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A CONSTRUCTIVE INTEGRAL EQUIVALENT TO THE INTEGRAL OF KURZWEIL

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Abstract. We slightly modify the definition of the Kurzweil integral and prove that it still gives the same integral.

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0. Introduction

We prove the equivalence of a multidimensional constructive integral with the multidimensional Kurzweil or Generalized Riemann integral [2] working in a general Banach-space valued context. The regular integral corresponding to the former, however, is *not equivalent* to the Mawhin integral [3].

1. Definitions and terminology

Let R be a compact interval of $\mathbb{R}^n$ with sides parallel to the coordinate axes. Any finite set of closed nonoverlapping subintervals of R is called a partition of R. A pair $d = (\xi_i, J_i)$ is a tagged division of R if (J_i) is a partition of R with $\bigcup J_i = R$ and $\xi_i \in J_i$ for every i. We denote by TD_R the set of all tagged divisions of R. Let $d = (\xi_i, J_i)$ and $d' = (\eta_j, I_j)$ belong to TD_R. We say that d' refines and write $d' \leqslant d$ if for given j there exists i such that $I_j \subset J_i$ (see [1], p. 41). A gauge of a subset E of R is a function $\delta\colon E \to]0, \infty[$. We say that $d = (\xi_i, J_i) \in TD_R$ is δ-fine if $J_i \subset \{t \in R;\ |t - \xi_i| < \delta(\xi_i)\}$ for every i. Given a gauge δ of R, there exists a δ-fine $d \in TD_R$ (Cousin's Lemma).

By int(A) and cl(A) we mean respectively the interior and the closure of a set $A \subset \mathbb{R}^n$ and we write $\partial(A) = \mathrm{cl}(A) \setminus \mathrm{int}(A)$.

In what follows X denotes a Banach space.

Definition 1.1. A function $f\colon R \to X$ is Kurzweil integrable (we write $f \in K(R, X)$) and $I \in X$ is its integral (we write $I = {}^{K}\!\int_R f$) if for every $\varepsilon > 0$, there is a gauge δ of R such that for every δ-fine $d = (\xi_i, J_i) \in TD_R$,

$$\Big\| \sum_i f(\xi_i)|J_i| - I \Big\| < \varepsilon. \tag{1}$$

Definition 1.2. We say that $f\colon R \to X$ is K^*-integrable (we write $f \in K^*(R, X)$) and that $I \in X$ is its integral (we write $I = {}^{K^*}\!\int_R f$) if for every $\varepsilon > 0$, there is a gauge δ of R and there is a δ-fine $d \in TD_R$ such that for every δ-fine $d' = (\xi_i, J_i) \in TD_R$ with $d' \leqslant d$, (1) holds.

Definition 1.3. We say that $f\colon R \to X$ is K^{**}-integrable (we write $f \in K^{**}(R, X)$) and that $I \in X$ is its integral (we write $I = {}^{K^{**}}\!\int_R f$) if for every $\varepsilon > 0$, there is a gauge δ of R and there exists $d \in TD_R$ (not necessarily δ-fine) such that for every δ-fine $d' = (\xi_i, J_i) \in TD_R$ with $d' \leqslant d$, (1) holds.

Remark. It is immediate that $K(R, X) \subset K^*(R, X)$ and $K(R, X) \subset K^{**}(R, X)$. Besides, $K^{**}(R, X) = K^*(R, X)$ and the integrals coincide when defined.

2. The main result

Theorem 2.1. *$K(R, X) = K^*(R, X) = K^{**}(R, X)$ and the integrals coincide.*

P r o o f. We prove the result for the two-dimensional case. When $n > 2$, the proof follows analogous steps. By the above Remark, it is enough to show that $K^*(R, X) \subset K(R, X)$.

Let $f \in K^*(R, X)$. Then given $\varepsilon > 0$, there exists a gauge δ and there exists a δ-fine $d = (\zeta_j, L_j) \in TD_R$ such that for every δ-fine $d' = (\xi_i, J_i) \in TD_R$ with $d' \leqslant d$,

$$\Big\| \sum_i f(\xi_i)|J_i| - {}^{K^*}\!\!\int_R f \Big\| < \varepsilon \tag{2}$$

Let us define another gauge δ' of R as follows:

(i) for every $\xi \in R$, let $\delta'(\xi) < \delta(\xi)$.

Let $\xi \in L_m$. Then,

(ii) if $\xi \in \mathrm{int}(L_m)$, let $\delta'(\xi) < \mathrm{dist}\{\xi, R \setminus L_m\}$;

(iii) if $\xi \in \partial(L_m)$ and $\xi \neq \zeta_j$ for every j, let $\delta'(\xi) < \min\{|\xi - \zeta_j|$, for every $j\}$;

(iv) if $\xi \in \partial(L_m)$ and $\xi = \zeta_j$ for some j, let $\delta'(\xi) < \min\{|\xi - \zeta_j|$ for every j such that $\xi \neq \zeta_j\}$ and $\delta'(\xi) < 1/2 \min\{h^j$ for every j such that $\xi = \zeta_j\}$, where h^j denotes the smallest side of the interval L_j.

Now, if $d_1 = (\eta_k, I_k) \in TD_R$ is δ'-fine, then it satisfies the following conditions:

(v) d_1 is δ-fine;

(vi) if $\eta_k \in \operatorname{int}(L_m)$, then $I_k \subset L_m$;

(vii) if $\eta_k \in \partial(L_m)$, then η_k belongs to at most three other intervals L_j's, $j \neq m$. Consider the set of indices $A_m = \{j;\ \eta_k \in L_j$ and $L_j \cap L_m \neq \emptyset\}$ and let n_m be the number of elements of A_m. Then $2 \leqslant n_m \leqslant 4$. Divide the interval I_k into n_m subintervals such that each new interval is contained in one and only one of the intervals L_j, $j \in A$. Hence, η_k belongs to each new interval and can be regarded as the tag of each of these intervals. Clearly $I_k = \bigcup\limits_{j \in Am} (L_j \cap I_k)$; $L_j \cap I_k$, $j \in A_m$ are nonoverlapping and therefore $|I_k| = \sum\limits_{j \in A_m} |L_j \cap I_k|$. Hence we can consider without loss of generality that d_1 is such that given k, there exists j such that $I_k \subset L_j$, since the Riemann sum with respect to the new d_1 is equal to the Riemann sum with respect to the original d_1. Thus, $d_1 \leqslant d$ and by (2) it follows that

$$\left\| \sum_k f(\eta_k)|I_k| - {}^{K^*}\!\!\int_R f \right\| < \varepsilon. \tag{3}$$

Hence, for every $\varepsilon > 0$, there is a gauge δ' of R such that for every δ'-fine $d_1 = (\eta_k, I_k) \in TD_R$, (3) holds. Then $f \in K(R, X)$ with ${}^{K}\!\!\int_R f = {}^{K^*}\!\!\int_R f$ and the proof is complete. $\square$

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