Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 2, 385–399

Persistent URL: `http://dml.cz/dmlcz/127727`

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LIAPUNOV-TYPE INEQUALITY FOR DELAY-DIFFERENTIAL EQUATIONS OF THIRD ORDER

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(Received May 27, 1999)

Abstract. A Liapunov-type inequality for a class of third order delay-differential equations is derived.

Keywords: Liapunov-type inequality, oscillatory solution, third order delay-differential equation

MSC 2000: 34C10

1.

It is well-known ([5]) that, for a solution $y(t)$ of

$$y'' + p(t)y = 0 \tag{1}$$

with $y(a) = 0 = y(b)(a < b)$ and $y(t) \neq 0$, $t \in (a, b)$, we have

$$(b - a)\int_a^b |p(t)|\,\mathrm{d}t > 4. \tag{2}$$

Clearly, this provides an implicit lower bound on the distance between the zeros a and b of y. It can also be used to obtain a lower bound on the first positive eigen-value of the Sturm-Liouville problem

$$\begin{aligned} &y'' + (p(t) + \lambda q(t))y = 0,\\ &y(a) = y(b) = 0. \end{aligned}$$

Further, the inequality (2) has found applications in the study of various properties of solutions of (1) ([6, 8]).

In recent years, many authors have generalized this inequality in different directions. The inequality (2) has been generalized to second order nonlinear differential equations by Eliason [2], to delay-differential equations of second order by Eliason [3, 4] and by Dahiya and Singh [1], and to higher order ordinary differential equations by Pachpatte [6]. However, the work in [6] is not applicable to third order differential equations. In an earlier work [7], the authors obtained

$$(b-a)^2 \int_a^b |p(t)| \, \mathrm{d}t > 4 \tag{3}$$

for a solution $y(t)$ of

$$y''' + p(t)y = 0 \tag{4}$$

with $y(a) = y(b) = 0 (a < b)$ and $y(t) \neq 0$, $t \in (a, b)$, provided there exists $a\, d \in [a, b]$ such that $y''(d) = 0$; otherwise,

$$(a'-a)^2 \int_a^{a'} |p(t)| \, \mathrm{d}t > 4, \tag{5}$$

where a, b, a' $(a < b < a')$ are consecutive zeros of $y(t)$ such that $y(t) \neq 0$ for $t \in (a, b)$ and $y(t) \neq 0$ for $t \in (b, a')$. In [3], S. B. Eliason obtained a lower bound for the distance between two consecutive zeros of a solution of the second order delay-differential equation of the form

$$y''(t) + p(t)|y(t)|^{\mu} \operatorname{sgn} y(t) + m(t)|y(g(t))|^{\nu} \operatorname{sgn} y(g(t)) = 0, \quad t \geqslant 0,$$

where p and m are non-negative functions on $[0, \infty)$, $\mu > 0$, $\nu > 0$ and g is a strictly increasing real-valued function on $[0, \infty)$.

In this paper we generalize the inequalities (3) and (5) to third order delay-differential equations of the form

$$y'''(t) + p(t)|y(t)|^{\mu} \operatorname{sgn} y(t) + m(t)|y(t-\tau)|^{\nu} \operatorname{sgn} y(t-\tau) = 0, \tag{6}$$

where p, $m \in C([0, \infty), \mathbb{R})$, $\mu > 0$, $\nu > 0$ and $\tau \geqslant 0$. For simplicity, we take $t - \tau$ in place of $g(t)$. If $p(t) \geqslant 0$ and $m(t) \geqslant 0$, then the bounds obtained in this paper are better than (3) or (5). These bounds will be derived in Section 2. In Section 3, we obtain such inequalities without any sign restriction on p and m.

2.

Consider Eq. (6) with $p(t) \geqslant 0$ and $m(t) \geqslant 0$, $t \geqslant 0$. Let $y(t)$ be a solution of (6) with $y(a) = y(b) = 0$ and $y(t) > 0$ for $t \in (a, b)$. This is called a '*positive semi-cycle*' by Eliason [3]. Let $y(t) \leqslant 0$ on $[a, b]^-$, where

$$[a, b]^- = \{t \in \mathbb{R}\colon\ t \leqslant a \text{ and } t \geqslant x - \tau \text{ for some } x \in [a, b]\}.$$

The existence of such $x \in [a, b]$ is obvious. For example, $[a, b]^- = [a - \tau, a]$ if $x = a$ and $[a, b]^- = \{a\}$ if $x = a + \tau$. Let

$$y'(t^*) = \max\{y'(t)\colon\ t \in [a, b]\}.$$

Hence $t^* \in [a, b)$ and $y'(t^*) > 0$. We consider two cases, viz. $y''(t^*) = 0$ and $y''(t^*) \neq 0$. It may be noted that $t^* \in (a, b)$ implies $y''(t^*) = 0$. If $t^* = a$, then $y''(t^*) = 0$ or $y''(t^*) \neq 0$. In the latter case, we consider two *semi-cycles*, viz. $y(a') = y(a) = y(b) = 0$ $(a' < a < b)$, $y(t) < 0$ for $t \in (a', a)$ and $y(t) > 0$ for $t \in (a, b)$. Hence there exist $c_1 \in (a', a)$ and $c_2 \in (a, b)$ such that $y'(c_1) = y'(c_2) = 0$ and $y'(t) > 0$ for $t \in (c_1, c_2)$. Thus there exists a $t_1 \in (c_1, c_2)$ such that $y''(t_1) = 0$.

