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ON FUZZY B-ALGEBRAS

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Abstract. The fuzzification of (normal) B-subalgebras is considered, and some related properties are investigated. A characterization of a fuzzy B-algebra is given.

Keywords: normal B-subalgebra, fuzzy (normal) B-algebra, upper level cut

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1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([4, 5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2, 3] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They showed that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Recently, the present authors ([6]) have introduced a new notion, called a BH-algebra, which is a generalization of $BCH/BCI/BCK$-algebras. They also defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. The third author together with J. Neggers ([9]) introduced and investigated a class of algebras, viz., the class of B-algebras, which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras, and which seems to have rather nice properties without being excessively complicated otherwise. J. R. Cho and H. S. Kim ([1]) discussed further relations between B-algebras and other classes of algebras, such as quasigroups. It is well known that every group determines a B-algebra, called a *group-derived* B-algebra. It is natural to consider

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the problem whether or not all B-algebras are so group-derived. It is proved that this is not the case in general, and thus that this class of algebras contains the class of groups indirectly via the group-derived principle (see [8]). In this paper we consider the fuzzification of (normal) B-subalgebras in B-algebras and investigate some related properties. We give a characterization of a fuzzy B-algebra.

2. Preliminaries

A *B-algebra* is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * z = x * (z * (0 * y))$

for all x, y, z in X. A non-empty subset N of a B-algebra X is called a *B-subalgebra* of X if $x * y \in N$ for any $x, y \in N$. A non-empty subset N of a B-algebra X is said to be *normal* if $(x * a) * (y * b) \in N$ whenever $x * y \in N$ and $a * b \in N$. Note that any normal subset N of a B-algebra X is a B-subalgebra of X, but the converse need not be true (see [10]). A non-empty subset N of a B-algebra X is called a *normal B-subalgebra* of X if it is both a B-subalgebra and normal.

Lemma 2.1 ([9]). *If X is a B-algebra, then $x*y = x*(0*(0*y))$ for all $x, y \in X$.*

Example 2.2 ([9]). Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by

$$x * y := \frac{n(x - y)}{n + y}.$$

Then $(X; *, 0)$ is a B-algebra.

Example 2.3 ([9]). Let Z be the group of integers under usual addition and let $\alpha \notin Z$. We adjoin the special element α to Z. Let $X := Z \cup \{\alpha\}$. Define $\alpha + 0 = \alpha$, $\alpha + n = n - 1$ where $n \neq 0$ in Z and $\alpha + \alpha$ is an arbitrary element in X. Define a mapping $\varphi\colon X \to X$ by $\varphi(\alpha) = 1$, $\varphi(n) = -n$ where $n \in Z$. If we define a binary operation "$*$" on X by $x * y := x + \varphi(y)$, then $(X; *, 0)$ is a non-group derived B-algebra.

3. Fuzzy B-algebras

In what follows, let X denote a B-algebra unless otherwise specified.

Definition 3.1. A fuzzy set μ in X is called a *fuzzy B-algebra* if it satisfies the inequality

$$\mu(x * y) \geqslant \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

Example 3.2. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X; *, 0)$ is a B-algebra (see [10, Example 3.5]). Define a fuzzy set $\mu\colon X \to [0, 1]$ by $\mu(0) = \mu(3) = 0.7 > 0.1 = \mu(x)$ for all $x \in X \setminus \{0, 3\}$. Then μ is a fuzzy B-algebra.

Proposition 3.3. *Every fuzzy B-algebra μ satisfies the inequality $\mu(0) \geqslant \mu(x)$ for all $x \in X$.*

Proof. Since $x * x = 0$ for all $x \in X$, we have $\mu(0) = \mu(x * x) \geqslant \min\{\mu(x), \mu(x)\} = \mu(x)$ for all $x \in X$. □

For any elements x and y of X, let us write $\prod^{n} x * y$ for $x * (\ldots * (x * (x * y)))$ where x occurs n times.

Proposition 3.4. *Let a fuzzy set μ in X be a fuzzy B-algebra and let $n \in \mathbb{N}$. Then*

(i) $\mu\Big(\prod^{n} x * x\Big) \geqslant \mu(x)$ *whenever n is odd,*

(ii) $\mu\Big(\prod^{n} x * x\Big) = \mu(x)$ *whenever n is even,*

for all $x \in X$.

