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ULTRA \textit{LI}-IDEALS IN LATTICE IMPLICATION ALGEBRAS

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Abstract. We define an ultra \textit{LI}-ideal of a lattice implication algebra and give equivalent conditions for an \textit{LI}-ideal to be ultra. We show that every subset of a lattice implication algebra which has the finite additive property can be extended to an ultra \textit{LI}-ideal.

Keywords: lattice implication algebra, (ultra) \textit{LI}-ideal, finite additive property

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INTRODUCTION

In order to research a logical system whose propositional value is given in a lattice, Y. Xu [5] proposed the concept of lattice implication algebras, and discussed some of their properties. Also, in [4], Y. Xu discussed the homomorphisms between lattice implication algebras. Y. Xu and K.Y. Qin [6] introduced the notion of filters in a lattice implication algebra, and investigated their properties. In [1], Y.B. Jun et al. proposed the concept of an \textit{LI}-ideal of a lattice implication algebra and discussed the relationship between filters and \textit{LI}-ideals, and studied how to generate an \textit{LI}-ideal by a set. This paper is devoted to the discussion of ultra \textit{LI}-ideals of lattice implication algebras. We give equivalent conditions for an \textit{LI}-ideal to be ultra. We show that every subset of a lattice implication algebra which has the finite additive property can be extended to an ultra \textit{LI}-ideal.
Preliminaries

By a lattice implication algebra we mean a bounded lattice \((L, \lor, \land, 0, 1)\) with order-reversing involution \(\neg\) and a binary operation \(\rightarrow\) satisfying the following axioms:

\begin{align*}
(I1) & \quad x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z), \\
(I2) & \quad x \rightarrow x = 1, \\
(I3) & \quad x \rightarrow y = y' \rightarrow x', \\
(I4) & \quad x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y, \\
(I5) & \quad (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x, \\
(L1) & \quad (x \lor y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z), \\
(L2) & \quad (x \land y) \rightarrow z = (x \rightarrow z) \lor (y \rightarrow z)
\end{align*}

for all \(x, y, z \in L\).

In the sequel the binary operation \(\rightarrow\) will be denoted by juxtaposition. We can define a partial ordering \(\leq\) on a lattice implication algebra \(L\) by \(x \leq y\) if and only if \(xy = 1\).

In a lattice implication algebra \(L\), the following relations hold (see [5]):

\begin{align*}
(1) & \quad 0x = 1, \quad 1x = x \quad \text{and} \quad x1 = 1, \\
(2) & \quad x' = x0, \\
(3) & \quad xy \leq (yz)(xz), \\
(4) & \quad x \lor y = (xy)y, \\
(5) & \quad x \leq y \quad \text{implies} \quad yz \leq xz \quad \text{and} \quad zx \leq zy.
\end{align*}

In a lattice implication algebra \(L\), if we denote \((xy)'\) by \(x \times y\) and \(x' y\) by \(x + y\), then the following relations are easily proved:

\begin{align*}
(6) & \quad x + y = y + x, \\
(7) & \quad (x + y) + z = x + (y + z), \\
(8) & \quad x + y \geq x \lor y, \\
(9) & \quad x \times y = y \times x, \\
(10) & \quad (x \times y) \times z = x \times (y \times z), \\
(11) & \quad x \times y \leq x \land y.
\end{align*}

A subset \(A\) of a lattice implication algebra \(L\) is called an LI-ideal of \(L\) (see [1]) if it satisfies

\begin{align*}
(LI1) & \quad 0 \in A, \\
(LI2) & \quad (xy)' \in A \quad \text{and} \quad y \in A \quad \text{imply} \quad x \in A \quad \text{for all} \quad x, y \in L.
\end{align*}

An LI-ideal \(A\) of a lattice implication algebra \(L\) is said to be proper if \(A \neq L\).

**Theorem 2.1.** ([1, Theorem 2.2]) Let \(A\) be an LI-ideal of a lattice implication algebra \(L\) and let \(x \in A\). If \(y \leq x\), then \(y \in A\) for all \(y \in L\).

Let \(A\) be a subset of a lattice implication algebra \(L\). Then the least LI-ideal containing \(A\) is called the LI-ideal generated by \(A\), denoted by \(\langle A \rangle\).
The next statement gives a description of the elements of $\langle A \rangle$.

**Theorem 2.2.** ([1, Theorem 2.9]) If $A$ is a non-empty subset of a lattice implication algebra $L$, then

$$\langle A \rangle = \{ x \in L \mid a'_n(...)a'_1x'(...) = 1 \text{ for some } a_1, ..., a_n \in A \}.$$  

**Ultra LI-ideals**

We start by providing a characterization of LI-ideals.

**Proposition 3.1.** Let $A$ be a subset of a lattice implication algebra $L$. Then $A$ is an LI-ideal of $L$ if and only if the following implications hold:

- (i) $x \in A$ and $y \leq x$ imply $y \in A$,
- (ii) $x \in A$ and $y \in A$ imply $x + y \in A$.

