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ON REGULARITIES AND FREDHOLM THEORY

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Abstract. We investigate the relationship between the regularities and the Fredholm theory in a Banach algebra.

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1. Introduction

Regularities are introduced and studied in [12] and [15] to give an axiomatic theory for spectra in literature which do not fit into the axiomatic theory of Żelazko [22]. In this note we investigate the relationship between the regularities and the Fredholm theory in a Banach algebra.

All algebras in this paper are complex and unital. Denote by A^{-1} the group of invertible elements in a Banach algebra A and by $\sigma(a, A) = \{\lambda \in \mathbb{C} \mid a - \lambda \notin A^{-1}\}$ the ordinary spectrum of $a \in A$. When no confusion can arise we write simply $\sigma(a)$. If $K \subset \mathbb{C}$, we use the symbol $\mathsf{acc}\, K$ to indicate the set of accumulation points of K and the symbol $\operatorname{iso} K$ for the set of isolated points of K. The topological boundary is denoted by ∂K and the closure by $\overline{K}$. If K is a bounded subset of $\mathbb{C}$ then ηK designates the connected hull of $\overline{K}$. By an ideal in A we mean a two sided ideal in A. An ideal J in A is said to be *inessential* [1, p. 106] if

$$a \in J \Longrightarrow \mathsf{acc}\, \sigma(a) \subset \{0\},$$

so that the spectrum of an element of J is either finite or a sequence converging to zero. If J is a closed inessential ideal in A then by a result of Aupetit [1, Theorem 5.7.4 (iii)] and [17, Theorem 5.3] we have

$$a \in A \Longrightarrow \mathsf{acc}\,\sigma(a) \subset \eta\sigma(a+J, A/J). \tag{1.1}$$

We will say a closed ideal J in A is *s-inessential* whenever

$$a \in A \Longrightarrow \mathsf{acc}\,\sigma(a) \subset \sigma(a+J, A/J).$$

The radical of A will be denoted by $\operatorname{Rad} A$ and A is said to be *semisimple* if $\operatorname{Rad} A = \{0\}$. A Banach algebra A is called *semiprime* if $0 \neq u \in A$ implies there exists $x \in A$ such that $uxu \neq 0$. All semisimple Banach algebras are semiprime. An element $a \in A$ is *quasinilpotent* if $\sigma(a) = \{0\}$. The set of these elements will be denoted by $\mathrm{QN}(A)$. Recall that if J is a closed ideal in A then $b \in A$ is called *Riesz* relative to J if $b + J \in \mathrm{QN}(A/J)$, see [2, Section R.1]. The set $\mathsf{kh}\, J$ is defined by $\mathsf{kh}\, J = \{b \in A \mid b + J \in \operatorname{Rad} A/J\}$. Clearly, this set is contained in the set of Riesz elements relative to J. An element $a \neq 0$ in a semiprime Banach algebra A is called *rank one* if there exists a linear functional τ_a on A such that $axa = \tau_a(x)a$ for all $x \in A$. For properties of these elements we refer to [19]. The *finite elements* of A, denoted by $\mathcal{F}(A)$, is the set of all $a \in A$ of the form $a = \sum_{i=1}^{n} a_i$ with each a_i a rank one element. In the case of a semiprime Banach algebra the set of finite elements coincides with the socle of A, i.e. $\operatorname{Soc} A = \mathcal{F}(A)$. By [19, Lemma 2.7] $\mathcal{F}(A)$ is an ideal in A.

We call an element $a \in A$ *regular* if it has a generalized inverse in A, $b \in A$ for which $a = aba$, and write

$$\widehat{A} = \{a \in A \mid a \in aAa\}$$

for the set of regular elements. These include both the left and right invertible elements,

$$A^{-1}_{\text{left}} \cup A^{-1}_{\text{right}} \subset \widehat{A} \tag{1.2}$$

as well as the idempotents $A^{\bullet} = \{a \in A \mid a^2 = a\}$. The *decomposably regular* elements are those which admit invertible generalized inverses; they are those elements which can be written as the product of an invertible and an idempotent:

$$A^{-1}A^{\bullet} = A^{\bullet}A^{-1} = \{a \in A \mid a \in aA^{-1}a\} \subset \widehat{A}.$$

It is then familiar [8, Theorem 7.3.4] that

$$A^{-1}A^{\bullet} = \widehat{A} \cap \overline{A^{-1}}. \tag{1.3}$$

For properties of the regular and decomposably regular elements we refer to [7], [8], [10].

