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THE TYPE SET FOR SOME MEASURES ON $\mathbb{R}^{2n}$ WITH n-DIMENSIONAL SUPPORT

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Abstract. Let $\varphi_1,\dots,\varphi_n$ be real homogeneous functions in $C^\infty(\mathbb{R}^n-\{0\})$ of degree $k\geqslant 2$, let $\varphi(x)=(\varphi_1(x),\dots,\varphi_n(x))$ and let μ be the Borel measure on $\mathbb{R}^{2n}$ given by

$$\mu(E)=\int_{\mathbb{R}^n}\chi_E(x,\varphi(x))\,|x|^{\gamma-n}\,\mathrm{d}x$$

where $\mathrm{d}x$ denotes the Lebesgue measure on $\mathbb{R}^n$ and $\gamma>0$. Let T_μ be the convolution operator $T_\mu f(x)=(\mu*f)(x)$ and let

$$E_\mu=\{(1/p,1/q)\colon\ \|T_\mu\|_{p,q}<\infty,\ \ 1\leqslant p,\,q\leqslant\infty\}.$$

Assume that, for $x\neq 0$, the following two conditions hold: $\det(\mathrm{d}^2\varphi(x)h)$ vanishes only at $h=0$ and $\det(\mathrm{d}\varphi(x))\neq 0$. In this paper we show that if $\gamma>n(k+1)/3$ then E_μ is the empty set and if $\gamma\leqslant n(k+1)/3$ then E_μ is the closed segment with endpoints $D=\bigl(1-\frac{\gamma}{n(k+1)},1-\frac{2\gamma}{n(k+1)}\bigr)$ and $D'=\bigl(\frac{2\gamma}{n(1+k)},\frac{\gamma}{n(1+k)}\bigr)$. Also, we give some examples.

Keywords: singular measures, convolution operators

MSC 2000: 42B20

1. Introduction

Let $\varphi_1,\dots,\varphi_n$ be real homogeneous functions in $C^\infty(\mathbb{R}^n-\{0\})$ of degree $k\geqslant 2$, let $\varphi(x)=(\varphi_1(x),\dots,\varphi_n(x))$, let $\gamma>0$ and let μ be the Borel measure on $\mathbb{R}^{2n}$ given by

$$\mu(E)=\int_{\mathbb{R}^n}\chi_E(x,\varphi(x))\,|x|^{\gamma-n}\,\mathrm{d}x,$$

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where $\mathrm{d}x$ denotes the Lebesgue measure on $\mathbb{R}^n$. Let T_μ be the convolution operator defined by $T_\mu f(x) = (\mu * f)(x)$ and let $\|T_\mu\|_{p,q}$ be the operator norm of T_μ from $L^p(\mathbb{R}^{2n})$ into $L^q(\mathbb{R}^{2n})$. The type set E_μ is the set defined by

$$E_\mu = \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \colon \; \|T_\mu\|_{p,q} < \infty, \;\; 1 \leqslant p, \, q \leqslant \infty \right\},$$

where the L^p spaces are taken with respect to the Lebesgue measure on $\mathbb{R}^{2n}$.

Since the adjoint T_μ^* is a convolution operator with a measure of the same kind, E_μ is symmetric with respect to the non principal diagonal. The Riesz Thorin theorem implies that E_μ is a convex set. On the other hand, it is a well known fact that E_μ lies below the principal diagonal $1/q = 1/p$. Also, a result of Oberlin (see e.g. [4], Theorem 1) says that

$$E_\mu \subset \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \colon \; \frac{1}{q} \geqslant \frac{2}{p} - 1 \right\}. \tag{1.1}$$

Thus, by the symmetry of E_μ, also

$$E_\mu \subset \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \colon \; \frac{1}{q} \geqslant \frac{1}{2p} \right\}. \tag{1.2}$$

The type set E_μ has been studied, for $\gamma = 2$ and under a suitable hypothesis on φ, in [2] covering a wide amount of cases. As there, if $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a twice continuously differentiable function, we say that $x \in \mathbb{R}^n$ is an elliptic point for φ if there exists $\lambda = \lambda_x > 0$ such that $|\det(\varphi''(x)h)| \geqslant \lambda |h|^n$ for all $h \in \mathbb{R}^n$ ([2], p. 152).

