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FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The method of quasilinearization is a well-known technique for obtaining approximate solutions of nonlinear differential equations. In this paper we apply this technique to functional differential problems. It is shown that linear iterations converge to the unique solution and this convergence is superlinear.

Keywords: quasilinearization, monotone iterations, superlinear convergence

MSC 2000: 34A45

1. Introduction

Consider the functional differential problem

$$\begin{cases} x'(t) = f(t, x(t), x_t), \quad t \in J = [0, T], \\ x_0 = \Phi_0, \end{cases} \tag{1}$$

where $f \in C(J \times \mathbb{R} \times C, \mathbb{R})$, $\Phi_0 \in C$, $C = C(J_0, \mathbb{R})$ with $J_0 = [-\tau, 0]$ for $\tau > 0$, and for any $t \in J$, $x_t \in C$ is defined by $x_t(s) = x(t+s)$ for $s \in J_0$. According to the above notation $x_0 \in C$ and $x_0(s) = x(s)$, $s \in J_0$. It means that in this case the initial condition $x_0 = \Phi_0$ means that $x(s) = \Phi(s)$ on J_0, where the function Φ is given and continuous on J_0.

The differential equation from problem (1) is of a very general type. It includes as special cases, for example, ordinary differential equations if $\tau = 0$, differential-difference equations, and integro-differential equations, too.

The method of quasilinearization gives linear iterations which converge monotonically to the unique solution of the initial value problem. Recently, this method has been extended so as to be applicable to a much larger class of nonlinear problems (see for example [7]). In this paper we extend this method to functional differential problems of type (1). If f does not depend on the second variable the method of quasilinearization is considered in [7].

2. Lemmas

A function $v \in C(\bar{J}, \mathbb{R}) \cap C^1(J, \mathbb{R})$, $\bar{J} = [-\tau, T]$ is said to be a lower solution of problem (1) if

$$\begin{cases} v'(t) \leqslant f(t, v(t), v_t), & t \in J, \\ \quad v_0 \leqslant \Phi_0, \end{cases}$$

and an upper solution of (1) if the inequalities are reversed.

Lemma 1. *Assume that* $f \in C(J \times \mathbb{R} \times C, \mathbb{R})$ *and*

1^0 f_x *exists and* $f_x(t, u, v) \leqslant K$, $K > 0$ *for* $(t, u, v) \in \Omega_0$, *where*

$$\Omega_0 = \{(t, u, v)\colon\ t \in J,\ u \in \mathbb{R},\ v \in C \text{ and } y_0(t) \leqslant u \leqslant z_0(t),\ y_{0,t} \leqslant v \leqslant z_{0,t}\},$$

2^0 *the Fréchet derivative* f_Φ *exists and is a linear operator satisfying*

(a) $f_\Phi(t, u, \Phi)\Psi \leqslant L \int_{-\tau}^0 \Psi(s)\,\mathrm{d}s$ *if* $\Psi > 0$ *for* $L > 0$, *and* $L + \mathrm{e}^{-L\tau} > 1 + K$,

(b) *if* $v_1, v_2 \in C$ *and* $v_1 \leqslant v_2$, *then*

$$f_\Phi(t, u, v)v_1 \leqslant f_\Phi(t, u, v)v_2 \text{ for } (t, u, v) \in \Omega_0,$$

3^0 $p \in C(\bar{J}, \mathbb{R}) \cap C^1(J, \mathbb{R})$, $(t, u, v) \in \Omega_0$, *and*

$$\begin{cases} p'(t) \leqslant f_x(t, u, v)p(t) + f_\Phi(t, u, v)p_t,\ t \in J, \\ p(s) \leqslant 0 \text{ on } J_0. \end{cases}$$

Then $p(t) \leqslant 0$ *on* J.

Proof. For $\varepsilon > 0$ put $\bar{v}(t) = \varepsilon \mathrm{e}^{Lt}$, $t \in \bar{J}$. Indeed, $\bar{v}_t > 0$, $t \in J$. Moreover, basing on 1^0 and $2^0(a)$, we obtain

$$\begin{aligned} f_x(t, u, v)\bar{v}(t) + f_\Phi(t, u, v)\bar{v}_t &\leqslant K\bar{v}(t) + L\int_{-\tau}^0 \bar{v}(t+s)\,\mathrm{d}s \\ &= K\varepsilon \mathrm{e}^{Lt} + L\varepsilon \mathrm{e}^{Lt}\int_{-\tau}^0 \mathrm{e}^{Ls}\,\mathrm{d}s = \varepsilon \mathrm{e}^{Lt}[K + 1 - \mathrm{e}^{-L\tau}]. \end{aligned}$$