2. Regularities

In this section we gather basic information on regularities as developed in [12].

2.1. Definition [12, Definition 1.2]**.** A nonempty subset $\mathcal{R}$ of a Banach algebra A is called a regularity if

1. $a \in A$ and $n \in \mathbb{N}$ then $a \in \mathcal{R} \Leftrightarrow a^n \in \mathcal{R}$,
2. a, b, c, d are mutually commuting elements of A and $ac+bd = 1$ then $ab \in \mathcal{R} \Leftrightarrow a \in \mathcal{R}$ and $b \in \mathcal{R}$.

2.2. Proposition [12, Proposition 1.3]**.** *Let $\mathcal{R}$ be a regularity in a Banach algebra A.*

1) *If $a, b \in A$, $ab = ba$ and $a \in A^{-1}$ then $ab \in \mathcal{R} \Leftrightarrow a \in \mathcal{R}$ and $b \in \mathcal{R}$.*
2) *$A^{-1} \subset \mathcal{R}$.*

A regularity $\mathcal{R}$ in A defines a mapping $\widetilde{\sigma}_{\mathcal{R}}$ from A into subsets of $\mathbb{C}$ by $\widetilde{\sigma}_{\mathcal{R}}(a) = \{\lambda \in \mathbb{C} \mid a-\lambda \notin \mathcal{R}\}$ $(a \in A)$. This mapping will be called the *spectrum corresponding to* $\mathcal{R}$. When no confusion can arise we will write $\widetilde{\sigma}(a)$. For results on the spectrum arising from the regularities $\mathcal{R}_5$ and $\mathcal{R}_6$, [12, p. 111], we refer to [13].

Consider the following condition:

(P1) $ab \in \mathcal{R} \Leftrightarrow a \in \mathcal{R}$ and $b \in \mathcal{R}$ for all commuting elements $a, b \in A$.

Clearly a nonempty subset $\mathcal{R}$ of A satisfying (P1) is a regularity.

3. Subalgebras

In this section we investigate how the spectrum corresponding to a regularity depends on the algebra. For the regularity A^{-1} of invertible elements this dependence is familiar [21, Theorem VII.2.6] and [4].

3.1. Theorem. *Let A and B be Banach algebras such that $1 \in B \subset A$. Suppose $\mathcal{R}_A$ is a regularity in A and $\mathcal{R}_B$ is a regularity in B such that $\mathcal{R}_{\mathcal{B}} \subset \mathcal{R}_{\mathcal{A}}$.*

1) *Then $\widetilde{\sigma}_{\mathcal{R}_{\mathcal{A}}}(b, A) \subset \widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B)$ for every $b \in B$.*
2) *If $\partial\mathcal{R}_{\mathcal{B}} \cap \mathcal{R}_{\mathcal{A}} = \emptyset$ then $\partial\widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B) \subset \widetilde{\sigma}_{\mathcal{R}_{\mathcal{A}}}(b, A)$ for all $b \in B$ such that $\widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B) \neq \emptyset$.*

P r o o f. 1) Let $b \in B$. If $\lambda \notin \widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B)$ then $b - \lambda \in \mathcal{R}_{\mathcal{B}} \subset \mathcal{R}_{\mathcal{A}}$ and so $\lambda \notin \widetilde{\sigma}_{\mathcal{R}_{\mathcal{A}}}(b, A)$.

2) Let $b \in B$ and $\lambda \in \partial\widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B)$. Then there is a sequence (λ_n) in $\mathbb{C} \setminus \widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B)$ such that $\lambda_n \to \lambda$ and a sequence (μ_n) in $\widetilde{\sigma}_{\mathcal{R}_{\mathcal{B}}}(b, B)$ such that $\mu_n \to \lambda$. Then $(b-\lambda_n)$ is a sequence in $\mathcal{R}_{\mathcal{B}}$ such that $b - \lambda_n \to b - \lambda$ and $(b - \mu_n)$ is a sequence in $B \setminus \mathcal{R}_{\mathcal{B}}$ such that $b - \mu_n \to b - \lambda$. Consequently, $b - \lambda \in \partial\mathcal{R}_{\mathcal{B}}$ and since $\partial\mathcal{R}_{\mathcal{B}} \cap \mathcal{R}_{\mathcal{A}} = \emptyset$ it follows that $b - \lambda \notin \mathcal{R}_{\mathcal{A}}$ and so $\lambda \in \widetilde{\sigma}_{\mathcal{R}_{\mathcal{A}}}(b, A)$. □

