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ON REGULARITIES AND FREDHOLM THEORY

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Abstract. We investigate the relationship between the regularities and the Fredholm theory in a Banach algebra.

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1. Introduction

Regularities are introduced and studied in [12] and [15] to give an axiomatic theory for spectra in literature which do not fit into the axiomatic theory of Želazko [22]. In this note we investigate the relationship between the regularities and the Fredholm theory in a Banach algebra.

All algebras in this paper are complex and unital. Denote by $A^{-1}$ the group of invertible elements in a Banach algebra $A$ and by $\sigma(a, A) = \{\lambda \in \mathbb{C} \mid a - \lambda \notin A^{-1}\}$ the ordinary spectrum of $a \in A$. When no confusion can arise we write simply $\sigma(a)$. If $K \subset \mathbb{C}$, we use the symbol acc $K$ to indicate the set of accumulation points of $K$ and the symbol iso $K$ for the set of isolated points of $K$. The topological boundary is denoted by $\partial K$ and the closure by $\overline{K}$. If $K$ is a bounded subset of $\mathbb{C}$ then $\eta K$ designates the connected hull of $\overline{K}$. By an ideal in $A$ we mean a two sided ideal in $A$. An ideal $J$ in $A$ is said to be inessential [1, p. 106] if

$$a \in J \implies \text{acc } \sigma(a) \subset \{0\},$$

so that the spectrum of an element of $J$ is either finite or a sequence converging to zero. If $J$ is a closed inessential ideal in $A$ then by a result of Aupetit [1, Theo-
We will say a closed ideal $J$ in $A$ is *s-inessential* whenever
\[ a \in A \implies \text{acc } \sigma(a) \subset \eta\sigma(a + J, A/J). \]

The radical of $A$ will be denoted by $\text{Rad}A$ and $A$ is said to be *semisimple* if $\text{Rad}A = \{0\}$. A Banach algebra $A$ is called *semiprime* if $0 \neq u \in A$ implies there exists $x \in A$ such that $ux \neq 0$. All semisimple Banach algebras are semiprime. An element $a \in A$ is *quasinilpotent* if $\sigma(a) = \{0\}$. The set of these elements will be denoted by $\text{QN}(A)$. Recall that if $J$ is a closed ideal in $A$ then $b \in A$ is called *Riesz* relative to $J$ if $b + J \in \text{QN}(A/J)$, see [2, Section R.1]. The set $\text{kh}J$ is defined by $\text{kh}J = \{b \in A \mid b + J \in \text{Rad}A/J\}$. Clearly, this set is contained in the set of Riesz elements relative to $J$. An element $a \neq 0$ in a semiprime Banach algebra $A$ is called *rank one* if there exists a linear functional $\tau_a$ on $A$ such that $axa = \tau_a(x)a$ for all $x \in A$. For properties of these elements we refer to [19]. The *finite elements* of $A$, denoted by $\mathcal{F}(A)$, is the set of all $a \in A$ of the form $a = \sum_{i=1}^{n} a_i$ with each $a_i$ a rank one element. In the case of a semiprime Banach algebra the set of finite elements coincides with the socle of $A$, i.e. $\text{Soc}A = \mathcal{F}(A)$. By [19, Lemma 2.7] $\mathcal{F}(A)$ is an ideal in $A$.

We call an element $a \in A$ *regular* if it has a generalized inverse in $A$, $b \in A$ for which $a = aba$, and write \( \widehat{A} = \{a \in A \mid a \in aAa\} \) for the set of regular elements. These include both the left and right invertible elements,
\[ A^{-1}_{\text{left}} \cup A^{-1}_{\text{right}} \subset \widehat{A} \]
as well as the idempotents $A^\bullet = \{a \in A \mid a^2 = a\}$. The *decomposably regular* elements are those which admit invertible generalized inverses; they are those elements which can be written as the product of an invertible and an idempotent:
\[ A^{-1}A^\bullet = A^\bullet A^{-1} = \{a \in A \mid a \in aA^{-1}a\} \subset \widehat{A}. \]

It is then familiar [8, Theorem 7.3.4] that
\[ A^{-1}A^\bullet = \widehat{A} \cap \overline{A^{-1}}. \]

For properties of the regular and decomposably regular elements we refer to [7], [8], [10].
2. Regularities

In this section we gather basic information on regularities as developed in [12].