Note that using the above relation and 2^0 (a), we get

$$\begin{aligned}\bar{v}'(t) &= \varepsilon L\mathrm{e}^{Lt} - f_x(t,u,v)\bar{v}(t) - f_\Phi(t,u,v)\bar{v}_t + f_x(t,u,v)\bar{v}(t) + f_\Phi(t,u,v)\bar{v}_t\\ &\geqslant f_x(t,u,v)\bar{v}(t) + f_\Phi(t,u,v)\bar{v}_t + \varepsilon L\mathrm{e}^{Lt} - \varepsilon\mathrm{e}^{Lt}[K+1-\mathrm{e}^{-L\tau}]\\ &= f_x(t,u,v)\bar{v}(t) + f_\Phi(t,u,v)\bar{v}_t + \varepsilon\mathrm{e}^{Lt}[L-K-1+\mathrm{e}^{-L\tau}]\\ &> f_x(t,u,v)\bar{v}(t) + f_\Phi(t,u,v)\bar{v}_t, \quad t\in J.\end{aligned}$$

Note that $p(0)\leqslant 0<\bar{v}(0)$ and $p(s)<\bar{v}(s)$, $s\in J_0$. We show that $p(t)<\bar{v}(t)$ on J. Suppose that it is not true. Then there exists $t_1\in(0,T]$ such that $p(t_1)=\bar{v}(t_1)$ and $p(t)<\bar{v}(t)$ on $[-\tau,t_1)$, so $p_t<\bar{v}_t$ on $[0,t_1)$. For each $h>0$ sufficiently small, we see that $p(t_1-h)-p(t_1)<\bar{v}(t_1-h)-\bar{v}(t_1)$. Hence $p'(t_1)\geqslant\bar{v}'(t_1)$.

Moreover,

$$\begin{aligned}f_x(t_1,u,v)p(t_1)+f_\Phi(t_1,u,v)p_{t_1} &\geqslant p'(t_1)\geqslant\bar{v}'(t_1)\\ &> f_x(t_1,u,v)\bar{v}(t_1)+f_\Phi(t_1,u,v)\bar{v}_{t_1}\\ &= f_x(t_1,u,v)p(t_1)+f_\Phi(t_1,u,v)\bar{v}_{t_1}\\ &\geqslant f_x(t_1,u,v)p(t_1)+f_\Phi(t_1,u,v)p_{t_1}.\end{aligned}$$

It is a contradiction. Hence $p(t)<\bar{v}(t)$ on J. If now $\varepsilon\to 0$, then we obtain $p(t)\leqslant 0$ on J. The proof is complete. □

Lemma 2. *Assume that*

1^0 $f_1,f_2\in C(J,\mathbb{R})$, $f\in C(J\times\mathbb{R}\times C,\mathbb{R})$,

2^0 *the Fréchet derivative* f_Φ *exists and is a linear operator satisfying the condition*

$$|f_\Phi(t,u,v)\Psi|\leqslant L\int_{-\tau}^{0}|\Psi(s)|\,\mathrm{d}s,\ L>0 \textit{ for } (t,u,v)\in\Omega_0 \textit{ and } \Psi\in C.$$

Then for $(t,u,v)\in\Omega_0$, *the problem*

$$\begin{cases} y'(t)=f_1(t)y(t)+f_\Phi(t,u,v)y_t+f_2(t), & t\in J,\\ y_0=\Phi_0\end{cases} \tag{2}$$

has a unique solution $y\in C(\bar{J},\mathbb{R})\cap C^1(J,\mathbb{R})$.

P r o o f. Note that, for $t\in J$, problem (2) is equivalent to

$$y(t)=\Phi(0)+\int_0^t \mathrm{e}^{\int_s^t f_1(r)\,\mathrm{d}r}[f_\Phi(s,u,v)y_s+f_2(s)]\,\mathrm{d}s\equiv Ay(t).$$

We will show that A is a contraction mapping. Let us define a norm by

$$|y|_* = \max_{t\in J}\big[|y(t)|\mathrm{e}^{-Mt}\big] \ \text{ with } \ M \geqslant N + L\tau,$$

where $|f_1(t)| \leqslant N$. Put

$$\overline{\Omega} = \{y\colon\ y \in C(\bar{J},\mathbb{R}) \cap C^1(J,\mathbb{R}),\ \ y_0 = \Phi_0\}.$$