The above theorem applies to the regularity $\mathcal{R}_2 = A^{-1}$ [12, p. 111] of invertible elements: Let A and B be Banach algebras such that $1 \in B \subset A$. Then in general $B^{-1} \subset A^{-1}$ and if B is a closed subalgebra of A then it is well known that $\partial B^{-1} \cap A^{-1} = \emptyset$ [21, p. 398]. The proof of the next result follows from the definition of a regularity and will be omitted.

3.2. Proposition. *Let A and B be Banach algebras such that $1 \in B \subset A$. If $\mathcal{R}_\mathcal{A}$ is a regularity in A and $\mathcal{R}_\mathcal{B}$ is a regularity in B then $\mathcal{R}_\mathcal{A} \cap \mathcal{R}_\mathcal{B}$ is a regularity in B.*

3.3. Corollary. *Let A and B be Banach algebras such that $1 \in B \subset A$. If $\mathcal{R}_\mathcal{A}$ is a regularity in A then $\mathcal{R}_\mathcal{A} \cap \mathcal{B}$ is a regularity in B.*

For the regularity of invertible elements it is well known that if A is a C^* algebra and if B is a closed C^* subalgebra of A then $B^{-1} = A^{-1} \cap B$, see the proof of Theorem VII.6.5 in [21]. The proof of the next result follows from Corollary 3.3 and Theorem 3.1.1) and will be omitted.

3.4. Proposition. *Let A and B be Banach algebras such that $1 \in B \subset A$. Suppose $\mathcal{R}_\mathcal{A}$ is a regularity in A. Then $\widetilde{\sigma}_{\mathcal{R}_\mathcal{A}}(b, A) = \widetilde{\sigma}_{\mathcal{R}_\mathcal{A} \cap \mathcal{B}}(b, B)$ for every $b \in B$.*

4. The radical

We provide a characterization of the radical in a Banach algebra involving a regularity in the algebra. The radical $\operatorname{Rad} A$ of A is the intersection of all maximal left (or right) ideals of A and it is familiar [1, Theorem 3.1.3] that

$$\operatorname{Rad} A = \{a \in A \mid 1 - Aa \subset A^{-1}\}.$$

It can also be shown that

$$\operatorname{Rad} A = \{a \in A \mid Aa \subset \mathrm{QN}(A)\}.$$

4.1. Proposition. *If $\mathcal{R}$ is a regularity in a Banach algebra A then* $\operatorname{Rad} A = \{a \in A \mid \mathcal{R}a \subset \mathrm{QN}(A)$.

Proof. Since $\mathcal{R} \subset A$ it follows that $\operatorname{Rad} A \subset \{a \in A \mid \mathcal{R}a \subset \mathrm{QN}(A)\}$. To prove the nontrivial inclusion suppose $a \in \{a \in A \mid \mathcal{R}a \subset \mathrm{QN}(A)\}$. Let $d \in A$. Since A is a complex Banach algebra, $A = A^{-1} + A^{-1}$ and so $d = d_1 + d_2$ with $d_i \in A^{-1}$ $(i = 1, 2)$. Since $A^{-1} \subset \mathcal{R}$ by Proposition 2.2.2), it follows from our assumption that $d_1 a, (1 - d_1 a)^{-1} d_2 a \in \mathrm{QN}(A)$ and so $1 - da = (1 - d_1 a)(1 - (1 - d_1 a)^{-1} d_2 a) \in A^{-1}$. We have shown that $a \in \{a \in A \mid 1 - Aa \subset A^{-1}\}$. □

Since A^{-1} is a regularity it follows at once from the above proposition that $\operatorname{Rad} A = \{a \in A \mid A^{-1}a \subset \mathrm{QN}(A)\}$. This result was proved in [18, Remark 4] by different methods.

