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SEMIREGULARITY OF CONGRUENCES IMPLIES  
CONGRUENCE MODULARITY AT 0

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*Abstract.* We introduce a weakened form of regularity, the so called semiregularity, and we show that if every diagonal subalgebra of  $\mathcal{A} \times \mathcal{A}$  is semiregular then  $\mathcal{A}$  is congruence modular at 0.

*Keywords:* regularity, modularity, semiregularity, modularity at 0

*MSC 2000:* 08A30, 08B10

Recall that an algebra  $\mathcal{A}$  is *regular* if for every two congruences  $\Theta, \Phi \in \text{Con } \mathcal{A}$  the following holds

$$\text{if } [a]_{\Theta} = [a]_{\Phi} \text{ for some } a \in A \text{ then } \Theta = \Phi.$$

Note that this condition can be rewritten in the form:

$$\text{if } [a]_{\Theta} = [a]_{\Phi} \text{ for some } a \in A \text{ then } [b]_{\Theta} = [b]_{\Phi} \text{ for each } b \in A.$$

This formulation was used in [2] for introducing local regularity. At first we say that an algebra  $\mathcal{A}$  has 0 if 0 is a nullary (term) operation of  $\mathcal{A}$ . An algebra  $\mathcal{A}$  with 0 is *locally regular* if for each  $\Theta, \Phi \in \text{Con } \mathcal{A}$  the following holds:

$$\text{if } [a]_{\Theta} = [a]_{\Phi} \text{ for some } a \in \mathcal{A} \text{ then } [0]_{\Theta} = [0]_{\Phi}.$$

The paper [2] contains examples of locally regular algebras and two characterizations of varieties of these algebras.

It was shown in [1] that if every subalgebra of the direct power  $\mathcal{A} \times \mathcal{A}$  is regular then  $\mathcal{A}$  is congruence modular, i.e. the congruence lattice  $\text{Con } \mathcal{A}$  is modular.

The concept of congruence modularity was weakened in [3] as follows:

An algebra  $\mathcal{A}$  with 0 is *congruence modular at 0* if for each  $\Theta, \Phi, \Psi \in \text{Con } \mathcal{A}$  with  $\Psi \subseteq \Phi$  the following holds

$$[0]_{\Phi \cap (\Theta \vee \Psi)} = [0]_{(\Phi \cap \Theta) \vee \Psi}.$$

Let  $\mathcal{A} = (A, F)$  be an algebra. Denote by  $\omega_A = \{\langle a, a \rangle; a \in \mathcal{A}\}$  the so called *diagonal* of  $\mathcal{A}$ , i.e. the least congruence on  $\mathcal{A}$ . A subalgebra  $\mathcal{B}$  of the direct square  $\mathcal{A} \times \mathcal{A}$  is called a *diagonal subalgebra* whenever  $\omega_A \subseteq \mathcal{B}$ .

Let us consider the conditions

- (i) Every diagonal subalgebra of  $\mathcal{A} \times \mathcal{A}$  is regular.
- (ii) Every diagonal subalgebra of  $\mathcal{A} \times \mathcal{A}$  is locally regular (with respect to the term  $(0, 0)$ ).

We can ask whether there exists an intermediate property between the conditions (i) and (ii) which ensures the congruence modularity at 0 for  $\mathcal{A}$ .

**Definition.** Let  $\mathcal{A}$  be an algebra with 0. We say that a diagonal subalgebra  $\mathcal{B}$  of  $\mathcal{A} \times \mathcal{A}$  is *semiregular* if for every  $\alpha, \beta \in \text{Con } \mathcal{B}$  the following holds: if  $[(a, a)]_\alpha = [(a, a)]_\beta$  for some  $a \in A$  then  $[(0, a)]_\alpha = [(0, a)]_\beta$  whenever  $(0, a) \in \mathcal{B}$ .

Now let us deal with the condition

- (iii) Every diagonal subalgebra of  $\mathcal{A} \times \mathcal{A}$  is semiregular.

Then we have

$$(i) \Rightarrow (iii) \Rightarrow (ii)$$

(since the element  $(0, 0)$  is contained in each diagonal subalgebra).

Applying an approach similar to that of [1] for regularity, we will show the connection between semiregularity and modularity at 0. For this, we need the following

**Lemma.** *Let every diagonal subalgebra of  $\mathcal{A} \times \mathcal{A}$  be semiregular, let  $\Psi, \Phi \in \text{Con } \mathcal{A}$  and  $R$  be a reflexive and compatible relation on  $\mathcal{A}$ . If  $\Psi \subseteq \Phi$  and  $\Phi \cap R \subseteq \Psi$  then*

$$[\langle x_1, x_2 \rangle \in R, \langle 0, y_2 \rangle \in R, \langle x_1, 0 \rangle \in \Phi, \langle x_2, y_2 \rangle \in \Psi] \Rightarrow \langle x_1, 0 \rangle \in \Psi.$$