Convolution operators associated with fractional measures on $\mathbb{R}^2$ supported on the graph of the parabola (t, t^2) have been studied in [1] by M. Christ, using a Littlewood Paley decomposition of the operator.

Our aim is to obtain an explicit description of E_μ, for a homogeneous and smooth φ as above, under the following assumptions.

1) The first differential $\mathrm{d}\varphi(x)$ is invertible for all $x \in \mathbb{R}^n - \{0\}$.

2) Every $x \neq 0$ is an elliptic point for φ.

To this end we will adapt Christ's arguments to our actual setting, using some results obtained in [2].

Finally, we will prove some facts concerning the two dimensional quadratic polynomial case.

Throughout the paper c will denote a positive constant not necessarily the same at each occurrence.

2. Preliminaries

Let η be a function in $C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\eta) \subset \{x \in \mathbb{R}^n : \frac{1}{4} \leqslant |x| \leqslant 2\}$, $0 \leqslant \eta \leqslant 1$ and $\sum\limits_{j\in\mathbb{Z}} \eta(2^j x) = 1$ if $x \neq 0$. For $j \in \mathbb{Z}$, let μ_j be the Borel measure on $\mathbb{R}^{2n}$ defined by

$$\mu_j(E) = \int_{\mathbb{R}^n} \chi_E(x, \varphi(x))\eta(2^j x)\,|x|^{\gamma-n}\,\mathrm{d}x$$

and let T_{μ_j} be the associated convolution operator.

For $t > 0$, $(x, y) \in \mathbb{R}^{2n}$ and for $f\colon \mathbb{R}^{2n} \to C$, we set $t \bullet (x, y) = (tx, t^k y)$ and $(t \bullet f)(x, y) = f(t \bullet (x, y))$. So $\|t \bullet f\|_q = t^{-\frac{n(k+1)}{q}} \|f\|_q$, $1 \leqslant q < \infty$, and $\|t \bullet f\|_\infty = \|f\|_\infty$. A standard homogeneity argument gives

Lemma 2.1. *Let $1 \leqslant p,\ q \leqslant \infty$. Then*

$$\|T_{\mu_j}\|_{p,q} = 2^{\left(-\gamma - \frac{n(k+1)}{q} + \frac{n(k+1)}{p}\right) j} \|T_{\mu_0}\|_{p,q}$$

for all $j \in \mathbb{Z}$. Moreover, if T_μ is bounded from $L^p(\mathbb{R}^{2n})$ into $L^q(\mathbb{R}^{2n})$ then $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n(k+1)}$.

Proof. For $(x, y) \in \mathbb{R}^{2n}$ a change of variable gives

$$\begin{aligned} T_{\mu_0}(2^{-j} \bullet f)(x, y) &= \int_{\mathbb{R}^n} (2^{-j} \bullet f)\big(x - w, y - \varphi(w)\big)\eta(w)\,|w|^{\gamma-n}\,\mathrm{d}w \\ &= 2^{jn} \int_{\mathbb{R}^n} f\big(2^{-j}x - z, 2^{-jk}y - \varphi(z)\big)\eta(2^j z)|2^j z|^{\gamma-n}\,\mathrm{d}z \\ &= 2^{j\gamma}(2^{-j} \bullet T_{\mu_j} f)(x, y). \end{aligned}$$

So

$$\|T_{\mu_j}\|_{p,q} = 2^{\left(-\gamma - \frac{n(k+1)}{q} + \frac{n(k+1)}{p}\right) j} \|T_{\mu_0}\|_{p,q}$$

and the first assertion of the lemma follows. On the other hand, if T_μ is bounded then $\sup\limits_{j\in\mathbb{Z}} \|T_{\mu_j}\|_{p,q} < \infty$ and so $-\gamma - \frac{n(k+1)}{q} + \frac{n(k+1)}{p} = 0$. $\square$