Then for $y, \bar{y} \in \overline{\Omega}$ we have

$$\begin{aligned}
|Ay - A\bar{y}|_* &= \max_{t\in J} \mathrm{e}^{-Mt} \int_0^t \mathrm{e}^{\int_s^t f_1(r)\,\mathrm{d}r} |f_\Phi(s,u,v)[y_s - \bar{y}_s]|\,\mathrm{d}s \\
&\leqslant \max_{t\in J} \mathrm{e}^{-Mt} \int_0^t \mathrm{e}^{N(t-s)} L \int_{-\tau}^0 |y(s+r) - \bar{y}(s+r)|\,\mathrm{d}r\,\mathrm{d}s \\
&\leqslant L|y-\bar{y}|_* \max_{t\in J} \mathrm{e}^{-Mt} \int_0^t \mathrm{e}^{N(t-s)} \int_{-\tau}^0 \mathrm{e}^{M(s+r)}\,\mathrm{d}r\,\mathrm{d}s \\
&\leqslant L\tau|y-\bar{y}|_* \max_{t\in J} \mathrm{e}^{-(M-N)t} \int_0^t \mathrm{e}^{(M-N)s}\,\mathrm{d}s \\
&= \frac{L\tau}{M-N}|y-\bar{y}|_*\big[1 - \mathrm{e}^{-(M-N)T}\big] \leqslant \big[1 - \mathrm{e}^{-(M-N)T}\big]|y-\bar{y}|_*.
\end{aligned}$$

Problem (2) has a unique solution, because $b \equiv 1 - \mathrm{e}^{-(M-N)T} < 1$. The proof is complete. □

Theorem 1. *Assume that $f \in C(J \times \mathbb{R} \times C, \mathbb{R})$ and*

1^0 *$y_0, z_0 \in C(\bar{J},\mathbb{R}) \cap C^1(J,\mathbb{R})$ are lower and upper solutions of problem* (1) *and $y_0(t) \leqslant z_0(t)$ on J,*

2^0 *f_x and f_{xx} exist, are continuous and*

- (a) *$f_x(t,u,v) \leqslant K$ for $(t,u,v) \in \Omega_0$,*
- (b) *if $v_1, v_2 \in C$, and $y_{0,t} \leqslant v_1 \leqslant v_2 \leqslant z_{0,t}$, then $f_x(t,u,v_1) \leqslant f_x(t,u,v_2)$ for $t \in J$, $u \in \mathbb{R}$, $y_0(t) \leqslant u \leqslant z_0(t)$,*
- *(c) $f_{xx}(t,u,v) \geqslant 0$ for $(t,u,v) \in \Omega_0$,*

3^0 *the Fréchet derivative f_Φ exists and is a linear operator satisfying*

- (a) *$|f_\Phi(t,u,\Phi)v| \leqslant L\int_{-\tau}^0 |v(s)|\,\mathrm{d}s$, $L > 0$ for $(t,u,\Phi) \in \Omega_0$, $v \in C$ with the condition $L + \mathrm{e}^{-L\tau} > 1 + K$,*
- (b) *$f(t,u,v_2) \geqslant f(t,u,v_1) + f_\Phi(t,u,v_1)(v_2 - v_1)$ for $t \in J$, $u \in \mathbb{R}$, $v_1, v_2 \in C$ and such that $y_0(t) \leqslant u \leqslant z_0(t)$, $y_{0,t} \leqslant v_1 \leqslant v_2 \leqslant z_{0,t}$,*
- (c) *if $v_1 \leqslant v_2$, $v_1, v_2 \in C$ then $f_\Phi(t,u,v)v_1 \leqslant f_\Phi(t,u,v)v_2$ for $(t,u,v) \in \Omega_0$,*

(d) *if* $u, \bar{u} \in \mathbb{R}$, $v, \bar{v}, V \in C$, $V \geqslant 0$, *then*

$$f_\Phi(t,u,v)V \geqslant f_\Phi(t,\bar{u},\bar{v})V \text{ for } t \in J,\ \ y_0(t) \leqslant \bar{u} \leqslant u \leqslant z_0(t),$$
$$y_{0,t} \leqslant \bar{v} \leqslant v \leqslant z_{0,t},$$

4^0 *there exist constants* $L_1, L_2, L_3 > 0$ *and* $\alpha, \beta \in [0,1]$ *such that the conditions*

$$|f_x(t,u,v_1) - f_x(t,u,v_2)| \leqslant L_1|v_1 - v_2|_0^\alpha,$$
$$|f_\Phi(t,u_1,v_1) - f_\Phi(t,u_2,v_2)| \leqslant L_2|u_1 - u_2| + L_3|v_1 - v_2|_0^\beta$$

hold for $t \in J$, $u, u_1, u_2 \in \mathbb{R}$, $v_1, v_2 \in C$ *with* $|v|_0 = \max\limits_{s\in[-\tau,0]} |v(s)|$.