Theorem 2.1. (i) *If $t^* \in (a, b)$, then*

$$2 \leqslant (b - t^*)^{-1}\Big\{[y'(t^*)]^{\mu-1}\int_{t^*}^{b}(b-\theta)^2(\theta-a)^{\mu}p(\theta)\,\mathrm{d}\theta + [y'(t^*)]^{\nu-1}\int_{A}^{B}(b-\theta)^2(\theta-\tau-a)^{\nu}m(\theta)\,\mathrm{d}\theta\Big\}, \tag{7}$$

where

$$A = \begin{cases} t^*, & \text{if } a + \tau \leqslant t^* \\ a + \tau, & \text{if } t^* < a + \tau < b \\ b, & \text{if } b \leqslant a + \tau \end{cases}$$

and $B = b$.

(ii) *If $t^* = a$ is such that $y''(t^*) = 0$, then (7) holds.*

(iii) *Let $t^* = a$ be such that $y''(t^*) \neq 0$. If $c_1 - a' \geqslant \tau$, then (7) holds.*

Proof. (i) Integrating (6) from t^* to t $(t^* < t \leqslant b)$ we obtain

$$y''(t) + \int_{t^*}^{t} p(s)y^{\mu}(s)\,\mathrm{d}s + \int_{t^*}^{t} m(s)|y(s-\tau)|^{\nu}\operatorname{sgn} y(s-\tau)\,\mathrm{d}s = 0.$$

Further integration from t^* to t yields

$$y'(t) = y'(t^*) - \int_{t^*}^{t} (t-\theta)p(\theta)y^{\mu}(\theta)d\theta \\ - \int_{t^*}^{t} (t-\theta)m(\theta)|y(\theta-\tau)|^{\nu} \operatorname{sgn} y(\theta-\tau)\,\mathrm{d}\theta.$$

Integrating the above identity from t^* to b, we get

$$0 > -y(t^*) = y'(t^*)(b-t^*) - \int_{t^*}^{b} \left(\int_{t^*}^{t} (t-\theta)p(\theta)y^{\mu}(\theta)\,\mathrm{d}\theta \right) \mathrm{d}t \\ - \int_{t^*}^{b} \left(\int_{t^*}^{t} (t-\theta)m(\theta)|y(\theta-\tau)|^{\nu} \operatorname{sgn} y(\theta-\tau)d\theta \right) \mathrm{d}t,$$

that is,

$$(b-t^*)y'(t^*) < \frac{1}{2}\int_{t^*}^{b} (b-\theta)^2 p(\theta)y^{\mu}(\theta)\,\mathrm{d}\theta \\ + \frac{1}{2}\int_{t^*}^{b} (b-\theta)^2 m(\theta)|y(\theta-\tau)|^{\nu} \operatorname{sgn} y(\theta-\tau)\,\mathrm{d}\theta. \tag{8}$$

We consider two cases, viz. $a+\tau \geqslant b$ or $a+\tau < b$. If $a+\tau \geqslant b$, then $\theta-\tau < a$ for $t^* < \theta < b$ and hence

$$\int_{t^*}^{b} (b-\theta)^2 m(\theta)|y(\theta-\tau)|^{\nu} \operatorname{sgn} y(\theta-\tau)d\theta \leqslant 0,$$

since $y(t) \leqslant 0$ on $[a,b]^-$. Further,

$$\int_{t^*}^{b} (b-\theta)^2 m(\theta)y^{\nu}(aV(\theta-\tau))d\theta = y^{\nu}(a)\int_{t^*}^{b} (b-\theta)^2 m(\theta)\,\mathrm{d}\theta = 0,$$

where $uVz = \max\{u,z\}$. Hence

$$\int_{t^*}^{b} (b-\theta)^2 m(\theta)|y(\theta-\tau)|^{\nu} \operatorname{sgn} y(\theta-\tau)d\theta \leqslant \int_{t^*}^{b} (b-\theta)^2 m(\theta)y^{\nu}(aV(\theta-\tau))d\theta.$$

Let $a+\tau < b$. If $a+\tau \leqslant t^*$, then $a \leqslant \theta-\tau$ and hence

$$\int_{t^*}^{b} (b-\theta)^2 m(\theta)|y(\theta-\tau)|^{\nu} \operatorname{sgn} y(\theta-\tau)\,\mathrm{d}\theta \\ = \int_{t^*}^{b} (b-\theta)^2 m(\theta)y^{\nu}(\theta-\tau)d\theta \\ = \int_{t^*}^{b} (b-\theta)^2 m(\theta)y^{\nu}(aV(\theta-\tau))\,\mathrm{d}\theta.$$