Remark 2.2. Let D be the intersection, in the $\left(\frac{1}{p}, \frac{1}{q}\right)$ plane, of the lines $\frac{1}{q} = \frac{2}{p} - 1$, $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n(k+1)}$ and let D' be its symmetric with respect to the non principal diagonal. So $D = \left(1 - \frac{\gamma}{n(k+1)}, 1 - \frac{2\gamma}{n(k+1)}\right)$ and $D' = \left(\frac{2\gamma}{n(k+1)}, \frac{\gamma}{n(k+1)}\right)$. Then (1.1), (1.2) and Lemma 2.1 imply that E_μ is the empty set for $\gamma > n(k+1)/3$ and that, for $\gamma \leqslant n(k+1)/3$, E_μ is contained in the closed segment with endpoints D and D'.

Let ν_0 be the Borel measure given by $\nu_0(E)=\int\chi_E(w,\varphi(w))\eta(w)\,\mathrm{d}w$. Then Theorem 3 in [2] and a compactness argument imply that $\left(\frac{2}{3},\frac{1}{3}\right)\in E_{\nu_0}$. Now $T_{\mu_0}f\leqslant cT_{\nu_0}f$ for $f\geqslant 0$, thus $\left(\frac{2}{3},\frac{1}{3}\right)\in E_{\mu_0}$. Since $(1,1)\in E_{\mu_0}$, the Riesz Thorin theorem implies that if $\gamma\leqslant n(k+1)/3$ then D belongs to E_{μ_0}. Moreover, for these γ, if p_D, q_D are given by $D=\left(\frac{1}{p_D},\frac{1}{q_D}\right)$, Lemma 2.1 says that there exists c independent of j such that

$$\|T_{\mu_j}\|_{p_D,q_D}\leqslant c \tag{2.3}$$

for all $j\in\mathbb{Z}$.

3. L^p-L^q estimates

In order to study E_μ, we will assume in this section that φ satisfies the hypotheses 1) and 2) stated in the introduction.

We modify, to our actual setting, Christ's arguments developed in [1], involving a Littlewood Paley decomposition of the operator. Decompositions of this kind have been used also in [6] to study fractional measures supported on curves and in [3] to study fractional measures supported on the graphs of holomorphic functions of one complex variable.

Let us consider the Fourier transform $\widehat{\mu}_0$. For $\xi=(\xi_1,\dots,\xi_{2n})\in\mathbb{R}^{2n}$ we put $\xi'=(\xi_1,\dots,\xi_n)$, $\xi''=(\xi_{n+1},\dots,\xi_{2n})$, then

$$\widehat{\mu}_0(\xi)=\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}\langle\xi',w\rangle-\mathrm{i}\langle\xi'',\varphi(w)\rangle}\eta(w)\,|w|^{\gamma-n}\,\mathrm{d}w.$$

For a fixed ξ, let $\Phi(w)=\langle\xi',w\rangle+\langle\xi'',\varphi(w)\rangle$, $w\in\mathbb{R}^n$. Suppose that Φ has a critical point w belonging to the support of η, then $\xi_j+\sum\limits_{k=1}^{n}\xi_{n+k}\frac{\partial\varphi_k}{\partial w_j}(w)=0$ for $j=1,\dots,n$. Now, the jacobian matrix of φ is continuous with a continuous inverse, hence there exist two positive constants c_1,c_2 independent of ξ such that ξ belongs to the interior of the cone $\Gamma_0=\{\xi\in\mathbb{R}^{2n}:\ c_1|\xi''|\leqslant|\xi'|\leqslant c_2|\xi''|\}$.