Then there exist monotone sequences $\{y_n\}$, $\{z_n\}$ *which converge uniformly to the unique solution* x *of problem* (1) *on* J *and that convergence is superlinear.*

P r o o f. Let $y_0(t) \leqslant \bar{u} \leqslant u \leqslant z_0(t)$, $y_{0,t} \leqslant \bar{v} \leqslant v \leqslant z_{0,t}$. Then, by the mean value theorem and 3^0 (b), we have

$$\begin{aligned} f(t,u,v) - f(t,\bar{u},\bar{v}) &= f(t,u,v) - f(t,\bar{u},v) + f(t,\bar{u},v) - f(t,\bar{u},\bar{v}) \\ &\geqslant f_x(t,\xi,v)(u-\bar{u}) + f_\Phi(t,\bar{u},\bar{v})(v-\bar{v}) \end{aligned}$$

with $\bar{u} < \xi < u$. Hence, by 2^0 (b), (c), we have

$$f(t,u,v) - f(t,\bar{u},\bar{v}) \geqslant f_x(t,\bar{u},\bar{v})(u-\bar{u}) + f_\Phi(t,\bar{u},\bar{v})(v-\bar{v}). \tag{3}$$

Let $y_{n+1,0} = \Phi_0$, $z_{n+1,0} = \Phi_0$ and

$$\begin{aligned} y'_{n+1}(t) &= f(t,y_n,y_{n,t}) + f_x(t,y_n,y_{n,t})[y_{n+1}(t) - y_n(t)] \\ &\quad + f_\Phi(t,y_n,y_{n,t})[y_{n+1,t} - y_{n,t}], \\ z'_{n+1}(t) &= f(t,z_n,z_{n,t}) + f_x(t,y_n,y_{n,t})[z_{n+1}(t) - z_n(t)] \\ &\quad + f_\Phi(t,y_n,y_{n,t})[z_{n+1,t} - z_{n,t}] \end{aligned}$$

for $t \in J$, $n = 0,1,\ldots$. Note that the elements y_{n+1}, z_{n+1} are well defined by Lemma 2.

Indeed, $y_0(t) \leqslant z_0(t)$, $t \in J$, by 1^0. Now we are going to show that

$$y_0(t) \leqslant y_1(t) \leqslant z_1(t) \leqslant z_0(t), \quad t \in J. \tag{4}$$

Put $p = y_0 - y_1$ on $\bar{J}$, so $p(s) = y_0(s) - y_1(s) \leqslant \Phi(s) - \Phi(s) = 0$, $s \in J_0$. Then

$$\begin{aligned} p'(t) &\leqslant f(t,y_0,y_{0,t}) - f(t,y_0,y_{0,t}) - f_x(t,y_0,y_{0,t})[y_1(t) - y_0(t)] \\ &\quad - f_\Phi(t,y_0,y_{0,t})[y_{1,t} - y_{0,t}] \\ &= f_x(t,y_0,y_{0,t})p(t) + f_\Phi(t,y_0,y_{0,t})p_t. \end{aligned}$$

By Lemma 1 we have $p(t) \leqslant 0$ on J showing that $y_0(t) \leqslant y_1(t)$ on J. By the same argument we can show that $z_1(t) \leqslant z_0(t)$ on J. Next, we let $p = y_1 - z_1$ on $\bar{J}$, so $p(s) = 0$ on J_0. By relation (3) we have

$$\begin{aligned} p'(t) &= f(t, y_0, y_{0,t}) + f_x(t, y_0, y_{0,t})[y_1(t) - y_0(t)] + f_\Phi(t, y_0, y_{0,t})[y_{1,t} - y_{0,t}] \\ &\quad - f(t, z_0, z_{0,t}) - f_x(t, y_0, y_{0,t})[z_1(t) - z_0(t)] - f_\Phi(t, y_0, y_{0,t})[z_{1,t} - z_{0,t}] \\ &\leqslant - f_x(t, y_0, y_{0,t})[z_0(t) - y_0(t)] - f_\Phi(t, y_0, y_{0,t})[z_{0,t} - y_{0,t}] \\ &\quad + f_x(t, y_0, y_{0,t})[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ &\quad + f_\Phi(t, y_0, y_{0,t})[y_{1,t} - y_{0,t} - z_{1,t} + z_{0,t}] \\ &= f_x(t, y_0, y_{0,t})p(t) + f_\Phi(t, y_0, y_{0,t})p_t. \end{aligned}$$

By Lemma 1, $p(t) \leqslant 0$ on J, so $y_1(t) \leqslant z_1(t)$ on J. It proves that (4) holds.