Let X be a complex Banach space and let $\mathcal{T}$ be a subset of X satisfying $\alpha\mathcal{T} \subset \mathcal{T}$ for all $0 \neq \alpha \in \mathbb{C}$. Following [14] let $P(\mathcal{T}) = \{x \in X \mid x + \mathcal{T} \subset \mathcal{T}\}$. If A is a Banach algebra and $\mathcal{R}$ a regularity in A then by [14, Lemma 2.1] $P(\mathcal{R})$ is a linear subspace of A and if $\mathcal{R}$ is an open subset of A then $P(\mathcal{R})$ is closed in A. If in addition A is a commutative Banach algebra then by Proposition 2.2 $A^{-1}\mathcal{R} \subset \mathcal{R}$ and $\mathcal{R}A^{-1} \subset \mathcal{R}$. In view of [14, Lemma 2.3] $P(\mathcal{R})$ is an ideal in A.

4.2. Theorem. *Let $\mathcal{R}$ be a regularity in a Banach algebra A such that $\partial A^{-1} \cap \mathcal{R} = \emptyset$. Then*

1) $\partial\sigma(a, A) \subset \widetilde{\sigma}_{\mathcal{R}}(a, A) \subset \sigma(a, A)$ *for all* $a \in A$.
2) $\operatorname{acc} \widetilde{\sigma}_{\mathcal{R}}(a, A) \subset \operatorname{acc} \sigma(a, A)$.
3) $\eta\sigma(a, A) = \eta\widetilde{\sigma}_{\mathcal{R}}(a, A)$.
4) $P(\mathcal{R}) \subset \operatorname{Rad} A$.

P r o o f. 1) Let $A = B$ in Theorem 3.1 and employ Proposition 2.2.2).

2) Follows from 1).

3) By 1) and the fact that the spectrum is closed it follows that $\overline{\widetilde{\sigma}_{\mathcal{R}}(a, A)} \subset \sigma(a, A)$ and so $\eta\widetilde{\sigma}_{\mathcal{R}}(a, A) = \eta\overline{\widetilde{\sigma}_{\mathcal{R}}(a, A)} \subset \eta\sigma(a, A)$, see the remarks preceding Lemma 1.1 in [11]. It also follows from 1) that $\partial\sigma(a, A) \subset \overline{\widetilde{\sigma}_{\mathcal{R}}(a, A)}$ and so by [11, Theorem 1.2] $\sigma(a, A) \subset \eta\overline{\widetilde{\sigma}_{\mathcal{R}}(a, A)} = \eta\widetilde{\sigma}_{\mathcal{R}}(a, A)$. Consequently, $\eta\sigma(a, A) \subset \eta\widetilde{\sigma}_{\mathcal{R}}(a, A)$. If we combine these remarks we obtain $\eta\sigma(a, A) = \eta\widetilde{\sigma}_{\mathcal{R}}(a, A)$.

4) Since $\mathcal{R}$ is a regularity it follows from Proposition 2.2 that $\alpha\mathcal{R} \subset \mathcal{R}$ for every $0 \neq \alpha \in \mathbb{C}$. Since $A^{-1} \subset \mathcal{R}$, by Proposition 2.2.2), and since A^{-1} is an open subset of A it follows from our assumption and Lemma 2.2 in [14] that $P(\mathcal{R}) \subset P(A^{-1}) = \operatorname{Rad} A$ [14, Theorem 2.5]. $\square$

We mention illustrations of the above theorem: If A is a Banach algebra then for the regularities $\mathcal{R}_i$ $(i = 2, 3, 4, 5, 6)$ [12, p. 111] it is familiar that $\partial A^{-1} \cap \mathcal{R}_i = \emptyset$, cf. [21, Theorem VII.2.5] and [3, Proposition].

5. Perturbation results

In this section we study the behaviour of elements belonging to a regularity under perturbations by rank one elements, inessential elements and Riesz elements.

5.1. Theorem. *Let A be a Banach algebra and suppose $\mathcal{R}$ is a regularity of A such that $\partial A^{-1} \cap \mathcal{R} = \emptyset$.*

1) *If J is a closed inessential ideal of $A, a \in A$ and $b \in J$ then* $\operatorname{acc} \widetilde{\sigma}_{\mathcal{R}}(a+b, A) \subset \eta \widetilde{\sigma}_{\mathcal{R}}(a, A)$.
2) *If J is a closed inessential ideal of $A, a \in A$ and b is Riesz relative to J with $ab = ba$ then* $\operatorname{acc} \widetilde{\sigma}_{\mathcal{R}}(a+b, A) \subset \eta \widetilde{\sigma}_{\mathcal{R}}(a, A)$.