Proof. Let $x \in X$ and assume that n is odd. Then $n = 2k - 1$ for some positive integer k. Observe that $\mu(x * x) = \mu(0) \geqslant \mu(x)$. Suppose that $\mu\Big(\prod^{2k-1} x * x\Big) \geqslant \mu(x)$

for a positive integer k. Then

$$\begin{aligned}\mu\Big(\prod^{2(k+1)-1} x * x\Big) &= \mu\Big(\prod^{2k+1} x * x\Big)\\ &= \mu\Big(\prod^{2k-1} x * (x * (x * x))\Big)\\ &= \mu\Big(\prod^{2k-1} x * x\Big) \qquad \text{[by (I), (II)]}\\ &\geqslant \mu(x),\end{aligned}$$

which proves (i). Similarly we obtain the second part. □

Proposition 3.5. *If a fuzzy set μ in X is a fuzzy B-algebra, then*
(fB1) $\mu(0 * x) \geqslant \mu(x)$,
(fB2) $\mu(x * (0 * y)) \geqslant \min\{\mu(x), \mu(y)\}$ *for all* $x, y \in X$.

Proof. For any $x, y \in X$ we have $\mu(0 * x) \geqslant \min\{\mu(0), \mu(x)\} \geqslant \mu(x)$ and

$$\begin{aligned}\mu(x * (0 * y)) &\geqslant \min\{\mu(x), \mu(0 * y)\}\\ &\geqslant \min\{\mu(x), \mu(y)\},\end{aligned}$$

proving the results. □

Since $x = 0 * (0 * x)$ (see [1, Lemma 3.5]), if μ is a fuzzy B-algebra, then $\mu(x) = \mu(0 * (0 * x)) \geqslant \min\{\mu(0), \mu(0 * x)\} = \mu(0 * x)$, i.e., $\mu(x) = \mu(0 * x)$ for any $x \in X$.

Theorem 3.6. *If a fuzzy set μ in X satisfies* (fB1) *and* (fB2), *then μ is a fuzzy B-algebra.*

Proof. Assume that a fuzzy set μ in X satisfies the conditions (fB1) and (fB2) and let $x, y \in X$. Then

$$\begin{aligned}\mu(x * y) &= \mu(x * (0 * (0 * y))) &&\text{[by Lemma 2.1]}\\ &\geqslant \min\{\mu(x), \mu(0 * y)\} &&\text{[by (fB2)]}\\ &\geqslant \min\{\mu(x), \mu(y)\}. &&\text{[by (fB1)]}\end{aligned}$$

Hence μ is a fuzzy B-algebra. □

4. Fuzzy normal B-algebras

Definition 4.1. A fuzzy set μ in X is said to be *fuzzy normal* if it satisfies the inequality

$$\mu((x * a) * (y * b)) \geqslant \min\{\mu(x * y), \mu(a * b)\}$$

for all $a, b, x, y \in X$.

Example 4.2. If we define a fuzzy set $\nu\colon X \to [0,1]$ by $\nu(0) = \nu(1) = \nu(2) = 0.8$ and $\nu(3) = \nu(4) = \nu(5) = 0.3$ in Example 3.2, then ν is a fuzzy normal set in X.

Example 4.3. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(X; *, 0)$ is a B-algebra ([8]). If we define a map $\mu\colon X \to [0,1]$ by $\mu(0) > \mu(2) > \mu(1) = \mu(3)$ then μ is a fuzzy normal set in X. Moreover, if we define a map $\sigma\colon X \to [0,1]$ by $\sigma(0) = \sigma(2) > \sigma(1) = \sigma(3)$, then σ is also a fuzzy normal set in X.

The next result, which we propose to discuss, will be used repeatedly in this paper.

Theorem 4.4. *Every fuzzy normal set μ in X is a fuzzy B-algebra.*

Proof. For any $x, y \in X$, since μ is fuzzy normal, we have

$$\mu(x * y) = \mu((x * y) * (0 * 0)) \geqslant \min\{\mu(x * 0), \mu(y * 0)\} = \min\{\mu(x), \mu(y)\}.$$

Hence μ is a fuzzy B-algebra. □

Remark 4.5. The converse of Theorem 4.4 is not true. For example, the fuzzy B-algebra μ in Example 3.2 is not fuzzy normal, since

$$\mu((2 * 5) * (4 * 1)) = \mu(2) < \mu(3) = \min\{\mu(2 * 4), \mu(5 * 1)\}.$$

Definition 4.6. A fuzzy set μ in X is called a *fuzzy normal B-algebra* if it is a fuzzy B-algebra which is fuzzy normal.