Let m_0 be a function belonging to $C^\infty(\mathbb{R}^{2n}-\{0\})$ homogeneous of degree zero with respect to the Euclidean dilations on $\mathbb{R}^{2n}$ such that $\operatorname{supp}(m_0)\subset\Gamma_0$ and let $m_j(y)=m_0(2^{-j}\bullet y)$. Moreover, modifying if necessary c_1 and c_2, m_0 can be chosen such that $\{m_j\}_{j\in\mathbb{Z}}$ is a C^∞ partition of the unity in $\mathbb{R}^{2n}$ minus the subspaces $\xi'=0$, $\xi''=0$. Let Q_j be the operator with the multiplier m_j and let C_0 be a large constant such that $\widetilde{m}_j=\sum\limits_{|i-j|\leqslant C_0}m_i$ is identically one on $2^j\bullet\Gamma_0$. We define $\widetilde{Q}_j=\sum\limits_{|i-j|\leqslant C_0}Q_i$. Let $h\in C_c^\infty(\mathbb{R}^{2n})$ be identically one in a neighbourhood of the origin. Taking account of Proposition 4 in [8] p. 341 and of the above observation about the critical points of Φ, we note that

$$\widehat{\mu}_0(1-h)(1-\widetilde{m}_0) \in S(\mathbb{R}^{2n}). \tag{3.1}$$

Let $h_j(y) = h(2^{-j} \bullet y)$ and let P_j be the Fourier multiplier operator with the symbol h_j. We will need the following three lemmas. They are proved for the case $n = 2$ in [3] (Remarks 2.11, 2.12 and 2.13). The same proofs hold, with the obvious changes, for an arbitrary n.

Lemma 3.2. *Let $\{\sigma_j\}_{j\in\mathbb{Z}}$ be a sequence of positive measures on $\mathbb{R}^{2n}$, and let $T_j f = \sigma_j * f$ for $f \in S(\mathbb{R}^{2n})$. Suppose $1 < p \leqslant 2$ and $p \leqslant q < \infty$. If there exists $A > 0$ such that $\sup\limits_{j\in\mathbb{Z}} \|T_j\|_{p,q} \leqslant A$, $\Big\| \sum\limits_{-J\leqslant j\leqslant J} T_j P_j \Big\|_{p,q} \leqslant A$ and $\Big\| \sum\limits_{-J\leqslant j\leqslant J} T_j(I-P_j)(I-\widetilde{Q}_j)\Big\|_{p,q} \leqslant A$ for all $J \in \mathbb{N}$, then there exists $c > 0$ independent of A, J and $\{\sigma_j\}_{j\in\mathbb{Z}}$, such that*

$$\Big\| \sum_{-J\leqslant j\leqslant J} T_j \Big\|_{p,q} \leqslant cA.$$

Lemma 3.3. *The kernel of the convolution operator*

$$\sum_{-J\leqslant j\leqslant J} T_{\mu_j}(I-P_j)(I-\widetilde{Q}_j)$$

belongs to weak-$L^{\frac{n(k+1)}{n(k+1)-\gamma}}(\mathbb{R}^{2n})$ with the weak constant independent of J.

Lemma 3.4. *The kernel of the convolution operator $\sum\limits_{-J\leqslant j\leqslant J} T_{\mu_j} P_j$ belongs to weak-$L^{\frac{n(k+1)}{n(k+1)-\gamma}}(\mathbb{R}^{2n})$ with the weak constant independent of J.*

Theorem 3.5. *If $\gamma \leqslant n(k+1)/3$ then E_μ is the closed segment with endpoints D and D'.*

P r o o f. Taking into account the considerations stated in the introduction, it is enough to check that $D \in E_\mu$. Lemmas 3.3, 3.4 and weak Young's inequality imply that there exists A independent of J such that

$$\Big\| \sum_{-J\leqslant j\leqslant J} T_{\mu_j} P_j \Big\|_{p_D,q_D} \leqslant A \quad \text{and} \quad \Big\| \sum_{-J\leqslant j\leqslant J} T_{\mu_j}(I-P_j)(I-\widetilde{Q}_j)\Big\|_{p_D,q_D} \leqslant A.$$