**Proof.** If $A$ is an LI-ideal of $L$, then (i) holds by Theorem 2.1. Let $x, y \in A$. Then

$$((x + y)y)' = ((x'y)y)' = (x' \lor y)' = x \land y' \leq x.$$  

It follows from Theorem 2.1 that $((x + y)y)' \in A$ and hence $x + y \in A$ by (LI2). Conversely, let $A$ be a subset of $L$ satisfying the conditions (i) and (ii). Since $0 \leq x$ for all $x \in L$ and hence for all $x \in A$, it follows from (i) that $0 \in A$. Suppose $(xy)' \in A$ and $y \in A$. Then $(xy)' + y \in A$ by (ii), and

$$(xy)' + y = ((xy)'y)'y = (xy)y = x \lor y \geq x.$$  

Using (i), we have $x \in A$ which proves (LI2), completing the proof. \hfill \square

**Theorem 3.2.** If $A$ is a subset of a lattice implication algebra $L$, then

$$\langle A \rangle = \{ x \in L \mid x \leq a_1 + a_2 + ... + a_n \text{ for some } a_1, ..., a_n \in A \}.$$  

**Proof.** By Theorem 2.2 it is sufficient to show that

$$x \leq a_1 + a_2 + ... + a_n \iff a'_n(...)a'_1x'(...) = 1.$$  

We will prove (3.1) by induction on $n$. If $n = 1$, then

$$x \leq a_1 \iff xa_1 = 1 \iff a'_1x' = 1;$$
hence (3.1) holds for \( n = 1 \). Suppose (3.1) is true for \( n = k \), i.e.,
\[
x \leq a_1 + a_2 + \ldots + a_k \iff a_k'(...) a_k' \ldots = 1.
\]
Then
\[
x \leq a_1 + a_2 + \ldots + a_k + a_{k+1} = a_k' + a_1 + a_2 + \ldots + a_k
\]
\[
\iff x \leq a_k' + a_1 + a_2 + \ldots + a_k = a_k' a_{k+1}
\]
\[
\iff (a_1 + a_2 + \ldots + a_k) a_{k+1} \leq (a_1 + a_2 + \ldots + a_k) a_{k+1}
\]
\[
\iff (a_1 + a_2 + \ldots + a_k) a_{k+1} \leq a_k' a_{k+1}
\]
\[
\iff (a_1 + a_2 + \ldots + a_k) a_{k+1} \leq a_k' a_{k+1}
\]
\[
\iff a_k' (a_{k-1}' \ldots (a_1' a_{k+1}') \ldots ) = 1
\]
\[
\iff a_k' (a_k' \ldots (a_1' a_{k+1}') \ldots ) = 1,
\]
which shows that (3.1) holds for \( n = k + 1 \). This completes the proof. \( \square \)

**Definition 3.3.** A subset \( A \) of a lattice implication algebra \( L \) is said to have the finite additive property if \( a_1 + a_2 + \ldots + a_n \neq 1 \) for any finite members \( a_1, a_2, \ldots, a_n \) of \( A \).

The following corollary is an immediate consequence of Theorem 3.2.

**Corollary 3.4.** For a subset \( A \) of a lattice implication algebra \( L \), \( \langle A \rangle \) is a proper LI-ideal of \( L \) if and only if \( A \) has the finite additive property.

**Definition 3.5.** An LI-ideal \( A \) of a lattice implication algebra \( L \) is said to be ultra if for every \( x \in L \), the following equivalence holds:

\[
(3.2) \quad x \in A \iff x' \not\in A.
\]

**Theorem 3.6.** Let \( A \) be a subset of a lattice implication algebra \( L \). Then \( A \) is an ultra LI-ideal of \( L \) if and only if \( A \) is a maximal proper LI-ideal of \( L \).

**Proof.** Suppose that \( A \) is an ultra LI-ideal of \( L \). Since \( 0 \in A \), we have \( 1 = 0' \not\in A \), and hence \( A \) is proper. If \( B \) is an LI-ideal of \( L \) and \( A \nsubseteq B \), then there exists \( x \in L \) such that \( x \in B \) and \( x \not\in A \). By (3.2) we have \( x' \in A \nsubseteq B \), and so \( 1 = x + x' \in B \). It follows that \( B = L \) and \( B \) is not proper. Therefore \( A \) is a maximal proper LI-ideal of \( L \).

Conversely, assume that \( A \) is a maximal proper LI-ideal of \( L \). For each \( x \in L \), we claim that (3.2) is true. Assume \( x' \not\in A \) and let \( B = A \cup \{x\} \). Then \( B \) has the finite additive property. In fact, suppose \( y_1, \ldots, y_n \in B \). If \( y_1, \ldots, y_n \in A \), then \( y_1 + \ldots + y_n \neq 1 \) because \( A \) is proper. Now if there exists \( i \leq n \) such that \( y_i = x \), then
\[
y_1 + \ldots + y_n = x + y_1 + \ldots + y_{i-1} + y_{i+1} + \ldots + y_n.
\]
If $y_1 + \ldots + y_n = 1$ then $x'(y_1 + \ldots + y_i-1 + y_i+1 + \ldots + y_n) = 1$, i.e., $x' \leq y_1 + \ldots + y_i-1 + y_i+1 + \ldots + y_n$. Thus $x' \in A$ by Theorem 2.1, a contradiction. This proves that $B$ has the finite additive property. Using Corollary 3.4, $(B)$ is a proper LI-ideal of $L$. Since $A \subseteq (B)$ and $A$ is a maximal proper LI-ideal, we have $(B) = A$ and hence $x \in (B) = A$. Suppose $x \in A$. If $x' \in A$, then $1 = x + x' \in A$; hence $A$ is not a proper LI-ideal. This is a contradiction. Therefore $x' \notin A$ and the proof is complete. \hfill \square