If $t^* < a+\tau$, then

$$\begin{aligned}\int_{t^*}^{b} &(b-\theta)^2 m(\theta)|y(\theta-\tau)|^\nu \operatorname{sgn} y(\theta-\tau) d\theta \\ &= \int_{t^*}^{a+\tau} (b-\theta)^2 m(\theta)|y(\theta-\tau)|^\nu \operatorname{sgn} y(\theta-\tau) d\theta \\ &+ \int_{a+\tau}^{b} (b-\theta)^2 m(\theta)|y(\theta-\tau)|^\nu \operatorname{sgn} y(\theta-\tau)\, \mathrm{d}\theta \\ &\leqslant \int_{a+\tau}^{b} (b-\theta)^2 m(\theta) y^\nu(\theta-\tau) d\theta \\ &= \int_{a+\tau}^{b} (b-\theta)^2 m(\theta) y^\nu(a \vee (\theta-\tau))\, \mathrm{d}\theta \\ &= \int_{t^*}^{b} (b-\theta)^2 m(\theta) y^\nu(a \vee (\theta-\tau))\, \mathrm{d}\theta.\end{aligned}$$

Thus from (8) we get

$$\begin{aligned}(b-t^*)y'(t^*) < \frac{1}{2}\int_{t^*}^{b} &(b-\theta)^2 p(\theta) y^\mu(\theta)\, \mathrm{d}\theta \\ &+ \frac{1}{2}\int_{t^*}^{b} (b-\theta)^2 m(\theta) y^\nu(a \vee (\theta-\tau))\, \mathrm{d}\theta.\end{aligned} \tag{9}$$

If $a+\tau \leqslant t^*$, then $a+\tau < \theta$ and hence

$$\int_{t^*}^{b} (b-\theta)^2 m(\theta) y^\nu(a \vee (\theta-\tau))\, \mathrm{d}\theta = \int_{A}^{B} (b-\theta)^2 m(\theta) y^\nu(\theta-\tau)\, \mathrm{d}\theta.$$

If $t^* < a+\tau < b$, then

$$\begin{aligned}\int_{t^*}^{b} (b-\theta)^2 m(\theta) y^\nu(a \vee (\theta-\tau))\, \mathrm{d}\theta &= \int_{t^*}^{a+\tau} (b-\theta)^2 m(\theta) y^\nu(a \vee (\theta-\tau))\, \mathrm{d}\theta \\ &+ \int_{a+\tau}^{b} (b-\theta)^2 m(\theta) y^\nu(a \vee (\theta-\tau))\, \mathrm{d}\theta \\ &= y^\nu(a)\int_{t^*}^{a+\tau} (b-\theta)^2 m(\theta)\, \mathrm{d}\theta + \int_{a+\tau}^{b} (b-\theta)^2 m(\theta) y^\nu(\theta-\tau)\, \mathrm{d}\theta \\ &= \int_{a+\tau}^{b} (b-\theta)^2 m(\theta) y^\nu(\theta-\tau)\, \mathrm{d}\theta = \int_{A}^{B} (b-\theta)^2 m(\theta) y^\nu(\theta-\tau)\, \mathrm{d}\theta.\end{aligned}$$

If $b \leqslant a+\tau$, then

$$\begin{aligned}\int_{t^*}^{b} (b-\theta)^2 m(\theta) y^\nu(a \vee (\theta-\tau)) d\theta &= y^\nu(a)\int_{t^*}^{b} (b-\theta)^2 m(\theta)\, \mathrm{d}\theta \\ &= 0 = \int_{A}^{B} (b-\theta)^2 m(\theta) y^\nu(\theta-\tau)\, \mathrm{d}\theta.\end{aligned}$$

Hence (9) yields

$$
\begin{aligned}
(b-t^*)y'(t^*) &< \frac{1}{2}\int_{t^*}^{b}(b-\theta)^2 p(\theta)y^{\mu}(\theta)\,\mathrm{d}\theta \\
&\quad + \frac{1}{2}\int_{A}^{B}(b-\theta)^2 m(\theta)y^{\nu}(\theta-\tau)\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{t^*}^{b}(b-\theta)^2 p(\theta)\Big[\int_a^{\theta} y'(s)\,\mathrm{d}s\Big]^{\mu}\,\mathrm{d}\theta \\
&\quad + \frac{1}{2}\int_{A}^{B}(b-\theta)^2 m(\theta)\Big[\int_a^{\theta-\tau} y'(s)\,\mathrm{d}s\Big]^{\nu}\,\mathrm{d}\theta \\
&\leqslant \frac{1}{2}[y'(t^*)]^{\mu}\int_{t^*}^{b}(b-\theta)^2 p(\theta)(\theta-a)^{\mu}\,\mathrm{d}\theta \\
&\quad + \frac{1}{2}[y'(t^*)]^{\nu}\int_{A}^{B}(b-\theta)^2 m(\theta)(\theta-\tau-a)^{\nu}\,\mathrm{d}\theta,
\end{aligned}
$$

that is,

$$
\begin{aligned}
2 < (b-t^*)^{-1}\Big\{&[y'(t^*)]^{\mu-1}\int_{t^*}^{b}(b-\theta)^2 p(\theta)(\theta-a)^{\mu}\,\mathrm{d}\theta \\
&+ [y'(t^*)]^{\nu-1}\int_{A}^{B}(b-\theta)^2 m(\theta)(\theta-\tau-a)^{\nu}\,\mathrm{d}\theta\Big\}.
\end{aligned}
$$

Hence the first part of the theorem is proved.