Proof. 1) Suppose J is a closed inessential ideal of A and $b \in J$. It follows from 1.1 that

$$\operatorname{acc} \sigma(a+b, A) \subset \eta\sigma(a+b+J, A/J) = \eta\sigma(a+J, A/J) \subset \eta\sigma(a, A).$$

If we combine this with Theorem 4.2.2) and 3) we obtain $\operatorname{acc} \widetilde{\sigma}_{\mathcal{R}}(a+b, A) \subset \eta \widetilde{\sigma}_{\mathcal{R}}(a, A)$.

2) The proof of this statement follows exactly in the same way as 1) if we observe that $b + J \in \mathrm{QN}(A/J)$ and $a + J$ and $b + J$ commute in A/J implies that $\sigma(a+b+J, A/J) = \sigma(a+J, A/J)$. □

5.2. Corollary. *Let A be a Banach algebra and suppose $\mathcal{R}$ is a regularity of A such that $\partial A^{-1} \cap \mathcal{R} = \emptyset$. If $a \in A$ and $b \in \operatorname{Rad} A$ then* $\operatorname{acc} \widetilde{\sigma}_{\mathcal{R}}(a+b, A) \subset \eta \widetilde{\sigma}_{\mathcal{R}}(a, A)$.

5.3. Corollary. *Let A be a semisimple Banach algebra and suppose $\mathcal{R}$ is a regularity of A such that $\partial A^{-1} \cap \mathcal{R} = \emptyset$. If $a \in A$ and if $b \in A$ is rank one then* $\operatorname{acc} \widetilde{\sigma}_{\mathcal{R}}(a+b, A) \subset \eta \widetilde{\sigma}_{\mathcal{R}}(a, A)$.

Proof. If $b \in A$ is rank one, then it belongs to the inessential ideal $\mathcal{F}(A)$ of finite elements [19, Sections 2 and 3]. By [1, Corollary 5.7.6] the closure $\overline{\mathcal{F}(A)}$ of $\mathcal{F}(A)$ is also an inessential ideal. □

One can also provide a direct proof of Corollary 5.3 if one combines [9, Theorem 5] and Theorem 4.2.2) and 3).

5.4. Theorem. *Let A and B be Banach algebras and $T\colon A \to B$ a bounded homomorphism with closed range. If $\mathcal{R}$ is a regularity of A and $\mathcal{M}$ is a regularity of B with $\partial B^{-1} \cap \mathcal{M} = \emptyset$ then for each $a \in A$*

$$\bigcap_{Tb=0} \widetilde{\sigma}_{\mathcal{R}}(a+b, A) \subset \eta \widetilde{\sigma}_{\mathcal{M}}(Ta, B).$$

Proof. This follows from [5, Theorem 3], Proposition 2.2.2) and Theorem 4.2 3). □

For the spectrum and singular spectrum the results in this section are familiar: e.g. [13, Section 3], [5, Theorem 5], [17, Theorem 5.3] and [1, Theorem 5.7.4 (iii)].

6. Regular elements

It is well known [7, Examples 4.5 and 4.6] and [10, Examples 1 and 2] that the elements of $\widehat{A}$ and $A^{-1}A^{\bullet}$ do not multiply well and so in general neither $\widehat{A}$ nor $A^{-1}A^{\bullet}$ is a regularity in A. However, we have the following

6.1. Proposition [12, Lemma 2.8]**.** *Let a, b, c, d be mutually commuting elements in a Banach algebra A with $ac + bd = 1$. Then $ab \in \widehat{A}$ if and only if $a, b \in \widehat{A}$.*

6.2. Lemma. *Let A be a semiprime Banach algebra. Then $\mathcal{F}(A) \subset A^{-1}A^{\bullet} \subset \widehat{A}$.*

Proof. We prove first that $\mathcal{F}(A) \subset \widehat{A}$. If $u \in \mathcal{F}(\mathcal{A})$ then by [19, Theorem 3.4] there is an idempotent $p \in \mathcal{F}(A) \cap uA$ such that $u = pu$. Since $p \in uA$, we have $p = uv$ for some $v \in A$. Consequently, $u = uvu$ which proves that u is regular. This together with $\mathcal{F}(\mathcal{A})$ being an inessential ideal in A gives $\mathcal{F}(A) \subset A^{-1}A^{\bullet}$ [10, Theorem 7 (7.2)]. □