2.1. Definition [12, Definition 1.2]. A nonempty subset $\mathcal{R}$ of a Banach algebra $A$ is called a regularity if

1. $a \in A$ and $n \in \mathbb{N}$ then $a \in \mathcal{R} \iff a^n \in \mathcal{R},$
2. $a, b, c, d$ are mutually commuting elements of $A$ and $ac + bd = 1$ then $ab \in \mathcal{R} \iff a \in \mathcal{R}$ and $b \in \mathcal{R}.$

2.2. Proposition [12, Proposition 1.3]. Let $\mathcal{R}$ be a regularity in a Banach algebra $A.$

1) If $a, b \in A, ab = ba$ and $a \in A^{-1}$ then $ab \in \mathcal{R} \iff a \in \mathcal{R}$ and $b \in \mathcal{R}.$
2) $A^{-1} \subset \mathcal{R}.$

A regularity $\mathcal{R}$ in $A$ defines a mapping $\tilde{\sigma}_\mathcal{R}$ from $A$ into subsets of $\mathbb{C}$ by $\tilde{\sigma}_\mathcal{R}(a) = \{ \lambda \in \mathbb{C} \mid a - \lambda \notin \mathcal{R} \} \quad (a \in A).$ This mapping will be called the spectrum corresponding to $\mathcal{R}.$ When no confusion can arise we will write $\tilde{\sigma}(a).$ For results on the spectrum arising from the regularities $\mathcal{R}_5$ and $\mathcal{R}_6,$ [12, p. 111], we refer to [13].

Consider the following condition:

(P1) $ab \in \mathcal{R} \iff a \in \mathcal{R}$ and $b \in \mathcal{R}$ for all commuting elements $a, b \in A.$

Clearly a nonempty subset $\mathcal{R}$ of $A$ satisfying (P1) is a regularity.

3. Subalgebras

In this section we investigate how the spectrum corresponding to a regularity depends on the algebra. For the regularity $A^{-1}$ of invertible elements this dependence is familiar [21, Theorem VII.2.6] and [4].

3.1. Theorem. Let $A$ and $B$ be Banach algebras such that $1 \in B \subset A.$ Suppose $\mathcal{R}_A$ is a regularity in $A$ and $\mathcal{R}_B$ is a regularity in $B$ such that $\mathcal{R}_B \subset \mathcal{R}_A.$

1) Then $\tilde{\sigma}_{\mathcal{R}_A}(b, A) \subset \tilde{\sigma}_{\mathcal{R}_B}(b, B)$ for every $b \in B.$
2) If $\partial \mathcal{R}_B \cap \mathcal{R}_A = \emptyset$ then $\partial \tilde{\sigma}_{\mathcal{R}_B}(b, B) \subset \tilde{\sigma}_{\mathcal{R}_A}(b, A)$ for all $b \in B$ such that $\tilde{\sigma}_{\mathcal{R}_B}(b, B) \neq \emptyset.$

Proof. 1) Let $b \in B.$ If $\lambda \notin \tilde{\sigma}_{\mathcal{R}_B}(b, B)$ then $b - \lambda \in \mathcal{R}_B \subset \mathcal{R}_A$ and so $\lambda \notin \tilde{\sigma}_{\mathcal{R}_A}(b, A).$