By virtue of (2.3), Lemma 3.2, and of the fact that $T_\mu f \leqslant \sum\limits_{j\in\mathbb{Z}} T_{\mu_j} f$ for $f \geqslant 0$, the theorem follows. □

Now we consider a local version of the problem, that is to say the study of the type set corresponding to the convolution operator T_σ with the Borel measure given by

$$\sigma(E) = \int_{|x|\leqslant 1} \chi_E(x, \varphi(x))\, |x|^{\gamma - n}\, \mathrm{d}x$$

with $\gamma > 0$.

Theorem 3.6. *If $\gamma > n(k+1)/3$, then E_σ is the triangular region with vertices $(\frac{2}{3}, \frac{1}{3})$, $(0,0)$ and $(1,1)$.*

If $\gamma \leqslant n(k+1)/3$ then E_σ is the closed polygonal region with vertices D, D', $(0,0)$ and $(1,1)$.

Proof. $E_\mu \subset E_\sigma$. Since E_σ is a convex set symmetric with respect to the non principal diagonal and since σ is a finite measure, $(1,1)$ and $(0,0)$ belong to E_σ. On the other hand, the constrains (1.1) and (1.2) hold for E_σ. Moreover, Lemma 2.1 implies that if $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\sigma$, hence $\frac{1}{q} \geqslant \frac{1}{p} - \frac{\gamma}{n(k+1)}$. Thus the case $\gamma \leqslant n(k+1)/3$ follows from Theorem 3.5.

If $\gamma > n(k+1)/3$, $(\frac{2}{3}, \frac{1}{3})$ lies above the line $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n(k+1)}$ and we have noted in Section 2 that $(\frac{2}{3}, \frac{1}{3})$ belongs to E_{μ_0}, so Lemma 2.1 implies that $(\frac{2}{3}, \frac{1}{3}) \in E_\sigma$. $\square$

Example 3.7. Let us consider $\mathbb{R}^2 \simeq C$ and $\mathbb{R}^4 \simeq C^2$ via $(x_1, x_2) \to x_1 + \mathrm{i}x_2$ and $(x_1, x_2, x_3, x_4) \to (x_1 + \mathrm{i}x_2, x_3 + \mathrm{i}x_4)$, respectively. Let $a \in C - \{0\}$ and let $\varphi\colon C \to C$ be given by $\varphi(z) = az^k$, $k \geqslant 2$. So $\mathrm{d}\varphi(z)w = kaz^{k-1}w$ and $\mathrm{d}^2\varphi(z)(w, \widetilde{w}) = k(k-1)az^{k-2}w\widetilde{w}$ for $w, \widetilde{w} \in C$. So φ satisfies the assumptions 1) and 2) in the introduction. So, Theorem 3.5 says that for $0 < \gamma \leqslant 2(k+1)/3$, E_μ is the closed segment with endpoints $\left(1 - \frac{\gamma}{2(k+1)}, 1 - \frac{\gamma}{1+k}\right)$ and $\left(\frac{\gamma}{1+k}, \frac{\gamma}{2(k+1)}\right)$.

4. Quadratic functions in $\mathbb{R}^2$

As in [2], we consider quadratic functions $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\varphi(x) = \Phi(x, x)$ where $\Phi\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is a symmetric bilinear function. Two such functions φ and $\widetilde{\varphi}$ are equivalent if there exist linear authomorphisms α, β such that $\varphi(x) = \alpha(\widetilde{\varphi}(\beta(x)))$. Thus equivalent functions yield to the same E_μ. It is pointed in [2] that each equivalence class contains exactly one of the following canonical forms:

I) $\varphi(x) = (0, 0)$,
II) $\varphi(x) = \left(\frac{1}{2}x_1^2, 0\right)$,
III) $\varphi(x) = \left(\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2, 0\right)$,
IV) $\varphi(x) = \left(x_1 x_2, \frac{1}{2}x_2^2\right)$,

V) $\varphi(x) = \left(\frac{1}{2}x_1^2, \frac{1}{2}x_2^2\right)$,

VI) $\varphi(x) = \left(\frac{1}{2}(x_1^2 - x_2^2), x_1(ax_1 + x_2)\right)$, $0 \leqslant a < 1$.