6.3. Theorem. *Let A be a semiprime Banach algebra. Then $\widehat{A} + \mathcal{F}(A) \subset \widehat{A}$.*

Proof. By the last lemma $\mathcal{F}(A) \subset \widehat{A}$. The result now follows from [8, (7.3.2.6)]. □

This result was proved by Kordula and Müller [12, Lemma 2.9] in the algebra $\mathcal{L}(X)$ of bounded linear operators on a Banach space X by different methods if one recalls that in the algebra $\mathcal{L}(X)$ the ideal of finite elements coincides with the ideal of finite rank operators, see [19].

Let J be an ideal in A. We say $a \in A$ is *J-Fredholm* if $a + J$ is invertible in the quotient algebra A/J. Recall [12, p. 111] that $\mathcal{R}_7 = \{a \in A \mid a \text{ is } J\text{-Fredholm}\}$ is a set satisfying (P1) and is therefore a regularity in A.

6.4. Proposition. *Suppose J is an ideal in A such that $J \subset \widehat{A}$. Then $\mathcal{R}_7 \subset \widehat{A}$.*

Proof. If $a \in \mathcal{R}_7$ then a is J-Fredholm and so by 1.2, we have $a + J \in \widehat{A/J}$. Since $J \subset \widehat{A}$, it follows from [8, Theorem 7.3.3] that $a \in \widehat{A}$. □

6.5. Theorem. *If J is a closed s-inessential ideal in A such that $J \subset \widehat{A}$ then $\mathcal{R}_7 \subset A^{-1}A^{\bullet}$.*

Proof. By Proposition 6.4 we have that $\mathcal{R}_7 \subset \widehat{A}$. Also, if $a \in \mathcal{R}_7$ then $0 \notin \sigma(a + J, A/J)$. In view of J being s-inessential it follows that $a \in \overline{A^{-1}}$. By 1.3 we conclude $a \in A^{-1}A^{\bullet}$. □

6.6. Theorem. *Let A be a semisimple Banach algebra and let J be an inessential ideal in A. Then $J \cap \widehat{A} \subset \mathcal{F}(A)$.*

P r o o f. Suppose $a = aa'a$ for some a' in A. If $a \in J$ then in view of [16, Theorem 1.4] the idempotent $a'a \in J \subset \mathsf{kh}\,\mathcal{F}(A)$. By [20, Theorem 4.6] we have $a'a \in \mathcal{F}(A)$. Since $\mathcal{F}(A)$ is an ideal in A it follows that $a \in \mathcal{F}(A)$. □

This result was proved by Harte [7, Theorem 4.2 (4.2.1)] in the algebra $\mathcal{L}(X)$ of bounded linear operators on a Banach space X.

7. An example

In this section we provide an example of a regularity in a Banach algebra and investigate how this regularity is related to the set of decomposably regular elements.

An element $a \in A$ is said to be *almost invertible* if $0 \notin \mathsf{acc}\,\sigma(a)$ [6]. We have the following implications:

$$\text{invertible} \implies \text{almost invertible } J\text{-Fredholm} \implies J\text{-Fredholm.}$$

Let J be a closed ideal in a Banach algebra A. Denote

$$\mathcal{R}_0(J) = \{a \in A \mid a \text{ is almost invertible } J\text{-Fredholm}\}.$$

7.1. Proposition. *Suppose a closed ideal J in A is s-inessential. Then $\mathcal{R}_0(J)$ is a regularity in A.*

P r o o f. We prove that $\mathcal{R}_0(J)$ satisfies (P1). If $a, b \in \mathcal{R}_0(J)$ with $ab = ba$ then ab is J-Fredholm. Since $\sigma(ab) \subset \sigma(a) \cdot \sigma(b)$ it follows that $ab \in \mathcal{R}_0(J)$. Conversely, if $ab \in \mathcal{R}_0(J)$ then a and b are J-Fredholm because $ab = ba$. This together with J s-inessential gives $a, b \in \mathcal{R}_0(J)$. □

7.2. Corollary. $\widetilde{\sigma}_{\mathcal{R}_0(J)}(a) = \mathsf{acc}\,\sigma(a) \cup \sigma(a + J, A/J)$ *for every $a \in A$.*

P r o o f. This follows from the definition of $\mathcal{R}_0(J)$. □

We will prove later that $\mathcal{R}_0(J)$ is actually an open regularity, see Theorem 7.5. However, to prove a stronger result we need the following

7.3. Definition. Let J be a closed ideal in A and $a \in A$. We say that a is J-Browder if $a = x + y$ with $x \in A^{-1}$, $y \in J$ and $xy = yx$.