2) Let $b \in B$ and $\lambda \in \partial \tilde{\sigma}_{\mathcal{R}_B}(b, B).$ Then there is a sequence $(\lambda_n)$ in $\mathbb{C} \setminus \tilde{\sigma}_{\mathcal{R}_B}(b, B)$ such that $\lambda_n \to \lambda$ and a sequence $(\mu_n)$ in $\tilde{\sigma}_{\mathcal{R}_B}(b, B)$ such that $\mu_n \to \lambda.$ Then $(b - \lambda_n)$ is a sequence in $\mathcal{R}_B$ such that $b - \lambda_n \to b - \lambda$ and $(b - \mu_n)$ is a sequence in $B \setminus \mathcal{R}_B$ such that $b - \mu_n \to b - \lambda.$ Consequently, $b - \lambda \in \partial \mathcal{R}_B$ and since $\partial \mathcal{R}_B \cap \mathcal{R}_A = \emptyset$ it follows that $b - \lambda \notin \mathcal{R}_A$ and so $\lambda \in \tilde{\sigma}_{\mathcal{R}_A}(b, A).$ □
The above theorem applies to the regularity $R_2 = A^{-1}$ [12, p. 111] of invertible elements: Let $A$ and $B$ be Banach algebras such that $1 \in B \subset A$. Then in general $B^{-1} \subset A^{-1}$ and if $B$ is a closed subalgebra of $A$ then it is well known that $\partial B^{-1} \cap A^{-1} = \emptyset$ [21, p. 398]. The proof of the next result follows from the definition of a regularity and will be omitted.

3.2. Proposition. Let $A$ and $B$ be Banach algebras such that $1 \in B \subset A$. If $R_A$ is a regularity in $A$ and $R_B$ is a regularity in $B$ then $R_A \cap R_B$ is a regularity in $B$.

3.3. Corollary. Let $A$ and $B$ be Banach algebras such that $1 \in B \subset A$. If $R_A$ is a regularity in $A$ then $R_A \cap B$ is a regularity in $B$.

For the regularity of invertible elements it is well known that if $A$ is a $C^*$ algebra and if $B$ is a closed $C^*$ subalgebra of $A$ then $B^{-1} = A^{-1} \cap B$, see the proof of Theorem VII.6.5 in [21]. The proof of the next result follows from Corollary 3.3 and Theorem 3.1.1) and will be omitted.

3.4. Proposition. Let $A$ and $B$ be Banach algebras such that $1 \in B \subset A$. Suppose $R_A$ is a regularity in $A$. Then $\tilde{\sigma}_{R_A}(b, A) = \tilde{\sigma}_{R_A \cap B}(b, B)$ for every $b \in B$.

4. The radical

We provide a characterization of the radical in a Banach algebra involving a regularity in the algebra. The radical $\text{Rad } A$ of $A$ is the intersection of all maximal left (or right) ideals of $A$ and it is familiar [1, Theorem 3.1.3] that

$$\text{Rad } A = \{a \in A \mid 1 - Aa \subset A^{-1}\}. $$

It can also be shown that

$$\text{Rad } A = \{a \in A \mid Aa \subset \text{QN}(A)\}. $$

4.1. Proposition. If $R$ is a regularity in a Banach algebra $A$ then $\text{Rad } A = \{a \in A \mid Ra \subset \text{QN}(A)\}$.

Proof. Since $R \subset A$ it follows that $\text{Rad } A \subset \{a \in A \mid Ra \subset \text{QN}(A)\}$. To prove the nontrivial inclusion suppose $a \in \{a \in A \mid Ra \subset \text{QN}(A)\}$. Let $d \in A$. Since $A$ is a complex Banach algebra, $A = A^{-1} + A^{-1}$ and so $d = d_1 + d_2$ with $d_i \in A^{-1}$ ($i = 1, 2$). Since $A^{-1} \subset R$ by Proposition 2.2.2), it follows from our assumption that $d_1 a, (1 - d_1 a)^{-1} d_2 a \in \text{QN}(A)$ and so $1 - da = (1 - d_1 a)(1 - (1 - d_1 a)^{-1} d_2 a) \in A^{-1}$. We have shown that $a \in \{a \in A \mid 1 - Aa \subset A^{-1}\}$. □
Since $A^{-1}$ is a regularity it follows at once from the above proposition that \( \text{Rad } A = \{ a \in A \mid A^{-1}a \subset QN(A) \} \). This result was proved in [18, Remark 4] by different methods.

Let \( X \) be a complex Banach space and let \( T \) be a subset of \( X \) satisfying \( \alpha T \subset T \) for all \( 0 \neq \alpha \in \mathbb{C} \). Following [14] let \( P(T) = \{ x \in X \mid x + T \subset T \} \). If \( A \) is a Banach algebra and \( \mathcal{R} \) a regularity in \( A \) then by [14, Lemma 2.1] \( P(\mathcal{R}) \) is a linear subspace of \( A \) and if \( \mathcal{R} \) is an open subset of \( A \) then \( P(\mathcal{R}) \) is closed in \( A \). In addition, if \( A \) is a commutative Banach algebra then by Proposition 2.2 \( A^{-1} \mathcal{R} \subset \mathcal{R} \) and \( \mathcal{R} A^{-1} \subset \mathcal{R} \). In view of [14, Lemma 2.3] \( P(\mathcal{R}) \) is an ideal in \( A \).