(ii) In this case the proof is similar and hence is omitted.

(iii) We claim that $t_1 \notin (a, c_2)$. If not, then $a < t_1 < c_2$. If $y''(t) < 0$ for $t \in (a, t_1)$ and $y''(t) < 0$ for $t \in (t_1, c_2)$, then $y'''(t_1) = 0$, a contradiction. Hence $y''(t) > 0$ for $t \in (a, t_1)$. Since $y'(t)$ is increasing in (a, t_1), we have $y'(t_1) > y'(a) = y'(t^*)$, a contradiction due to the definition of $y'(t^*)$. Then our claim holds. Hence $t_1 \in (c_1, a]$. The case $t_1 = a = t^*$ has been discussed earlier. Thus we take $t_1 \in (c_1, a)$. Integrating (6) from t_1 to t ($a < t \leqslant b$) we obtain

$$
\begin{aligned}
y''(t) &+ \int_{t_1}^{a}[p(s)|y(s)|^{\mu}\operatorname{sgn} y(s) + m(s)|y(s-\tau)|^{\nu}\operatorname{sgn} y(s-\tau)]\,\mathrm{d}s \\
&+ \int_a^t [p(s)y^{\mu}(s) + m(s)|y(s-\tau)|^{\nu}\operatorname{sgn} y(s-\tau)]\,\mathrm{d}s = 0.
\end{aligned}
$$

Since $c_1 - a' \geqslant \tau$, we have

$$
y''(t) + \int_a^t [p(s)y^{\mu}(s) + m(s)|y(s-\tau)|^{\nu}\operatorname{sgn} y(s-\tau)]\,\mathrm{d}s > 0.
$$

Integrating the above inequality from a to t ($a < t \leqslant b$) we obtain

$$y'(t) > y'(a) - \int_a^t (t-s)[p(s)y^\mu(s) + m(s)|y(s-\tau)|^\nu \operatorname{sgn} y(s-\tau)]\,\mathrm{d}s.$$

Further integration from a to b yields

$$(b-a)y'(a) < \frac{1}{2}\int_a^b (b-s)^2[p(s)y^\mu(s) + m(s)|y(s-\tau)|^\nu \operatorname{sgn} y(s-\tau)]\,\mathrm{d}s.$$

Since $a = t^*$, then proceeding as in case (i) of the proof we obtain (7). Thus the theorem is proved. □

Remark 1. If we consider a *negative semi-cycle*, viz. $y(a) = 0 = y(b)$, $y(t) < 0$ for $t \in (a,b)$, then we get

$$\begin{aligned}2 \leqslant (b-t^*)^{-1}\biggl\{|y'(t^*)|^{\mu-1}\int_{t^*}^b (b-s)^2(s-a)^\mu p(s)\,\mathrm{d}s \\ + |y'(t^*)|^{\nu-1}\int_A^B (b-s)^2(s-\tau-a)^\nu m(s)\,\mathrm{d}s\biggr\},\end{aligned}$$

where A and B are the same as in (7). In this case, we assume $y(t) \geqslant 0$ on $[a,b]^-$ and $y'(t^*) = \min\{y'(t)\colon\ t \in [a,b]\}$.

Thus $a \leqslant t^* < b$ and $y'(t^*) < 0$.

Remark 2. If $A = b$, then $\int_A^B m(\theta)\,\mathrm{d}\theta = 0$. If $p(t) \equiv 0$ in this case, then the inequality (7) does not hold.

Remark 2′. In the case of second order delay equations it is possible (see [3, p. 184]) to predict the location of t^* in some situations. However, such a prediction is not possible in the case of third order delay equations.

Remark 3. For an equation of the form

$$\text{(10)}\qquad y'''(t) + q(t)y'(t) + p(t)|y(t)|^\mu \operatorname{sgn} y(t) + m(t)|y(t-\tau)|^\nu \operatorname{sgn} y(t-\tau) = 0,$$

where p, m, μ, ν and τ are the same as in (6) and $q \in C([0,\infty),\mathbb{R})$, we may have a theorem similar to Theorem 2.1. Indeed, if $p(t) \geqslant 0$, $q(t) \geqslant 0$ and $m(t) \geqslant 0$, then the following theorem holds.

Theorem 2.2. (i) *If* $t^* \in (a,b)$, *then*

$$(11)\qquad 2 \leqslant (b-t^*)^{-1}\Big\{\int_{t^*}^{b}(b-s)^2 q(s)\,\mathrm{d}s + [y'(t^*)]^{\mu-1}\int_{t^*}^{b}(b-s)^2(s-a)^{\mu}p(s)\,\mathrm{d}s + [y'(t^*)]^{\nu-1}\int_{A}^{B}(b-s)^2(s-\tau-a)^{\nu}m(s)\,\mathrm{d}s\Big\},$$

where A *and* B *are the same as in* (7).

(ii) *If* $t^* = a$ *is such that* $y''(t^*) = 0$, *then* (11) *holds.*

Remark 4. A result similar to Theorem 2.1 (iii) does not hold for (10) since $y'(t) > 0$ for $t \in (c_1, a]$.