**Proof.** Let every diagonal subalgebra of  $\mathcal{A} \times \mathcal{A}$  be semiregular and  $\Theta, \Psi, R$  satisfy the assumptions. Of course,  $R$  is a diagonal subalgebra of  $\mathcal{A} \times \mathcal{A}$ . Introduce the following two congruences  $\alpha, \beta$  on  $R$ :

$$\begin{aligned} \langle (x_1, x_2), (y_1, y_2) \rangle \in \alpha & \quad \text{if } \langle x_1, y_1 \rangle \in \Theta \quad \text{and} \quad \langle x_2, y_2 \rangle \in \Psi, \\ \langle (x_1, x_2), (y_1, y_2) \rangle \in \beta & \quad \text{if } \langle x_1, y_1 \rangle \in \Psi \quad \text{and} \quad \langle x_2, y_2 \rangle \in \Psi. \end{aligned}$$

Since  $\Psi \subseteq \Phi$ , we have  $\beta \subseteq \alpha$ . First we prove  $[(y_2, y_2)]_\alpha = [(y_2, y_2)]_\beta$ . Suppose  $(z_1, z_2) \in [(y_2, y_2)]_\beta$  for some  $\langle z_1, z_2 \rangle \in R$ . Then  $\langle y_2, z_1 \rangle \in \Phi$ ,  $\langle y_2, z_2 \rangle \in \Psi \subseteq \Phi$  thus also  $\langle z_1, z_2 \rangle \in \Phi$ , i.e.  $\langle z_1, z_2 \rangle \in \Phi \cap R \subseteq \Psi$ .

Together with  $\langle y_2, z_2 \rangle \in \Psi$  this gives  $\langle y_2, z_1 \rangle \in \Psi$  thus  $\langle (y_2, y_2), (z_1, z_1) \rangle \in \alpha$  proving our equality. Since  $R$  is semiregular, this implies

$$[(0, y_2)]_\alpha = [(0, y_2)]_\beta.$$

By the assumption,  $\langle 0, x_1 \rangle \in \Phi$ ,  $\langle y_2, x_2 \rangle \in \Psi$ , i.e.  $\langle x_1, x_2 \rangle \in [(0, y_2)]_\alpha = [(0, y_2)]_\beta$  thus also  $\langle x_1, 0 \rangle \in \Psi$ .  $\square$

**Theorem.** *If every diagonal subalgebra of  $\mathcal{A} \times \mathcal{A}$  is semiregular then  $\mathcal{A}$  is congruence modular at 0.*

**Proof.** Let every diagonal subalgebra of  $\mathcal{A} \times \mathcal{A}$  be semiregular,  $\Theta, \Phi, \Psi \in \text{Con } \mathcal{A}$  and  $\Psi \subseteq \Phi$ . To prove congruence modularity at 0 we need only to show that

$$[0]_{\Phi \cap (\Psi \vee \Theta)} \subseteq [0]_{\Psi \vee (\Phi \cap \Theta)}.$$

Denote by  $R_0 = \Theta$  and define inductively

$$R_{k+1} = R_k \cdot \Psi \cdot \Theta \quad \text{for } k = 0, 1, 2, \dots$$

Hence, we need to prove

$$(*) \quad [0]_{\Phi \cap R_k} \subseteq [0]_{\Psi \vee (\Phi \cap \Theta)}$$

for every  $k = 0, 1, 2, \dots$

For  $k = 0$  this holds trivially. Suppose that  $(*)$  holds for some  $k \geq 0$  and let us prove it for  $k + 1$ . Let  $a \in [0]_{\Phi \cap R_{k+1}}$ . Then there exist  $b, c \in A$  such that

$$\langle a, 0 \rangle \in \Phi, \quad \langle a, b \rangle \in R_k, \quad \langle b, c \rangle \in \Psi, \quad \langle c, 0 \rangle \in \Theta.$$

However,  $\Theta \subseteq R$  gives  $\langle 0, c \rangle \in R_k$ .

Set  $\Psi^* = \Psi \vee (\Phi \cap \Theta)$ . Then  $\Psi^* \subseteq \Phi$  and  $\Phi \cap R_k \subseteq \Psi^*$ . Evidently,  $R_k$  is a diagonal subalgebra of  $\mathcal{A} \times \mathcal{A}$ . In account of the Lemma, we obtain  $\langle a, 0 \rangle \in \Psi^*$  which proves  $(*)$  for  $k + 1$ . By induction, we have shown that  $\mathcal{A}$  is congruence modular at 0.  $\square$

### References

- [1] *S. Bulman-Fleming, A. Day and W. Taylor*: Regularity and modularity of congruences. *Algebra Universalis* 4 (1974), 58–60.
- [2] *I. Chajda*: Locally regular varieties. *Acta Sci. Math. (Szeged)* 64 (1998), 431–435.
- [3] *I. Chajda and R. Halaš*: Congruence modularity at 0. *Discuss. Math., Algebra and Stochast. Methods* 17 (1997), 57–65.

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