### 4.2. Theorem

Let \( \mathcal{R} \) be a regularity in a Banach algebra \( A \) such that \( \partial A^{-1} \cap \mathcal{R} = \emptyset \). Then

1) \( \partial \sigma(a, A) \subset \tilde{\sigma}_\mathcal{R}(a, A) \subset \sigma(a, A) \) for all \( a \in A \).
2) \( \text{acc} \tilde{\sigma}_\mathcal{R}(a, A) \subset \text{acc} \sigma(a, A) \).
3) \( \eta \sigma(a, A) = \eta \tilde{\sigma}_\mathcal{R}(a, A) \).
4) \( P(\mathcal{R}) \subset \text{Rad } A \).

**Proof.**

1) Let \( A = B \) in Theorem 3.1 and employ Proposition 2.2.2).

2) Follows from 1).

3) By 1) and the fact that the spectrum is closed it follows that \( \tilde{\sigma}_\mathcal{R}(a, A) \subset \sigma(a, A) \) and so \( \eta \tilde{\sigma}_\mathcal{R}(a, A) = \eta \tilde{\sigma}_\mathcal{R}(a, A) \subset \eta \sigma(a, A) \), see the remarks preceding Lemma 1.1 in [11]. It also follows from 1) that \( \partial \sigma(a, A) \subset \overline{\sigma_\mathcal{R}(a, A)} \) and so by [11, Theorem 1.2] \( \sigma(a, A) \subset \overline{\eta \mathcal{R}(a, A)} = \eta \tilde{\sigma}_\mathcal{R}(a, A) \). Consequently, \( \eta \sigma(a, A) \subset \eta \tilde{\sigma}_\mathcal{R}(a, A) \).

4) Since \( \mathcal{R} \) is a regularity it follows from Proposition 2.2 that \( \alpha \mathcal{R} \subset \mathcal{R} \) for every \( 0 \neq \alpha \in \mathbb{C} \). Since \( A^{-1} \subset \mathcal{R} \), by Proposition 2.2.2), and since \( A^{-1} \) is an open subset of \( A \) it follows from our assumption and Lemma 2.2 in [14] that \( P(\mathcal{R}) \subset P(A^{-1}) = \text{Rad } A \) [14, Theorem 2.5]. \( \square \)

We mention illustrations of the above theorem: If \( A \) is a Banach algebra then for the regularities \( \mathcal{R}_i (i = 2, 3, 4, 5, 6) \) [12, p. 111] it is familiar that \( \partial A^{-1} \cap \mathcal{R}_i = \emptyset \), cf. [21, Theorem VII.2.5] and [3, Proposition].

### 5. Perturbation results

In this section we study the behaviour of elements belonging to a regularity under perturbations by rank one elements, inessential elements and Riesz elements.

#### 5.1. Theorem

Let \( A \) be a Banach algebra and suppose \( \mathcal{R} \) is a regularity of \( A \) such that \( \partial A^{-1} \cap \mathcal{R} = \emptyset \).
1) If \( J \) is a closed inessential ideal of \( A \), \( a \in A \) and \( b \in J \) then \( \text{acc} \tilde{\sigma}_R(a + b, A) \subset \eta \tilde{\sigma}_R(a, A) \).

2) If \( J \) is a closed inessential ideal of \( A \), \( a \in A \) and \( b \) is Riesz relative to \( J \) with \( ab = ba \) then \( \text{acc} \tilde{\sigma}_R(a + b, A) \subset \eta \tilde{\sigma}_R(a, A) \).