In the sequel we consider the '*positive left-quarter cycle case*', viz. $y(a) = 0$, $y'(c) = 0$ and $y(t) > 0$ for $t \in (a,c]$, where $y(t)$ is a solution of (6). We continue to assume that $p(t) \geqslant 0$ and $m(t) \geqslant 0$. Let $y(t) \leqslant 0$ for $t \in [a,c]^-$, where

$$[a,c]^- = \{t \in \mathbb{R}\colon\ t \leqslant a \quad \text{and} \quad t \geqslant x-\tau \quad \text{for some} \quad x \in [a,c]\}.$$

Let $y'(t^*) = \max\{y'(t)\colon\ a \leqslant t \leqslant c\}$. Hence $a \leqslant t^* < c$ and $y'(t^*) > 0$. If $t^* \in (a,c)$, then $y''(t^*) = 0$. If $t^* = a$ is such that $y''(t^*) \neq 0$, then we consider a *negative semi-cycle* left to the *positive quarter cycle*, viz. $y(a') = y(a) = 0$, $y(t) < 0$ for $t \in (a', a)$, $y'(c) = 0$ and $y(t) > 0$ for $t \in (a,c]$, where $a' < a < c$. Then there exists a $c_1 \in (a', a)$ such that $y'(c_1) = 0$. Hence there is $t_1 \in (c_1, c)$ such that $y''(t_1) = 0$. As in '*semi-cycle*' case, we can show that $t_1 \in (c_1, a]$. We have the following theorem,

Theorem 2.3. (i) *If* $t^* \in (a,c)$, *then*

$$(12)\quad 1 \leqslant [y'(t^*)]^{\mu-1}\int_{t^*}^{c}(c-t)(t-a)^{\mu}p(t)\,\mathrm{d}t + [y'(t^*)]^{\nu-1}\int_{A}^{B}(c-t)(t-\tau-a)^{\nu}m(t)\,\mathrm{d}t,$$

where

$$A = \begin{cases} t^*, & \text{if}\quad a+\tau \leqslant t^* \\ a+\tau & \text{if}\quad t^* < a+\tau < c \\ c, & \text{if}\quad c \leqslant a+\tau \end{cases}$$

and $B = c$.

(ii) *If* $t^* = a$ *is such that* $y''(t^*) = 0$, *then* (12) *holds.*

(iii) *Let* $t^* = a$ *be such that* $y''(t^*) \neq 0$. *If* $c_1 - a' \geqslant \tau$, *then* (12) *holds.*

P r o o f. (i) Integrating (6) successively, first from t^* to t ($t^* < t < c$) and then from t^* to c, we obtain

$$y'(t^*) = \int_{t^*}^{c} (c-t)p(t)y^{\mu}(t)\,\mathrm{d}t + \int_{t^*}^{c} (c-t)m(t)|y(t-\tau)|^{\nu}\operatorname{sgn} y(t-\tau)\,\mathrm{d}t.$$

If $a+\tau \geqslant c$, then

$$\int_{t^*}^{c} (c-t)m(t)|y(t-\tau)|^{\nu}\operatorname{sgn} y(t-\tau)\,\mathrm{d}t \leqslant 0 = \int_{t^*}^{c} (c-t)m(t)y^{\nu}(aV(t-\tau))\,\mathrm{d}t.$$

Let $a+\tau < c$. If $a+\tau \leqslant t^*$, then

$$\begin{aligned}\int_{t^*}^{c} (c-t)m(t)|y(t-\tau)|^{\nu}\operatorname{sgn} y(t-\tau)\,\mathrm{d}t &= \int_{t^*}^{c} (c-t)m(t)y^{\nu}(t-\tau)\,\mathrm{d}t\\ &= \int_{t^*}^{c} (c-t)m(t)y^{\nu}(aV(t-\tau))\,\mathrm{d}t.\end{aligned}$$

If $a+\tau > t^*$, then

$$\begin{aligned}\int_{t^*}^{c} (c-t)m(t)|y(t-\tau)|^{\nu}\operatorname{sgn} y(t-\tau)\,\mathrm{d}t &\leqslant \int_{a+\tau}^{c} (c-t)m(t)y^{\nu}(t-\tau)\,\mathrm{d}t\\ &= \int_{a+\tau}^{c} (c-t)m(t)y^{\nu}(aV(t-\tau))\,\mathrm{d}t\\ &= \int_{t^*}^{c} (c-t)m(t)y^{\nu}(aV(t-\tau))\,\mathrm{d}t.\end{aligned}$$

Hence

$$y'(t^*) \leqslant \int_{t^*}^{c} (c-t)p(t)y^{\mu}(t)\,\mathrm{d}t + \int_{t^*}^{c} (c-t)m(t)y^{\nu}(aV(t-\tau))\,\mathrm{d}t. \tag{13}$$

Proceeding as in the proof of Theorem 2.1 (i), we can show that

$$\int_{t^*}^{c} (c-t)m(t)y^{\nu}(aV(t-\tau))\,\mathrm{d}t = \int_{A}^{B} (c-t)m(t)y^{\nu}(t-\tau)\,\mathrm{d}t.$$