Then we have the following implications [6, 16]:

(7.4) invertible $\Longrightarrow$ almost invertible J-Fredholm $\Longrightarrow$ J-Browder $\Longrightarrow$ J-Fredholm.

If A and B are Banach algebras then the homomorphism $T\colon A \to B$ is said to have the *Riesz property* if its kernel $T^{-1}(0)$ is an inessential ideal. If J is a closed inessential ideal then the almost invertible J-Fredholm and J-Browder elements coincide [6, Theorem 1] or [17, Corollary 3.6].

7.5. Theorem. *Suppose J is a closed inessential ideal in A. Then $\mathcal{R}_0(J)$ is an open regularity in A.*

Proof. We prove that $\mathcal{R}_0(J)$ satisfies (P1). If $a, b \in \mathcal{R}_0(J)$ with $ab = ba$ then it follows as in the proof of Proposition 7.1 that $ab \in \mathcal{R}_0(J)$. Conversely, if $ab \in \mathcal{R}_0(J)$ then by 7.4 ab is J-Browder. In view of $ab = ba$ and J being inessential (meaning that the quotient map $A \to A/J$ has the Riesz property) it follows from [8, Theorem 7.7.6] that both a and b are J-Browder. By the remarks following 7.4 we have $a, b \in \mathcal{R}_0(J)$.

We prove finally that $\mathcal{R}_0(J)$ is open. Let $x \in \mathcal{R}_0(J)$ and let $\varepsilon > 0$ satisfy $\{\lambda \in \mathbb{C} \mid |\lambda| < 3\varepsilon\} \cap \sigma(x) \subset \{0\}$. Since $\sigma(\cdot)$ and $\sigma(\cdot, A/J)$ are both upper semicontinuous there exists $\delta > 0$ such that if $\|x - y\| < \delta$ then y is J-Fredholm,

$$\sigma(y) \subset \{\lambda \in \mathbb{C} \mid |\lambda| < \varepsilon\} \cup \{\lambda \in \mathbb{C} \mid |\lambda| > 2\varepsilon\}$$

and

$$\sigma(y + J, A/J) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \geqslant 2\varepsilon\}.$$

However, since J is inessential, $\sigma(y) \setminus \sigma(y + J, A/J)$ consists of isolated points and some of the holes of $\sigma(y + J, A/J)$ [4, Theorem 6.1]. Hence either $0 \notin \sigma(y)$ or $0 \in \mathsf{iso}\,\sigma(y)$ and so y is almost invertible. We have shown that $y \in \mathcal{R}_0(J)$. $\square$

The above theorem was proved in the operator algebra $\mathcal{L}(X)$ of bounded linear operators on a Banach space X by Kordula and Müller [12, Theorem 2.1].

7.6. Theorem. *Suppose J is a closed inessential ideal in a semisimple Banach algebra A. Then $\mathcal{R}_0(J) \subset A^{-1}A^{\bullet}$.*

Proof. If $a \in \mathcal{R}_0(J)$ then a is almost invertible and so $a \in \overline{A^{-1}}$. Since a is J-Fredholm and since $J \subset \mathsf{kh}\,\mathcal{F}(A)$ [16, Theorem 4.6] it follows that a is $\mathsf{kh}\,\mathcal{F}(A)$-Fredholm. In view of $\mathcal{F}(A)$ and $\mathsf{kh}\,\mathcal{F}(A)$ having the same set of idempotents, see the remark following Lemma 5.7.1 in [1], we have by [1, Theorem 5.7.2] that a is $\mathcal{F}(A)$-Fredholm. By Lemma 6.2 and Proposition 6.4 we obtain $a \in \widehat{A}$. It follows from 1.3 that $a \in A^{-1}A^{\bullet}$. $\square$

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