**Proof.** 1) Suppose \( J \) is a closed inessential ideal of \( A \) and \( b \in J \). It follows from 1.1 that

\[
\text{acc} \sigma(a + b, A) \subset \eta \sigma(a + b + J, A/J) = \eta \sigma(a + J, A/J) \subset \eta \sigma(a, A).
\]

If we combine this with Theorem 4.2.2) and 3) we obtain \( \text{acc} \tilde{\sigma}_R(a + b, A) \subset \eta \tilde{\sigma}_R(a, A) \).

2) The proof of this statement follows exactly in the same way as 1) if we observe that \( b + J \in QN(A/J) \) and \( a + J \) and \( b + J \) commute in \( A/J \) implies that \( \sigma(a + b + J, A/J) = \sigma(a + J, A/J) \). \( \square \)

5.2. **Corollary.** Let \( A \) be a Banach algebra and suppose \( \mathcal{R} \) is a regularity of \( A \) such that \( \partial A^{-1} \cap \mathcal{R} = \emptyset \). If \( a \in A \) and \( b \in \text{Rad} A \) then \( \text{acc} \tilde{\sigma}_R(a + b, A) \subset \eta \tilde{\sigma}_R(a, A) \).

5.3. **Corollary.** Let \( A \) be a semisimple Banach algebra and suppose \( \mathcal{R} \) is a regularity of \( A \) such that \( \partial A^{-1} \cap \mathcal{R} = \emptyset \). If \( a \in A \) and if \( b \in A \) is rank one then \( \text{acc} \tilde{\sigma}_R(a + b, A) \subset \eta \tilde{\sigma}_R(a, A) \).

**Proof.** If \( b \in A \) is rank one, then it belongs to the inessential ideal \( \mathcal{F}(A) \) of finite elements [19, Sections 2 and 3]. By [1, Corollary 5.7.6] the closure \( \overline{\mathcal{F}(A)} \) of \( \mathcal{F}(A) \) is also an inessential ideal. \( \square \)

One can also provide a direct proof of Corollary 5.3 if one combines [9, Theorem 5] and Theorem 4.2.2) and 3).

5.4. **Theorem.** Let \( A \) and \( B \) be Banach algebras and \( T: A \to B \) a bounded homomorphism with closed range. If \( \mathcal{R} \) is a regularity of \( A \) and \( \mathcal{M} \) is a regularity of \( B \) with \( \partial B^{-1} \cap \mathcal{M} = \emptyset \) then for each \( a \in A \)

\[
\bigcap_{Tb = 0} \tilde{\sigma}_R(a + b, A) \subset \eta \tilde{\sigma}_M(Ta, B).
\]

**Proof.** This follows from [5, Theorem 3], Proposition 2.2.2) and Theorem 4.2.3). \( \square \)

For the spectrum and singular spectrum the results in this section are familiar: e.g. [13, Section 3], [5, Theorem 5], [17, Theorem 5.3] and [1, Theorem 5.7.4 (iii)].
6. Regular elements

It is well known [7, Examples 4.5 and 4.6] and [10, Examples 1 and 2] that the elements of \( \hat{A} \) and \( A^{-1}A^\bullet \) do not multiply well and so in general neither \( \hat{A} \) nor \( A^{-1}A^\bullet \) is a regularity in \( A \). However, we have the following

6.1. Proposition [12, Lemma 2.8]. Let \( a, b, c, d \) be mutually commuting elements in a Banach algebra \( A \) with \( ac + bd = 1 \). Then \( ab \in \hat{A} \) if and only if \( a, b \in \hat{A} \).

6.2. Lemma. Let \( A \) be a semiprime Banach algebra. Then \( F(A) \subset A^{-1}A^\bullet \subset \hat{A} \).

Proof. We prove first that \( F(A) \subset \hat{A} \). If \( u \in F(A) \) then by [19, Theorem 3.4] there is an idempotent \( p \in F(A) \cap uA \) such that \( u = pu \). Since \( p \in uA \), we have \( p = uv \) for some \( v \in A \). Consequently, \( u = uvu \) which proves that \( u \) is regular. This together with \( F(A) \) being an inessential ideal in \( A \) gives \( F(A) \subset A^{-1}A^\bullet \) [10, Theorem 7 (7.2)].

6.3. Theorem. Let \( A \) be a semiprime Banach algebra. Then \( \hat{A} + F(A) \subset \hat{A} \).

Proof. By the last lemma \( F(A) \subset \hat{A} \). The result now follows from [8, (7.3.2.6)].