Example 4.7. The fuzzy sets discussed in Examples 4.2 and 4.3 are indeed fuzzy normal B-algebras.

Proposition 4.8. *If a fuzzy set μ in X is a fuzzy normal B-algebra, then $\mu(x*y)=\mu(y*x)$ for all $x,y\in X$.*

P r o o f. Let $x,y\in X$. Then

$$\begin{aligned}\mu(x*y)&=\mu((x*y)*(x*x)) && \text{[by (I), (II)]}\\ &\geqslant \min\{\mu(x*x),\mu(y*x)\} && \text{[since } \mu \text{ is fuzzy normal]}\\ &=\mu(y*x) && \text{[by Proposition 3.3]}.\end{aligned}$$

Interchanging x with y, we obtain $\mu(y*x)\geqslant\mu(x*y)$, which proves the proposition. □

The next result will be useful for characterizing the fuzzy normal B-algebras in the next section.

Theorem 4.9. *Let μ be a fuzzy normal B-algebra. Then the set*

$$X_\mu := \{x\in X \mid \mu(x)=\mu(0)\}$$

is a normal B-subalgebra of X.

P r o o f. It is sufficient to show that X_μ is normal. Let $a,b,x,y\in X$ be such that $x*y\in X_\mu$ and $a*b\in X_\mu$. Then $\mu(x*y)=\mu(0)=\mu(a*b)$. Since μ is fuzzy normal, it follows that

$$\mu((x*a)*(y*b))\geqslant\min\{\mu(x*y),\mu(a*b)\}=\mu(0).$$

Applying Proposition 3.3, we conclude that $\mu((x*a)*(y*b))=\mu(0)$, which shows that $(x*a)*(y*b)\in X_\mu$. This completes the proof. □

Theorem 4.10. *The intersection of any set of fuzzy normal B-algebras is also a fuzzy normal B-algebra.*

P r o o f. Let $\{\mu_\alpha \mid \alpha\in\Lambda\}$ be a family of fuzzy normal B-algebras and let $a,b,x,y\in X$. Then

$$\begin{aligned}\Big(\bigcap_{\alpha\in\Lambda}\mu_\alpha\Big)((x*a)*(y*b)) &= \inf_{\alpha\in\Lambda}\mu_\alpha((x*a)*(y*b))\\ &\geqslant \inf_{\alpha\in\Lambda}\big\{\min\{\mu_\alpha(x*y),\mu_\alpha(a*b)\}\big\}\\ &= \min\{\inf_{\alpha\in\Lambda}\mu_\alpha(x*y),\inf_{\alpha\in\Lambda}\mu_\alpha(a*b)\}\\ &= \min\Big\{\Big(\bigcap_{\alpha\in\Lambda}\mu_\alpha\Big)(x*y),\Big(\bigcap_{\alpha\in\Lambda}\mu_\alpha\Big)(a*b)\Big\},\end{aligned}$$

which shows that $\bigcap_{\alpha\in\Lambda}\mu_\alpha$ is a fuzzy normal set in X. Using Theorem 4.4, we conclude that $\bigcap_{\alpha\in\Lambda}\mu_\alpha$ is a fuzzy normal B-algebra. □

The union of any set of fuzzy B-algebras need not be a fuzzy B-algebra. For example, if we define a fuzzy set $\sigma\colon X \to [0,1]$ by $\sigma(0) = \sigma(4) = 0.8 > 0.2 = \sigma(1) = \sigma(2) = \sigma(3) = \sigma(5)$ in Example 3.2, then it is also a fuzzy B-algebra. Since

$$(\mu \cup \sigma)(3 * 4) = 0.2 \text{ and } \min\{(\mu \cup \sigma)(3), (\mu \cup \sigma)(4)\} = 0.7,$$

$\mu \cup \sigma$ is not a fuzzy B-algebra. Since every fuzzy normal B-algebra is a fuzzy B-algebra, the union of fuzzy normal B-algebras need not be a fuzzy normal B-algebra.

5. Characterization of fuzzy normal B-algebras

Theorem 5.1. *Let N be a non-empty subset of X and let μ_N be a fuzzy set in X defined by*

$$\mu_N(x) := \begin{cases} \alpha & \text{if } x \in N, \\ \beta & \text{otherwise} \end{cases}$$

for all $x \in X$ and α, $\beta \in [0,1]$ with $\alpha > \beta$. Then μ_N is a fuzzy normal B-algebra if and only if N is a normal B-subalgebra of X. Moreover, in this case, $X_{\mu_N} = N$.