Thus (13) yields

$$\begin{aligned}y'(t^*) &\leqslant \int_{t^*}^{c} (c-t)p(t)\Big(\int_a^t y'(s)\,\mathrm{d}s\Big)^{\mu}\mathrm{d}t + \int_{A}^{B} (c-t)m(t)\Big(\int_a^{t-\tau} y'(s)\,\mathrm{d}s\Big)^{\nu}\mathrm{d}t\\ &\leqslant [y'(t^*)]^{\mu}\int_{t^*}^{c} (c-t)p(t)(t-a)^{\mu}\,\mathrm{d}t + [y'(t^*)]^{\nu}\int_{A}^{B} (c-t)m(t)(t-\tau-a)^{\nu}\,\mathrm{d}t.\end{aligned}$$

That is,

$$1 \leqslant [y'(t^*)]^{\mu-1} \int_{t^*}^{c} (c-t)(t-a)^{\mu} p(t)\,\mathrm{d}t + [y'(t^*)]^{\nu-1} \int_{A}^{B} (c-t)(t-\tau-a)^{\nu} m(t)\,\mathrm{d}t.$$

Hence the proof of part (i) is complete. The proof of part (ii) is similar to that of part (i). One may proceed as in the proof of Theorem 2.1 (iii) to complete the present proof. □

Remark 5. For a *negative left-quarter cycle*, that is, when $y(a)=0$, $y'(c)=0$ and $y(t)<0$, $t\in(a,c]$, one gets

$$(14)\quad 1 \leqslant |y'(t^*)|^{\mu-1} \int_{t^*}^{c} (c-t)(t-a)^{\mu} p(t)\,\mathrm{d}t + |y'(t^*)|^{\nu-1} \int_{A}^{B} (c-t)(t-\tau-a)^{\nu} m(t)\,\mathrm{d}t,$$

where A and B are the same as in (12) and

$$y'(t^*) = \min\{y'(t)\colon\ a \leqslant t \leqslant c\}.$$

Here we assume $y(t) \geqslant 0$, $t \in [a,c]^-$.

Remark 6. If $p(t)\equiv 0$ and $A=c$, then (12) fails to hold.

Example 1. Consider

$$y'''(t) + 2y(t) + 2e^{\pi/2} y\Big(t-\frac{\pi}{2}\Big) = 0.$$

Clearly, $y(t)=e^t\sin t$ is a solution of the equation with $y(0)=0$, $y(\pi)=0$, $y(t)\geqslant 0$ for $t\in(0,\pi]$ and $y(t)\leqslant 0$ for $t\in[-\pi,0]=[0,\pi]^-$. Since here $t^*=\frac{\pi}{2}$ and $a+\tau=\frac{\pi}{2}$, we have $A=\frac{\pi}{2}$ and $B=\pi$. Further, $t^*\in(0,\pi)$ implies that $y''(t^*)=0$. Clearly,

$$\Big(\pi-\frac{\pi}{2}\Big)^{-1}\Big\{\int_{\pi/2}^{\pi} 2(\pi-s)^2 s\,\mathrm{d}s + \int_{\pi/2}^{\pi} 2(\pi-s)^2\Big(s-\frac{\pi}{2}\Big)e^{\pi/2}\,\mathrm{d}s\Big\} = \frac{\pi^3}{48}(5+e^{\pi/2}) > 2.$$

Thus (7) is verified.

Example 2. Clearly, $y(t)=e^t\cos t$ is a solution of

$$y'''(t) + 2y(t) + 2e^{5\pi/2} y\Big(t-\frac{5\pi}{2}\Big) = 0$$

with $y(-\frac{\pi}{2})=y(\frac{\pi}{2})=0$ and $y(t)>0$ for $-\frac{\pi}{2}<t<\frac{\pi}{2}$. Here $t^*=0\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $a+\tau=2\pi>\frac{\pi}{2}$ and hence $A=B=\frac{\pi}{2}$. Thus the second integral of (7) is zero. Clearly,

$$\frac{2}{\pi}\int_0^{\pi/2} 2\Big(\frac{\pi}{2}-\theta\Big)^2\Big(\theta+\frac{\pi}{2}\Big)d\theta = \frac{5\pi^3}{48} > 2$$

and hence (7) is true.

3.

In this section we apply another technique to obtain a lower bound of the distance between two or three consecutive zeros of a solution of (6). We do not assume that $p(t)$ and $m(t)$ are non-negative functions. However, the bound obtained in Section 2 is better than the bound obtained in this section. There are several instances where the analysis of Section 2 fails while the results of this section can be applied. These remarks will be illustrated through suitable examples.

Let $y(t)$ be a solution of (6) with $y(a) = y(b) = 0$ $(a < b)$ and $y(t) \neq 0$ for $a < t < b$. In some cases it is possible to find a $d \in [a,b]$ such that $y''(d) = 0$. If $y''(t) \neq 0$ for $t \in [a,b]$, then we consider three consecutive zeros of $y(t)$, viz. $y(a) = y(b) = y(a') = 0$ $(a < b < a')$ and $y(t) \neq 0$ for $t \in (a,b) \cup (b,a')$. Then there exist $c_1 \in (a,b)$ and $c_2 \in (b,a')$ such that $y'(c_1) = y'(c_2) = 0$. Hence we can find $d \in (c_1, c_2)$ such that $y''(d) = 0$. Since $y''(t) \neq 0$ for $t \in [a,b]$, we have $d \in (b, c_2)$. Thus we consider two cases, viz. $y''(d) = 0$ for some $d \in [a,b]$ and $y''(t) \neq 0$ for $t \in [a,b]$.

