb{Q}$. Then, for $A'_j = RC_j \subseteq A_j$, arguing as in Theorem 4, we find that $\varphi(A'_1 \oplus \ldots \oplus A'_n) = M'$ is a bracket R-module (the proof goes through because $\oplus_j C_j$ contains a nonzero element of $\operatorname{Ker}\varphi$). Since each J_j is a fractional ideal of R, there is a nonzero integer m such that $mJ_j \subseteq R$ for every j. It follows that the index of M' in M is bounded by m, and so M is quasi-isomorphic to $R \otimes G$ where $G = \varphi(\oplus_j C_j)$. □

3. Rational socles of bracket modules

In this section we will show that any group M quasi-isomorphic to a bracket R-module over a pid that is an IP ring, is characterized (up to quasi-isomorphism) by the ranks of its socles. Recall, given a torsion-free group G and a type (of a subgroup of $\mathbb{Q}$), the τ-socle of G, written $G(\tau)$, is the image of the natural map $\mathrm{Hom}(X, G) \otimes X \to G$ where X is any rank 1 group having type τ. Equivalently, $G(\tau) = \{x \in G \mid \text{type } x \geqslant \tau\}$.

When dealing with a pid R, one can replicate the study of modules over the integers; for example the notion of a type is obtained. For this reason, we will call the type of a subgroup of $\mathbb{Q}$, a *rational type*, and the quasi-isomorphism class of a rank one R-module, an *R-type*, to avoid duplicitous terms. Further, one may wish to delineate between socles $G(\tau)$ where τ is a rational type, and socles (defined analogously) where τ is an R-type. When τ is a rational type, $G(\tau)$ will be called a *rational socle*, and when τ is an R-type, $G(\tau)$ is an *R-socle*.

Theorem 5. *Let R be an IP ring that is a pid. For any rational type τ and any bracket group G, $(R \otimes G)(\tau)$ contains $R \otimes G(\tau)$ as a subgroup of finite index.*

P r o o f. As usual, G is an image of a completely decoposable group $C_1 \oplus \ldots \oplus C_n$ with $(1, \ldots, 1)$ purely generating the kernel. Then the module $M = R \otimes G$ is a bracket R-module that is the image of the completely decomposable R-module $RC_1 \oplus \ldots \oplus RC_n$ whose kernel is generated, as a pure R-submodule, by $(1, \ldots, 1)$.

The socles of a bracket group G were examined extensively in [6]; the essence of this work can be summarized in the following way: Given a subset I of $\{1, 2, \ldots, n\}$, let e_I represent the standard element in $C_1 \oplus \ldots \oplus C_n$ with 1's in components indexed by members of I, and 0's elsewhere. Assuming G is the epimorphic image of $C_1 \oplus \ldots \oplus C_n$ with kernel purely generated by $(1, \ldots, 1)$, let $\overline{e_I}$ denote the image of e_I in G. For any such $\overline{e_I}$, let $C_I = \{t \in \mathbb{Q} \mid t\overline{e_I} \in G\}$. It is shown, for any rational type τ, that $G(\tau) = \sum\bigl\{C_I \overline{e_I} \mid I \subseteq \{1, \ldots, n\} \text{ and } \overline{e_I} \text{ has rational type} \geqslant \tau\bigr\}$.

Let $M = R \otimes G$. A rank one R-module is of the form $R \otimes X$ for some subgroup X of $\mathbb{Q}$ by Lemma 1. We claim that the R-socle of M with respect to the R-type $R \otimes X$, coincides with the rational socle $M(\tau)$ where τ is the rational type of X. That is, we claim that the images of the natural maps $\mathrm{Hom}_R(R \otimes X, M) \otimes_R (R \otimes X) \to M$ and $\mathrm{Hom}(X, M) \otimes X \to M$ are the same. With τ_R equal to the R-type of $R \otimes X$, denote the image of the latter map by $M(\tau)$ and the image of the former, by $M(\tau_R)$.

If $y \in M(\tau)$ then y has rational type at least τ. Then for Y taken to be the pure subgroup of M generated by y, $R \otimes Y \cong R \cdot Y \subseteq M(\tau_R)$, and so $y \in M(\tau_R)$. Conversely, if $z \in M(\tau_R)$, then clearly z must have integral type at least τ since $R \otimes X$ is homogeneous of rational type τ as a group. So our claim has been established.

The description of the socles of bracket groups mentioned above, carries over to bracket modules over pid's. Specifically, using the notation above, given $I \subseteq \{1, \ldots, n\}$, let $A_I = \{t \in \mathbb{Q}R \mid t\overline{e_I} \in M\}$. Then, encorporating the results from [6] mentioned above for bracket modules over the pid R, $M(\tau) = M(\tau_R) = \sum\{A_I\overline{e_I} \mid R\text{-type } \overline{e_I} \geqslant \tau_R\} = \sum\{A_I\overline{e_I} \mid \overline{e_I} \text{ has rational type } \geqslant \tau\}$.

Let $I \subseteq \{1, \ldots, n\}$ be such that $\bar{e}_I$ has rational type at least τ. Then using the notation above, $A_I \cong RC_I$ so there is an integer $n_i \neq 0$ for which $n_I A_I \subseteq RC_I$ due to the fact that $RC_I \subseteq A_I$. With n equal to the product over all appropriate n_I's, n bounds the cokernel of $R \otimes G(\tau)$ in $(R \otimes G)(\tau)$ as desired. □

Theorem 6. *Let R be an IP ring which is a pid. The following are equivalent for two bracket R-modules M and N:*

(1) *M and N are quasi-isomorphic as groups (R-modules).*

(2) *For any rational type τ,* $\operatorname{rank} M(\tau) = \operatorname{rank} N(\tau)$.

(3) *There is a bracket group G such that both M and N are quasi-isomorphic to $R \otimes G$.*

P r o o f. The implication (1) → (2) holds generally and (3) → (1) is clear. By Theorem 5, if M is quasi-isomorphic to $R \otimes G$ for some bracket group G, then $\operatorname{rank} M(\tau) = (\operatorname{rank} R)(\operatorname{rank} G(\tau))$ for every rational type τ. By Theorem 4, M and N are quasi-isomorphic to $R \otimes G$ and $R \otimes H$ respectively, where G and H are certain bracket groups. It follows from (2) and from what we have just stated, that $\operatorname{rank} G(\tau) = \operatorname{rank} H(\tau)$ for every rational type τ. The main result in [7] is that bracket groups are characterized by the ranks of their rational socles. Therefore G and H are quasi-isomorphic, from which (3) is a consequence. □

The assumption of the last two Theorems that R is a pid is in place in order to apply the results from [6]. We feel the techniques from [6] carry over to more general domains with appropriate modifications of the statements of the results. If this is the case, the pid assumption can be removed from the last two Theorems.

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