Proof. Assume that μ_N is a fuzzy normal B-algebra. Let $a, b, x, y \in X$ be such that $x * y \in N$ and $a * b \in N$. Then

$$\mu_N((x * a) * (y * b)) \geqslant \min\{\mu_N(x * y), \mu_N(a * b)\} = \alpha$$

and so $\mu_N((x * a) * (y * b)) = \alpha$, which shows that $(x * a) * (y * b) \in N$. Hence N is a normal B-subalgebra of X. Conversely, suppose that N is a normal B-subalgebra of X and let $a, b, x, y \in X$. If $x * y \in N$ and $a * b \in N$, then $(x * a) * (y * b) \in N$ and so

$$\mu_N((x * a) * (y * b)) = \alpha = \min\{\mu_N(x * y), \mu_N(a * b)\}.$$

If $x * y \notin N$ or $a * b \notin N$, then clearly

$$\mu_N((x * a) * (y * b)) \geqslant \beta = \min\{\mu_N(x * y), \mu_N(a * b)\}.$$

This shows that μ_N is a fuzzy normal set. It follows from Theorem 4.4 that μ_N is a fuzzy normal B-algebra. Moreover, using Theorem 4.9 we have

$$X_{\mu_N} = \{x \in X \mid \mu_N(x) = \mu_N(0)\} = \{x \in X \mid \mu_N(x) = \alpha\} = N.$$

This completes the proof. □

Theorem 5.2. *Let μ be a fuzzy set in X. Then μ is a fuzzy normal B-algebra if and only if the set $U(\mu;\alpha) = \{x \in X \mid \mu(x) \geqslant \alpha\}$, called an upper level cut of μ, is a normal B-subalgebra of X for all $\alpha \in [0,1]$, where $U(\mu;\alpha) \neq \emptyset$.*

Proof. Let μ be a fuzzy normal B-algebra and assume that $U(\mu;\alpha) \neq \emptyset$ for all $\alpha \in [0,1]$. Let $a, b, x, y \in X$ be such that $x * y \in U(\mu;\alpha)$ and $a * b \in U(\mu;\alpha)$. Then

$$\mu((x * a) * (y * b)) \geqslant \min\{\mu(x * y), \mu(a * b)\} \geqslant \alpha$$

and thus $(x * a) * (y * b) \in U(\mu;\alpha)$. Hence $U(\mu;\alpha)$ is a normal B-subalgebra of X. Conversely, suppose that $U(\mu;\alpha)(\neq \emptyset)$ is a normal B-subalgebra of X for every $\alpha \in [0,1]$. Using Theorem 4.4, it is sufficient to show that μ is a fuzzy normal set in X. If there are $a_0, b_0, x_0, y_0 \in X$ such that

$$\mu((x_0 * a_0) * (y_0 * b_0)) < \min\{\mu(x_0 * y_0), \mu(a_0 * b_0)\},$$

then by taking $\alpha_0 := \frac{1}{2}\big(\mu((x_0 * a_0) * (y_0 * b_0)) + \min\{\mu(x_0 * y_0), \mu(a_0 * b_0)\}\big)$ we have

$$\mu((x_0 * a_0) * (y_0 * b_0)) < \alpha_0 < \min\{\mu(x_0 * y_0), \mu(a_0 * b_0)\}.$$

It follows that $x_0 * y_0 \in U(\mu;\alpha_0)$ and $a_0 * b_0 \in U(\mu;\alpha_0)$, but $(x_0 * a_0) * (y_0 * b_0) \notin U(\mu;\alpha_0)$, a contradiction. Hence μ is fuzzy normal, which proves the theorem. □

Theorem 5.3. *Let μ be a fuzzy normal B-algebra with $\mathrm{Im}(\mu) = \{\alpha_i \mid i \in \Lambda\}$ and $\mathcal{B} = \{U(\mu;\alpha_i) \mid i \in \Lambda\}$ where Λ is an arbitrary index set. Then*

(i) *there exists a unique $i_0 \in \Lambda$ such that $\alpha_{i_0} \geqslant \alpha_i$ for all $i \in \Lambda$;*