Theorem 3.1. (i) *If there exists a $d \in [a,b]$ such that $y''(d) = 0$, then*

$$\int_a^b [|p(t)| + |m(t)|]\,\mathrm{d}t \geqslant \frac{4M_1}{L(b-a)^2},$$

where $M_1 = \max\{|y(t)|\colon a \leqslant t \leqslant b\}$ and $L = \max\{M_1^{\mu}, M_2^{\nu}\}$ and $M_2 = \max\{|y(t)|\colon a - \tau \leqslant t \leqslant b\}$.

(ii) *If $y''(t) \neq 0$ for $t \in [a,b]$, then*

$$\int_a^{a'} [|p(t)| + |m(t)|]\,\mathrm{d}t \geqslant \frac{4M_3}{M(b-a)^2},$$

where $M_3 = \max\{|y(t)|\colon a \leqslant t \leqslant a'\}$, $M = \max\{M_3^{\mu}, M_4^{\nu}\}$ and

$$M_4 = \max\{|y(t)|\colon a - \tau \leqslant t \leqslant a'\}.$$

P r o o f. (i) There exists $c \in (a,b)$ such that $M_1 = |y(c)|$. Since

$$M_1 = \left|\int_a^c y'(t)\,\mathrm{d}t\right| \leqslant \int_a^c |y'(t)|\,\mathrm{d}t$$

and

$$M_1 = \left|\int_c^b y'(t)\,\mathrm{d}t\right| \leqslant \int_c^b |y'(t)|\,\mathrm{d}t,$$

then

$$2M_1 \leqslant \int_a^b |y'(t)|\,\mathrm{d}t.$$

Hence, using the Cauchy-Schwarz inequality, we obtain

$$4M_1^2 \leqslant \left(\int_a^b |y'(t)|\,\mathrm{d}t\right)^2 \leqslant (b-a)\int_a^b (y'(t))^2\,\mathrm{d}t = -(b-a)\int_a^b y(t)y''(t)\,\mathrm{d}t$$

$$\leqslant (b-a)\int_a^b |y(t)||y''(t)|\,\mathrm{d}t, \tag{15}$$

where integration by parts is performed. Integrating (6) from d to t if $d < t$ or from t to d if $t < d$, we obtain

$$y''(t) = -\int_d^t p(s)|y(s)|^{\mu}\operatorname{sgn} y(s)\,\mathrm{d}s - \int_d^t m(s)|y(s-\tau)|^{\nu}\operatorname{sgn} y(s-\tau)\,\mathrm{d}s.$$

Thus

$$\begin{aligned}|y''(t)| &\leqslant \left|\int_d^t |p(s)||y(s)|^{\mu}\,\mathrm{d}s\right| + \left|\int_d^t |m(s)||y(s-\tau)|^{\nu}\,\mathrm{d}s\right| \\ &\leqslant M_1^{\mu}\int_a^b |p(t)|\,\mathrm{d}t + M_2^{\nu}\int_a^b |m(t)|\,\mathrm{d}t \leqslant L\int_a^b [|p(t)| + |m(t)|]\,\mathrm{d}t.\end{aligned}$$

Hence (15) yields

$$4M_1^2 \leqslant (b-a)^2 M_1 L\int_a^b [|p(t)| + |m(t)|]\,\mathrm{d}t,$$

that is,

$$\int_a^b [|p(t)| + |m(t)|]\,\mathrm{d}t \geqslant \frac{4M_1}{L(b-a)^2}.$$

Thus part (i) of the theorem is proved.

(ii) In this case, there exists a $d \in (b, c_2)$ such that $y''(d) = 0$. Clearly, there is a $c \in (a,b) \cup (b,a')$ such that $M_3 = |y(c)|$. Proceeding as in part (i) of the proof, we obtain

$$4M_3^2 \leqslant (a'-a)\int_a^{a'} |y(t)||y''(t)|\,\mathrm{d}t$$

and

$$|y''(t)| \leqslant M\int_a^{a'} [|p(t)| + |m(t)|]\,\mathrm{d}t.$$

The required inequality is obtained from these two inequalities. Hence part (ii) of the theorem is proved. □

Remark 7. For Eq. (10) we get the following bounds:

(i) If $y''(d) = 0$ for some $d \in [a,b]$, then

$$4M_1 \leqslant (b-a)^2 \Big[M_1^{\mu} \int_a^b |p(t)| \, \mathrm{d}t + M_2^{\nu} \int_a^b |m(t)| \, \mathrm{d}t + 2M_1 q_1 + M_1 \int_a^b |q'(t)| \, \mathrm{d}t \Big],$$

where $q_1 = \max\{|q(t)| \colon \; t \in [a,b]\}$ and M_1, M_2 are the same as in Theorem 3.1 (i).