Now we prove that y_1, z_1 are lower and upper solutions, respectively, of problem (1). Relation (3) and conditions 2^0 (b), (c) and 3^0 (d) yield

$$\begin{aligned} y_1'(t) &= f(t, y_0, y_{0,t}) + f_x(t, y_0, y_{0,t})[y_1(t) - y_0(t)] + f_\Phi(t, y_0, y_{0,t})[y_{1,t} - y_{0,t}] \\ &\leqslant f(t, y_1, y_{1,t}) - f_x(t, y_0, y_{0,t})[y_1(t) - y_0(t)] - f_\Phi(t, y_0, y_{0,t})[y_{1,t} - y_{0,t}] \\ &\quad + f_x(t, y_0, y_{0,t})[y_1(t) - y_0(t)] + f_\Phi(t, y_0, y_{0,t})[y_{1,t} - y_{0,t}] \\ &= f(t, y_1, y_{1,t}) \end{aligned}$$

and

$$\begin{aligned} z_1'(t) &= f(t, z_0, z_{0,t}) + f_x(t, y_0, y_{0,t})[z_1(t) - z_0(t)] + f_\Phi(t, y_0, y_{0,t})[z_{1,t} - z_{0,t}] \\ &\geqslant f(t, z_1, z_{1,t}) + f_x(t, z_1, z_{1,t})[z_0(t) - z_1(t)] + f_\Phi(t, z_1, z_{1,t})[z_{0,t} - z_{1,t}] \\ &\quad + f_x(t, y_0, y_{0,t})[z_1(t) - z_0(t)] + f_\Phi(t, y_0, y_{0,t})[z_{1,t} - z_{0,t}] \\ &= f(t, z_1, z_{1,t}) + [f_x(t, z_1, z_{1,t}) - f_x(t, y_0, y_{0,t})][z_0(t) - z_1(t)] \\ &\quad + [f_\Phi(t, z_1, z_{1,t}) - f_\Phi(t, y_0, y_{0,t})][z_{0,t} - z_{1,t}] \\ &\geqslant f(t, z_1, z_{1,t}). \end{aligned}$$

The above proves that y_1, z_1 are lower and upper solutions of (1).

Let us assume that

$$y_0(t) \leqslant y_1(t) \leqslant \ldots \leqslant y_{k-1}(t) \leqslant y_k(t) \leqslant z_k(t) \leqslant z_{k-1}(t) \leqslant \ldots \leqslant z_1(t) \leqslant z_0(t), \\ t \in J,$$

and let y_k, z_k be lower and upper solutions of problem (1) for some $k \geqslant 1$. We shall prove that:

$$(5) \qquad y_k(t) \leqslant y_{k+1}(t) \leqslant z_{k+1}(t) \leqslant z_k(t), \quad t \in J.$$

Let $p = y_k - y_{k+1}$ on J. Then $p(s) = 0$ on J_0. Using the mean value theorem and the fact that y_k is a lower solution of problem (1), we obtain

$$\begin{aligned} p'(t) &\leqslant f(t, y_k, y_{k,t}) - f(t, y_k, y_{k,t}) - f_x(t, y_k, y_{k,t})[y_{k+1}(t) - y_k(t)] \\ &\quad - f_\Phi(t, y_k, y_{k,t})[y_{k+1,t} - y_{k,t}] \\ &= f_x(t, y_k, y_{k,t})p(t) + f_\Phi(t, y_k, y_{k,t})p_t. \end{aligned}$$

Lemma 1 yields $p(t) \leqslant 0$, so $y_k(t) \leqslant y_{k+1}(t)$ on J. Similarly, we can show that $z_{k+1}(t) \leqslant z_k(t)$ on J.

Now, if $p = y_{k+1} - z_{k+1}$ on J, then $p(s) = 0$, $s \in J_0$, and using relation (3) we get

$$\begin{aligned} p'(t) &= f(t, y_k, y_{k,t}) + f_x(t, y_k, y_{k,t})[y_{k+1}(t) - y_k(t)] + f_\Phi(t, y_k, y_{k,t})[y_{k+1,t} - y_{k,t}] \\ &\quad - f(t, z_k, z_{k,t}) - f_x(t, y_k, y_{k,t})[z_{k+1}(t) - z_k(t)] - f_\Phi(t, y_k, y_{k,t})[z_{k+1,t} - z_{k,t}] \\ &\leqslant - f_x(t, y_k, y_{k,t})[z_k(t) - y_k(t)] - f_\Phi(t, y_k, y_{k,t})[z_{k,t} - y_{k,t}] \\ &\quad + f_x(t, y_k, y_{k,t})[y_{k+1}(t) - y_k(t) - z_{k+1}(t) + z_k(t)] \\ &\quad + f_\Phi(t, y_k, y_{k,t})[y_{k+1,t} - y_{k,t} - z_{k+1,t} + z_{k,t}] \\ &= f_x(t, y_k, y_{k,t})p(t) + f_\Phi(t, y_k, y_{k,t})p_t. \end{aligned}$$

This yields $y_{k+1}(t) \leqslant z_{k+1}(t)$, $t \in J$, so inequality (5) holds.