This result was proved by Kordula and Müller [12, Lemma 2.9] in the algebra \( L(X) \) of bounded linear operators on a Banach space \( X \) by different methods if one recalls that in the algebra \( L(X) \) the ideal of finite elements coincides with the ideal of finite rank operators, see [19].

Let \( J \) be an ideal in \( A \). We say \( a \in A \) is \( J \)-Fredholm if \( a + J \) is invertible in the quotient algebra \( A/J \). Recall [12, p. 111] that \( R_7 = \{ a \in A \mid a \text{ is } J \text{-Fredholm} \} \) is a set satisfying (P1) and is therefore a regularity in \( A \).

6.4. Proposition. Suppose \( J \) is an ideal in \( A \) such that \( J \subset \hat{A} \). Then \( R_7 \subset \hat{A} \).

Proof. If \( a \in R_7 \) then \( a \) is \( J \)-Fredholm and so by 1.2, we have \( a + J \in \hat{A}/J \). Since \( J \subset \hat{A} \), it follows from [8, Theorem 7.3.3] that \( a \in \hat{A} \). 

6.5. Theorem. If \( J \) is a closed \( s \)-inessential ideal in \( A \) such that \( J \subset \hat{A} \) then \( R_7 \subset A^{-1}A^\bullet \).

Proof. By Proposition 6.4 we have that \( R_7 \subset \hat{A} \). Also, if \( a \in R_7 \) then \( 0 \notin \sigma(a + J, A/J) \). In view of \( J \) being \( s \)-inessential it follows that \( a \in \overline{A^{-1}} \). By 1.3 we conclude \( a \in A^{-1}A^\bullet \).
6.6. **Theorem.** Let $A$ be a semisimple Banach algebra and let $J$ be an inessential ideal in $A$. Then $J \cap \widehat{A} \subset \mathcal{F}(A)$.

**Proof.** Suppose $a = aa'a$ for some $a'$ in $A$. If $a \in J$ then in view of [16, Theorem 1.4] the idempotent $a'a \in J \subset \text{kh}\mathcal{F}(A)$. By [20, Theorem 4.6] we have $a'a \in \mathcal{F}(A)$. Since $\mathcal{F}(A)$ is an ideal in $A$ it follows that $a \in \mathcal{F}(A)$. □

This result was proved by Harte [7, Theorem 4.2(4.2.1)] in the algebra $\mathcal{L}(X)$ of bounded linear operators on a Banach space $X$.

7. **An example**

In this section we provide an example of a regularity in a Banach algebra and investigate how this regularity is related to the set of decomposably regular elements.

An element $a \in A$ is said to be *almost invertible* if $0 \notin \text{acc} \sigma(a)$ [6]. We have the following implications:

invertible $\implies$ almost invertible $\implies$ $J$-Fredholm $\implies$ $J$-Fredholm.

Let $J$ be a closed ideal in a Banach algebra $A$. Denote

$$\mathcal{R}_0(J) = \{a \in A \mid a \text{ is almost invertible } J\text{-Fredholm}\}.$$ 

**7.1. Proposition.** Suppose a closed ideal $J$ in $A$ is s-inessential. Then $\mathcal{R}_0(J)$ is a regularity in $A$.

**Proof.** We prove that $\mathcal{R}_0(J)$ satisfies (P1). If $a, b \in \mathcal{R}_0(J)$ with $ab = ba$ then $ab$ is $J$-Fredholm. Since $\sigma(ab) \subset \sigma(a) \cdot \sigma(b)$ it follows that $ab \in \mathcal{R}_0(J)$. Conversely, if $ab \in \mathcal{R}_0(J)$ then $a$ and $b$ are $J$-Fredholm because $ab = ba$. This together with $J$ s-inessential gives $a, b \in \mathcal{R}_0(J)$. □

**7.2. Corollary.** $\widetilde{\sigma}_{\mathcal{R}_0(J)}(a) = \text{acc} \sigma(a) \cup \sigma(a + J, A/J)$ for every $a \in A$.