(ii) If $y''(t) \neq 0$ for $t \in [a,b]$, then

$$4M_3 \leqslant (a'-a)^2 \Big[M_3^{\mu} \int_a^{a'} |p(t)| \, \mathrm{d}t + M_4^{\nu} \int_a^{a'} |m(t)| \, \mathrm{d}t + 2M_3 q_2 + M_3 \int_a^{a'} |q'(t)| \, \mathrm{d}t \Big]$$

where $q_2 = \max\{|q(t)| \colon \; a \leqslant t \leqslant a'\}$ and M_3, M_4 are the same as in Theorem 3.1 (ii).

Example 3. Consider

$$y''' + y = 0, \quad t \geqslant 0.$$

Clearly, $u(t) = e^{t/2} \cos \frac{\sqrt{3}}{2} t$ is a solution of the equation with zeros given by

$$t_n = \frac{2}{\sqrt{3}} \frac{(2n-1)\pi}{2}, \quad n = 1, 2, \ldots$$

For $a = \frac{\pi}{\sqrt{3}}$ and $b = \frac{3\pi}{\sqrt{3}}$, we notice that $u(t) < 0$ for $t \in (a,b)$ and

$$u'\Big(\frac{5\pi}{3\sqrt{3}}\Big) = \min \Big\{ u'(t) \colon \; \frac{\pi}{\sqrt{3}} \leqslant t \leqslant \frac{3\pi}{\sqrt{3}} \Big\}.$$

Since $\frac{\pi}{\sqrt{3}} < \frac{5\pi}{3\sqrt{3}} < \frac{3\pi}{\sqrt{3}}$, we have $u''(\frac{5\pi}{3\sqrt{3}}) = 0$. From Theorem 3.1 (i) it follows that

$$(b-a)^3 \geqslant 4 \quad \text{or} \quad b - a \geqslant (4)^{1/3}.$$

However, from Theorem 2.1 (i) (see Remark 1) we obtain

$$2 \leqslant \frac{3\sqrt{3}}{4\pi} \int_{5\pi/3\sqrt{3}}^{3\pi/\sqrt{3}} \Big(\frac{3\pi}{\sqrt{3}} - s \Big)^2 \Big(s - \frac{\pi}{\sqrt{3}} \Big) \, \mathrm{d}s = \frac{16\pi^3}{81\sqrt{3}}.$$

Thus $\frac{2\pi}{\sqrt{3}} \geqslant 3$, that is, $(b-a) \geqslant 3$.

Hence the bound obtained by applying Theorem 2.1 (i) is better than the corresponding bound obtained by applying Theorem 3.1 (i).

Example 4. Clearly, $y(t) = \sin t$ is a solution of the equation

$$y''' + y' = 0$$

with $y(0) = 0$, $y(\pi) = 0$, $y(t) > 0$ for $t \in (0, \pi)$. As

$$y'(0) = \max\{y'(t)\colon\ 0 \leqslant t \leqslant \pi\}$$

and $y''(0) = 0$, from Theorem 2.2 (ii) we obtain

$$2 \leqslant \frac{1}{\pi} \int_0^{\pi} (\pi - s)^2 \, \mathrm{d}s = \frac{\pi^2}{3}.$$

Hence $b - a = \pi > \sqrt{6}$. On the other hand, from Remark 7 (i) one gets $4 < 2\pi^2$ or $b - a = \pi > \sqrt{2}$. Clearly, $b - a > \sqrt{6} > \sqrt{2}$.

However, there are occasions when we fail when applying the analysis of Section 2 but the results of Section 3 can be applied.

Example 5. Consider

$$y'''(t) + y\Big(t - \frac{3\pi}{2}\Big) = 0, \quad t \geqslant 0.$$

We may see that $y(t) = \sin t$ is a solution of the equation with $y(0) = y(\pi) = 0$, $y(t) > 0$ for $t \in (0, \pi)$ and $y(t) < 0$ for $t \in (-\pi, 0)$. Clearly, $y'(0) = \max\{y'(t)\colon\ 0 \leqslant t \leqslant \pi\}$ and $y''(0) = 0$. Theorem 2.1 (ii) cannot be applied since $a + \tau = 0 + \frac{3\pi}{2} > \pi = b$ and hence $A = B = \pi$ (see Remark 2). However, by Theorem 3.1 (i), we get

$$\int_0^{\pi} \mathrm{d}t \geqslant \frac{4}{\pi^2}, \text{ that is, } \pi > (4)^{1/3}, \text{ which is true.}$$

Example 6. No result of Section 2 can be applied to

$$y'''(t) - y\Big(t - \frac{\pi}{2}\Big) = 0, \quad t > 0,$$

since $m(t) < 0$. Clearly, $y(t) = \cos t$ is a solution of the equation with $y(-\frac{\pi}{2}) = y(\frac{\pi}{2}) = 0$, $y(t) > 0$ for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $y''(-\frac{\pi}{2}) = 0$. From Theorem 3.1 (i) it follows that

$$\int_{-\pi/2}^{\pi/2} \mathrm{d}t \geqslant \frac{4}{\pi^2}, \quad \text{that is,} \quad \pi > (4)^{1/3}.$$

Remark 8. The distribution of zeros of solutions of second order differential equations was discussed in [8] and of third order differential equations in [7]. However, for higher order and delay-differential equations, the nature of distributions of zeros of solutions is not known.

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