Hence, by induction, we have

$$y_0(t) \leqslant y_1(t) \leqslant \ldots \leqslant y_n(t) \leqslant z_n(t) \leqslant \ldots \leqslant z_1(t) \leqslant z_0(t), \quad t \in J$$

for all n. Employing the standard techniques, it can be shown that the sequences $\{y_n\}$, $\{z_n\}$ converge uniformly and monotonically to solutions y and z of problem (1). Now, we are going to show that problem (1) has a unique solution. To prove it we assume that it has two solutions u and v. Set $p = u - v$. Then $p(0) = 0$, and

$$\begin{aligned} p(t) &= f(t, u, u_t) - f(t, v, u_t) + f(t, v, u_t) - f(t, v, v_t) \\ &= f_x(t, \xi, u_t)p(t) + \int_0^1 f_\Phi(t, v, su_t + (1-s)v_t)\,\mathrm{d}s\, p_t, \quad t \in J, \end{aligned} \tag{6}$$

where ξ is between u and v. By Lemma 2, equation (6) has a unique solution. Since $p(t) = 0$, $t \in \overline{J}$ is a solution of (6), hence $u = v$ on $\overline{J}$. This proves that the sequences $\{y_n\}$, $\{z_n\}$ converge to the unique solution x of problem (1).

We shall next show that the convergence of y_n, z_n to the unique solution x of problem (1) is superlinear. For this purpose, we consider

$$p_{n+1} = x - y_{n+1} \geqslant 0, \qquad q_{n+1} = z_{n+1} - x \geqslant 0 \quad t \in \overline{J}.$$

Note that $p_{n+1}(s) = q_{n+1}(s) = 0$ for $s \in J_0$. Using the mean value theorem, 2^0 (c), 3^0 (a) and 4^0, we get

$$\begin{aligned}
p'_{n+1}(t) &= f(t,x,x_t) - f(t,y_n,x_t) + f(t,y_n,x_t) - f(t,y_n,y_{n,t}) \\
&\quad - f_x(t,y_n,y_{n,t})[y_{n+1}(t) - y_n(t)] - f_\Phi(t,y_n,y_{n,t})[y_{n+1,t} - y_{n,t}] \\
&= f_x(t,\xi_1,x_t)p_n(t) + \int_0^1 f_\Phi(t,y_n,sx_t + (1-s)y_{n,t})p_{n,t}\,\mathrm{d}s \\
&\quad - f_x(t,y_n,y_{n,t})[p_n(t) - p_{n+1}(t)] - f_\Phi(t,y_n,y_{n,t})[p_{n,t} - p_{n+1,t}] \\
&\leqslant [f_x(t,x,x_t) - f_x(t,y_n,x_t) + f_x(t,y_n,x_t) - f_x(t,y_n,y_{n,t})]p_n(t) \\
&\quad + \int_0^1 [f_\Phi(t,y_n,sx_t + (1-s)y_{n,t}) - f_\Phi(t,y_n,y_{n,t})]p_{n,t}\,\mathrm{d}s \\
&\quad + f_x(t,y_n,y_{n,t})p_{n+1}(t) + f_\Phi(t,y_n,y_{n,t})p_{n+1,t} \\
&\leqslant [f_{xx}(t,\xi_2,x_t)p_n(t) + L_1|p_{n,t}|_0^\alpha]\,p_n(t) + f_x(t,y_n,y_{n,t})p_{n+1}(t) \\
&\quad + L_3\int_0^1 s^\beta|p_{n,t}|_0^{\beta+1}\,\mathrm{d}s + L\int_{-\tau}^0 p_{n+1,t}(s)\,\mathrm{d}s \\
&\leqslant [A_1p_n(t) + L_1|p_{n,t}|_0^\alpha]\,p_n(t) + A_2p_{n+1}(t) + L_3|p_{n,t}|_0^{\beta+1} \\
&\quad + L\int_{-\tau}^0 p_{n+1,t}(s)\,\mathrm{d}s,
\end{aligned}$$

where

$$y_n(t) < \xi_1,\ \xi_2 < x(t), \quad t \in J, \quad \text{and} \quad |f_{xx}| \leqslant A_1, \quad |f_x| \leqslant A_2 \ \text{on}\ \Omega_0.$$