(ii) $X_\mu = \bigcap\limits_{i\in\Lambda} U(\mu;\alpha_i) = U(\mu;\alpha_{i_0})$;

(iii) $X = \bigcup\limits_{i\in\Lambda} U(\mu;\alpha_i)$;

(iv) *the members of $\mathcal{B}$ form a chain;*

(v) *$\mathcal{B}$ contains all upper level cuts of μ if and only if μ attains its infimum on all normal B-subalgebras of X.*

Proof. (i) Since $\mu(0) \in \mathrm{Im}(\mu)$, there exists a unique $i_0 \in \Lambda$ such that $\mu(0) = \alpha_{i_0}$. It follows from Proposition 3.3 that $\mu(x) \leqslant \mu(0) = \alpha_{i_0}$ for all $x \in X$ so that $\alpha_{i_0} \geqslant \alpha_i$ for all $i \in \Lambda$.

(ii) We have

$$\begin{aligned} U(\mu;\alpha_{i_0}) &= \{x \in X \mid \mu(x) \geqslant \alpha_{i_0}\} = \{x \in X \mid \mu(x) = \alpha_{i_0}\} \\ &= \{x \in X \mid \mu(x) = \mu(0)\} = X_\mu. \end{aligned}$$

Since $\alpha_{i_0} \geqslant \alpha_i$ for all $i \in \Lambda$, it follows that $U(\mu;\alpha_{i_0}) \subseteq U(\mu;\alpha_i)$ for all $i \in \Lambda$. Hence $U(\mu;\alpha_{i_0}) \subseteq \bigcap\limits_{i\in\Lambda} U(\mu;\alpha_i)$ and so $U(\mu;\alpha_{i_0}) = \bigcap\limits_{i\in\Lambda} U(\mu;\alpha_i)$ because $i_0 \in \Lambda$.

(iii) Clearly $\bigcup_{i\in\Lambda} U(\mu;\alpha_i) \subseteq X$. For every $x \in X$ there exists $i(x) \in \Lambda$ such that $\mu(x) = \alpha_{i(x)}$. This implies $x \in U(\mu;\alpha_{i(x)}) \subseteq \bigcup_{i\in\Lambda} U(\mu;\alpha_i)$, which proves (iii).

(iv) Since either $\alpha_i \geqslant \alpha_j$ or $\alpha_i \leqslant \alpha_j$ for all $i, j \in \Lambda$, we have either $U(\mu;\alpha_i) \subseteq U(\mu;\alpha_j)$ or $U(\mu;\alpha_j) \subseteq U(\mu;\alpha_i)$ for all $i, j \in \Lambda$.

(v) Suppose $\mathcal{B}$ contains all upper level cuts of μ and let N be a normal B-subalgebra of X. If μ is constant on N, then we are done. Assume that μ is not constant on N. We distinguish the following two cases: (1) $N = X$ and (2) $N \subsetneq X$. For the case (1), we let $\beta = \inf\{\alpha_i \mid i \in \Lambda\}$. Then $\beta \leqslant \alpha_i$ and so $U(\mu;\alpha_i) \subseteq U(\mu;\beta)$ for all $i \in \Lambda$. Note that $X = U(\mu;0) \in \mathcal{B}$ because $\mathcal{B}$ contains all upper level cuts of μ. Hence there exists $j \in \Lambda$ such that $\alpha_j \in \mathrm{Im}(\mu)$ and $U(\mu;\alpha_j) = X$. It follows that $U(\mu;\beta) \supseteq U(\mu;\alpha_j) = X$ so that $U(\mu;\beta) = U(\mu;\alpha_j) = X$ because every upper level cut of μ is a normal B-subalgebra of X. Now it is sufficient to show that $\beta = \alpha_j$. If $\beta < \alpha_j$, then there exists $k \in \Lambda$ such that $\alpha_k \in \mathrm{Im}(\mu)$ and $\beta \leqslant \alpha_k < \alpha_j$. This implies that $U(\mu;\alpha_k) \supsetneq U(\mu;\alpha_j) = X$, a contradiction. Therefore $\beta = \alpha_j$. If the case (2) holds, consider the restriction μ_N of μ to N. By Theorem 5.1, μ_N is a fuzzy normal B-algebra. Let $\Lambda_N = \{i \in \Lambda \mid \mu(y) = \alpha_i \text{ for some } y \in N\}$ and $\mathcal{B}_N = \{U(\mu_N;\alpha_i) \mid i \in \Lambda_N\}$. Noticing that $\mathcal{B}_N$ contains all upper level cuts of μ_N, we conclude that there exists $z \in N$ such that $\mu_N(z) = \inf\{\mu_N(x) \mid x \in N\}$, which implies that $\mu(z) = \inf\{\mu(x) \mid x \in N\}$.