**Theorem 3.7.** Let $A$ be a subset of a lattice implication algebra $L$. If $A$ has the finite additive property, then there exists an ultra LI-ideal $B$ of $L$ containing $A$.

**Proof.** Let

$$\Omega = \{B \mid B \text{ is a proper } LI\text{-ideal of } L \text{ containing } A\}.$$ 

Then $(A) \in \Omega$ and hence $\Omega \neq \emptyset$. Suppose $B_1 \subseteq B_2 \subseteq \ldots$ is a chain of elements of $\Omega$ and let $C = \bigcup B_i$. Then (i) $A \subseteq C$, (ii) $1 \notin C$ (because $1 \notin B_i$ for all $i$), (iii) $0 \in C$, and (iv) if $(xy)', y \in C$ then there exists $i$ such that $(xy)', y \in B_i$ and so $x \in B_i \subseteq C$. This shows that $C$ is a proper LI-ideal of $L$ containing $A$ so that $C \in \Omega$. By Zorn’s lemma, $\Omega$ has a maximal element, say $D$, which is the desired ultra LI-ideal of $L$. \hfill \square

Since every proper LI-ideal has the finite additive property, we have the following corollary.

**Corollary 3.8.** Every proper LI-ideal of a lattice implication algebra can be extended to an ultra LI-ideal.

**Theorem 3.9.** Let $A$ be a proper LI-ideal of a lattice implication algebra $L$. Then $A$ is ultra if and only if for every $a, b \in L$, whenever $a \times b \in A$ then $a \in A$ or $b \in A$.

**Proof.** Suppose $A$ is ultra and let $a, b \in L$. If $a \times b \in A$, then $(a \times b)' \notin A$. Since $(a \times b)' = ((ab)')' = ab' = a' + b'$, it follows that $a' \notin A$ or $b' \notin A$ so that $a \in A$ or $b \in A$. Conversely, assume that for every $a, b \in L$, $a \in A$ or $b \in A$ whenever $a \times b \in A$. Then for each $x \in L$, if $x' \notin A$ then $x' \times x = (x'x)' = 1' = 0 \in A$, which implies that $x \in A$. Clearly if $x \in A$, then $x' \notin A$. This completes the proof. \hfill \square

**Theorem 3.10.** Let $f: L \to M$ be an implication homomorphism of lattice implication algebras satisfying $f(0) = 0$.

(i) If $B$ is an ultra LI-ideal of $M$, then $f^{-1}(B)$ is an ultra LI-ideal of $L$.

(ii) If $f$ is an isomorphism and if $A$ is an ultra LI-ideal of $L$, then $f(A)$ is an ultra LI-ideal of $M$.  

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Proof. (i) Clearly $0 \in f^{-1}(B)$. Let $x, y \in L$ be such that $(xy)' \in f^{-1}(B)$ and $y \in f^{-1}(B)$. Then $f(y) \in B$ and $(f(x)f(y))' = (f(xy))' = f((xy)') \in B$. Since $B$ is an $LI$-ideal of $M$, it follows from (LI2) that $f(x) \in B$ so that $x \in f^{-1}(B)$. Hence $f^{-1}(B)$ is an $LI$-ideal of $L$. For each $x \in L$, we have

\[ x \in f^{-1}(B) \iff f(x) \in B \iff f(x)' \notin B \iff x' \notin f^{-1}(B). \]

Hence $f^{-1}(B)$ is an ultra $LI$-ideal of $L$.

(ii) Note that $0 = f(0) \in f(A)$. Let $x, y \in M$ be such that $(xy)' \in f(A)$ and $y \in f(A)$. Then there exist $u \in L$ and $v \in A$ such that $f(u) = x$ and $f(v) = y$. It follows that

\[ f((uv)') = (f(uv))' = (f(u)f(v))' = (xy)' \in f(A) \]

so that $(uv)' \in A$. Using $v \in A$, we know that $u \in A$ and so $x = f(u) \in f(A)$. Thus $f(A)$ is an $LI$-ideal of $M$. For each $y \in M$, let $x \in L$ be such that $f(x) = y$. Then

\[ y \in f(A) \iff x = f^{-1}(y) \in A \iff x' \notin A \iff y' = (f(x))' = f(x') \notin f(A). \]

Therefore $f(A)$ is an ultra $LI$-ideal of $M$. This completes the proof. 

\[ \Box \]

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References


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