Put

$$\begin{aligned}
w'(t) &= [A_1p_n(t) + L_1|p_{n,t}|_0^\alpha]\,p_n(t) + A_2p_{n+1}(t) + L_3|p_{n,t}|_0^{\beta+1} \\
&\quad + L\int_{-\tau}^0 p_{n+1,t}(s)\,\mathrm{d}s, \quad t \in J,
\end{aligned}$$

and $w(0) = 0$. Note that $w'(t) \geqslant 0$ on J. Since $p_{n+1}(t) \leqslant w(t)$, $t \in J$, and w is nondecreasing in t, we obtain

$$\begin{aligned}
w(t) &= \int_0^t \big[A_1p_n^2(s) + L_1|p_{n,s}|_0^\alpha p_n(s) + L_3|p_{n,s}|_0^{\beta+1}\big]\,\mathrm{d}s \\
&\quad + A_2\int_0^t p_{n+1}(s)\,\mathrm{d}s + L\int_0^t\!\!\int_{-\tau}^0 p_{n+1,s}(r)\,\mathrm{d}r\,\mathrm{d}s \\
&\leqslant Dt + A_2\int_0^t w(s)\,\mathrm{d}s + L\int_0^t\!\!\int_{-\tau}^0 p_{n+1}(s+r)\,\mathrm{d}r\,\mathrm{d}s \\
&\leqslant Dt + A_2\int_0^t w(s)\,\mathrm{d}s + L\int_0^t\!\!\int_{-\tau}^0 w(s)\,\mathrm{d}r\,\mathrm{d}s = Dt + (A_2 + L\tau)\int_0^t w(s)\,\mathrm{d}s
\end{aligned}$$

where

$$D = \max_{t\in J}\bigl[A_1|p_n^2(t)| + L_1|p_{n,t}|_0^{\alpha}|p_n(t)| + L_3|p_{n,t}|_0^{\beta+1}\bigr].$$

Putting $u(t) = \int_0^t w(s)\,\mathrm{d}s$ we see that $u'(t) = w(t)$, $t \in J$, and $u(0) = 0$. By Gronwall's inequality for

$$u'(t) \leqslant Dt + (A_2 + L\tau)u(t), \quad u(0) = 0,$$

we have

$$u(t) \leqslant D\int_0^t s\mathrm{e}^{(A_2+L\tau)(t-s)}\,\mathrm{d}s, \quad t \in J.$$

Hence

$$\begin{aligned} p_{n+1}(t) &\leqslant w(t) \leqslant Dt + (A_2 + L\tau)u(t) \\ &\leqslant Dt + (A_2 + L\tau)D\int_0^t s\mathrm{e}^{(A_2+L\tau)(t-s)}\,\mathrm{d}s \\ &= Dt + (A_2 + L\tau)D\mathrm{e}^{(A_2+L\tau)t}\int_0^t s\mathrm{e}^{-(A_2+L\tau)s}\,\mathrm{d}s \leqslant BD, \end{aligned}$$

where

$$B = \frac{1}{A_2 + L\tau}\,\mathrm{e}^{(A_2+L\tau)T}.$$

Because $|p_n(t)| \leqslant |p_{n,t}|_0$, we finally obtain

$$\max_{t\in J}|p_{n+1}(t)| \leqslant BA_1\max_{t\in J}|p_{n,t}|_0^2 + BL_1\max_{t\in J}|p_{n,t}|_0^{\alpha+1} + BL_3\max_{t\in J}|p_{n,t}|_0^{\beta+1}.$$

Similarly,

$$\begin{aligned} q'_{n+1}(t) &= f(t, z_n, z_{n,t}) - f(t, x, z_{n,t}) + f(t, x, z_{n,t}) - f(t, x, x_t) \\ &\quad + f_x(t, y_n, y_{n,t})[z_{n+1}(t) - x(t) + x(t) - z_n(t)] \\ &\quad + f_\Phi(t, y_n, y_{n,t})[z_{n+1,t} - x_t + x_t - z_{n,t}] \\ &= f_x(t, \sigma_1, z_{n,t})q_n(t) + \int_0^1 f_\Phi(t, x, sz_{n,t} + (1-s)x_t)q_{n,t}\,\mathrm{d}s \\ &\quad + f_x(t, y_n, y_{n,t})[q_{n+1}(t) - q_n(t)] + f_\Phi(t, y_n, y_{n,t})[q_{n+1,t} - q_{n,t}] \\ &\leqslant [f_x(t, z_n, z_{n,t}) - f_x(t, y_n, z_{n,t}) + f_x(t, y_n, z_{n,t}) - f_x(t, y_n, x_t) \\ &\quad + f_x(t, y_n, x_t) - f_x(t, y_n, y_{n,t})]q_n(t) \\ &\quad + \int_0^1 [f_\Phi(t, x, sz_{n,t} + (1-s)x_t) - f_\Phi(t, y_n, sz_{n,t} + (1-s)x_t) \\ &\qquad + f_\Phi(t, y_n, sz_{n,t} + (1-s)x_t) - f_\Phi(t, y_n, x_t) + f_\Phi(t, y_n, x_t) \\ &\qquad - f_\Phi(t, y_n, y_{n,t})]q_{n,t}\,\mathrm{d}s \\ &\quad + f_x(t, y_n, y_{n,t})q_{n+1}(t) + f_\Phi(t, y_n, y_{n,t})q_{n+1,t}, \end{aligned}$$