Conversely, assume that μ attains its infimum on all normal B-subalgebras of X. Let $U(\mu;\alpha)$ be an upper level cut of μ. If $\alpha = \alpha_i$ for some $i \in \Lambda$, then clearly $U(\mu;\alpha) \in \mathcal{B}$. Assume that $\alpha \neq \alpha_i$ for all $i \in \Lambda$. Then there does not exist $x \in X$ such that $\mu(x) = \alpha$. Let $N = \{x \in X \mid \mu(x) > \alpha\}$. Let $a, b, x, y \in X$ be such that $x * y \in N$ and $a * b \in N$. Then $\mu(x * y) > \alpha$ and $\mu(a * b) > \alpha$. It follows that

$$\mu((x * a) * (y * b)) \geqslant \min\{\mu(x * y), \mu(a * b)\} > \alpha$$

so that $(x * a) * (y * b) \in N$. This shows that N is a normal B-subalgebra of X. By hypothesis, there exists $y \in N$ such that $\mu(y) = \inf\{\mu(x) \mid x \in N\}$. Now $\mu(y) \in \mathrm{Im}(\mu)$ implies $\mu(y) = \alpha_i$ for some $i \in \Lambda$. Hence we get $\inf\{\mu(x) \mid x \in N\} = \alpha_i$. Obviously $\alpha_i \geqslant \alpha$, and so $\alpha_i > \alpha$ by assumption. Note that there does not exist $z \in X$ such that $\alpha \leqslant \mu(z) < \alpha_i$. It follows that $U(\mu;\alpha) = U(\mu;\alpha_i) \in \mathcal{B}$. This concludes the proof. □

Theorem 5.4. *Let μ be a fuzzy set in X with a finite image $\mathrm{Im}(\mu) = \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$ where $\alpha_i < \alpha_j$ whenever $i > j$. Let $\{N_n \mid n = 0, 1, \ldots, k\}$ be a family of normal B-subalgebras of X such that*

(i) $N_0 \subset N_1 \subset \ldots \subset N_k = X$,

(ii) $\mu(\widetilde{N_n}) = \alpha_n$ *where* $\widetilde{N_n} = N_n \setminus N_{n-1}$ *and* $N_{-1} = \emptyset$ *for* $n = 0, 1, \ldots, k$.

Then μ is a fuzzy normal B-algebra.

P r o o f. According to Theorem 4.4, it is sufficient to show that μ is a fuzzy normal set in X. Let $a, b, x, y \in X$. If $x * y \in \widetilde{N_n}$ and $a * b \in \widetilde{N_n}$ for every n, then $(x * a) * (y * b) \in N_n$ since N_n is a normal B-subalgebra of X. Hence

$$\mu((x * a) * (y * b)) \geqslant \alpha_n = \min\{\mu(x * y), \mu(a * b)\}.$$

If $x * y \in \widetilde{N_n}$ and $a * b \in \widetilde{N_m}$ where $0 \leqslant m < n \leqslant k$, then $x * y \in N_n$ and $a * b \in N_m \subseteq N_n$. It follows that $(x * a) * (y * b) \in N_n$. Therefore

$$\mu((x * a) * (y * b)) \geqslant \alpha_n = \mu(x * y).$$

Since $m < n$ implies $\alpha_n < \alpha_m$, we have $\mu(a * b) = \alpha_m < \alpha_n$. Consequently,

$$\mu((x * a) * (y * b)) \geqslant \alpha_n = \min\{\mu(x * y), \mu(a * b)\}.$$

Similarly for the case $x * y \in \widetilde{N_m}$ and $a * b \in \widetilde{N_n}$ for $0 \leqslant m < n \leqslant k$, proving the result. □

We have introduced the notion of fuzzy (normal) B-algebras and discussed its characterization. This ideas could enable us to discuss the direct products of fuzzy (normal) B-algebras, fuzzy topological B-algebras, and offer a new construction of quotient B-algebras via fuzzy B-algebras. They also suggest possible problems to fuzzify the quotient B-algebras discussed in [10], and compare them with two fuzzified quotient B-algebras.

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