**Proof.** This follows from the definition of $\mathcal{R}_0(J)$. □

We will prove later that $\mathcal{R}_0(J)$ is actually an open regularity, see Theorem 7.5. However, to prove a stronger result we need the following

**7.3. Definition.** Let $J$ be a closed ideal in $A$ and $a \in A$. We say that $a$ is $J$-Browder if $a = x + y$ with $x \in A^{-1}$, $y \in J$ and $xy = yx$. 

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Then we have the following implications [6, 16]:

(7.4) invertible $\implies$ almost invertible $\implies$ $J$-Fredholm $\implies$ $J$-Browder $\implies$ $J$-Fredholm.

If $A$ and $B$ are Banach algebras then the homomorphism $T: A \to B$ is said to have the Riesz property if its kernel $T^{-1}(0)$ is an inessential ideal. If $J$ is a closed inessential ideal then the almost invertible $J$-Fredholm and $J$-Browder elements coincide [6, Theorem 1] or [17, Corollary 3.6].

7.5. **Theorem.** Suppose $J$ is a closed inessential ideal in $A$. Then $\mathcal{R}_0(J)$ is an open regularity in $A$.

**Proof.** We prove that $\mathcal{R}_0(J)$ satisfies (P1). If $a, b \in \mathcal{R}_0(J)$ with $ab = ba$ then it follows as in the proof of Proposition 7.1 that $ab \in \mathcal{R}_0(J)$. Conversely, if $ab \in \mathcal{R}_0(J)$ then by 7.4 $ab$ is $J$-Browder. In view of $ab = ba$ and $J$ being inessential (meaning that the quotient map $A \to A/J$ has the Riesz property) it follows from [8, Theorem 7.7.6] that both $a$ and $b$ are $J$-Browder. By the remarks following 7.4 we have $a, b \in \mathcal{R}_0(J)$.

We prove finally that $\mathcal{R}_0(J)$ is open. Let $x \in \mathcal{R}_0(J)$ and let $\varepsilon > 0$ satisfy $\{\lambda \in \mathbb{C} \mid |\lambda| < 3\varepsilon\} \cap \sigma(x) \subset \{0\}$. Since $\sigma(\cdot)$ and $\sigma(\cdot, A/J)$ are both upper semicontinuous there exists $\delta > 0$ such that if $\|x - y\| < \delta$ then $y$ is $J$-Fredholm,

$$\sigma(y) \subset \{\lambda \in \mathbb{C} \mid |\lambda| < \varepsilon\} \cup \{\lambda \in \mathbb{C} \mid |\lambda| > 2\varepsilon\}$$

and

$$\sigma(y + J, A/J) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \geq 2\varepsilon\}.$$ 

However, since $J$ is inessential, $\sigma(y) \setminus \sigma(y + J, A/J)$ consists of isolated points and some of the holes of $\sigma(y + J, A/J)$ [4, Theorem 6.1]. Hence either $0 \notin \sigma(y)$ or $0 \in \text{iso}\sigma(y)$ and so $y$ is almost invertible. We have shown that $y \in \mathcal{R}_0(J)$. □

The above theorem was proved in the operator algebra $\mathcal{L}(X)$ of bounded linear operators on a Banach space $X$ by Kordula and Müller [12, Theorem 2.1].

7.6. **Theorem.** Suppose $J$ is a closed inessential ideal in a semisimple Banach algebra $A$. Then $\mathcal{R}_0(J) \subset A^{-1}A^\ast$.

**Proof.** If $a \in \mathcal{R}_0(J)$ then $a$ is almost invertible and so $a \in \overline{A^{-1}}$. Since $a$ is $J$-Fredholm and since $J \subset \text{kh}\mathcal{F}(A)$ [16, Theorem 4.6] it follows that $a$ is $\text{kh}\mathcal{F}(A)$-Fredholm. In view of $\mathcal{F}(A)$ and $\text{kh}\mathcal{F}(A)$ having the same set of idempotents, see the remark following Lemma 5.7.1 in [1], we have by [1, Theorem 5.7.2] that $a$ is $\mathcal{F}(A)$-Fredholm. By Lemma 6.2 and Proposition 6.4 we obtain $a \in \hat{A}$. It follows from 1.3 that $a \in A^{-1}A^\ast$. □

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