$$
\begin{aligned}
q'_{n+1}(t) \leqslant{}& [f_{xx}(t,\sigma_2,z_{n,t})[q_n(t)+p_n(t)] + L_1|q_{n,t}|_0^{\alpha} + L_1|p_{n,t}|_0^{\alpha}]q_n(t) \\
&+ f_x(t,y_n,y_{n,t})q_{n+1}(t) \\
&+ \int_0^1 \big[L_2|p_n(t)| + L_3 s^{\beta}|q_{n,t}|_0^{\beta} + L_3|p_{n,t}|_0^{\beta}\big]\, q_{n,t}\,\mathrm{d}s + L\int_{-\tau}^0 q_{n+1,t}(s)\,\mathrm{d}s \\
\leqslant{}& A_1 q_n^2(t) + \tfrac{1}{2}A_1[q_n^2(t)+p_n^2(t)] + L_1|q_{n,t}|_0^{\alpha+1} + L_1|p_{n,t}|_0^{\alpha}|q_{n,t}|_0 + A_2 q_{n+1}(t) \\
&+ L_2|p_{n,t}|_0|\,q_{n,t}|_0 + L_3|q_{n,t}|_0^{\beta+1} + L_3|p_{n,t}|_0^{\beta}\,|q_{n,t}|_0 + L\int_{-\tau}^0 q_{n+1,t}(s)\,\mathrm{d}s \\
\leqslant{}& P + A_2 q_{n+1}(t) + L\int_{-\tau}^0 q_{n+1,t}(s)\,\mathrm{d}s,
\end{aligned}
$$

where $x(t) < \sigma_1 < z_n(t)$, $y_n(t) < \sigma_2 < z_n(t)$ and

$$
\begin{aligned}
P = \max_{t\in J}\Big[&\big(\tfrac{3}{2}A_1 + \tfrac{1}{2}L_2\big)|q_{n,t}|_0^2 + \tfrac{1}{2}(A_1+L_2)|p_{n,t}|_0^2 + L_1|\,q_{n,t}|_0^{\alpha+1} \\
&+ L_1|p_{n,t}|_0^{\alpha}|q_{n,t}|_0 + L_3|q_{n,t}|_0^{\beta+1} + L_3|p_{n,t}|_0^{\beta}|q_{n,t}|_0\Big].
\end{aligned}
$$

Put

$$
w'(t) = P + A_2 q_{n+1}(t) + L\int_{-\tau}^0 q_{n+1,t}(s)\,\mathrm{d}s, \quad w(0) = 0.
$$

Note that $q_{n+1}(t) \leqslant w(t)$ on J and w is nondecreasing in t. Hence we get

$$
\begin{aligned}
w(t) &= Pt + A_2\int_0^t q_{n+1}(s)\,\mathrm{d}s + L\int_0^t\int_{-\tau}^0 q_{n+1,s}(r)\,\mathrm{d}r\,\mathrm{d}s \\
&\leqslant Pt + (A_2 + L\tau)\int_0^t w(s)\,\mathrm{d}s.
\end{aligned}
$$

By Gronwall's inequality we have $w(t) \leqslant BP$, $t \in J$, and hence

$$
\begin{aligned}
\max_{t\in J}|q_{n+1}(t)| \leqslant{}& \tfrac{1}{2}B(3A_1+L_2)\max_{t\in J}|q_{n,t}|_0^2 + \tfrac{1}{2}B(A_1+L_2)\max_{t\in J}|p_{n,t}|_0^2 \\
&+ BL_1\max_{t\in J}|p_{n,t}|_0^{\alpha}|q_{n,t}|_0 + BL_1\max_{t\in J}|q_{n,t}|_0^{\alpha+1} \\
&+ BL_3\max_{t\in J}|q_{n,t}|_0^{\beta+1} + BL_3\max_{t\in J}\big(|p_{n,t}|_0^{\beta}|\,q_{n,t}|_0\big).
\end{aligned}
$$

The proof is complete. □

Remark 1. If $\alpha = \beta = 1$, then the convergence of sequences $\{y_n\}$, $\{z_n\}$ is quadratic.

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