In each case we have, as in Remark 2.2, that $E_\mu = \emptyset$ for $\gamma > 2$. In the first three cases, the support of the measure is contained in a hyperplane, so E_μ reduces to the empty set. In the fifth case, from [2] we obtain that $\left(\frac{2}{3}, \frac{1}{3}\right) \in E_{\nu_0} = E_{\mu_0}$. Lemma 2.1 implies that $\{T_{\mu_j}\}_{j\in\mathbb{Z}}$ is a sequence of operators uniformly bounded on D. Thus we can proceed as in the proof of Theorem 3.5 to obtain that, for $0 < \gamma \leqslant 2$, E_μ is the closed segment with endpoints D and D'. In the sixth case, a computation shows that φ satisfies the assumptions 1) and 2) stated in the introduction and so E_μ is the same closed segment. In the fourth case, since $\left(x_1x_2, \frac{1}{2}x_2^2\right)$ is equivalent to (x_1^2, x_1x_2) we will assume that $\varphi = (x_1^2, x_1x_2)$. In the local case we can obtain for this φ the following result:

Theorem 4.1. *Assume* $\varphi(x) = (x_1^2, x_1x_2)$.

a) If $\gamma \geqslant 3/2$, *then* E_σ *contains the closed triangular region with vertices* $(0,0)$, $(1,1)$ *and* $(\frac{5}{8}, \frac{3}{8})$. *Moreover, the point* $(\frac{5}{8}, \frac{3}{8})$ *is the lowest point of* E_σ *lying on the non principal diagonal.*

b) If $0 < \gamma < 3/2$, *then* E_σ *contains the closed polygonal region with vertices* $(0,0)$, $(1,1)$, $\left(1 - \frac{1}{4}\gamma, 1 - \frac{5}{12}\gamma\right)$ *and* $\left(\frac{5}{12}\gamma, \frac{1}{4}\gamma\right)$. *Moreover, the point* $\left(\frac{1}{2} + \frac{\gamma}{12}, \frac{1}{2} - \frac{\gamma}{12}\right)$ *is the lowest point of* E_σ *lying on the non principal diagonal.*

P r o o f. We take a rectangle $R \subset \{x \in \mathbb{R}^2 \colon |x| < 1\}$ of the form $[-\frac{1}{2}, \frac{1}{2}] \times [a,b]$, $a > 0$. We define the measure $\mu_R(E) = \int_R \chi_E(x_1, x_2, \varphi(x_1,x_2))\,\mathrm{d}x_1\,\mathrm{d}x_2$ and denote by T_R the corresponding convolution operator. We now define $t \circ (x_1, \ldots, x_4) = (tx_1, x_2, t^2x_3, tx_4)$ and $t \circ f(x) = f(t \circ x)$. It is easy to see that for $f \geqslant 0$ and $j \in \mathbb{N}$, $T_Rf(2^j \circ x) \leqslant 2^j T_R(2^j \circ f)(x)$, and so if T_R is bounded from $L^p(\mathbb{R}^4)$ into $L^q(\mathbb{R}^4)$, then $\frac{1}{q} \geqslant \frac{1}{p} - \frac{1}{4}$. Now, for $f \geqslant 0$, $T_Rf(x) \leqslant c_\gamma T_\sigma f(x)$, hence $E_\sigma \subset \left\{\left(\frac{1}{p}, \frac{1}{q}\right) \colon \frac{1}{q} \geqslant \frac{1}{p} - \frac{1}{4}\right\}$. Lemma 2.1 implies that $E_\sigma \subset \left\{\left(\frac{1}{p}, \frac{1}{q}\right) \colon \frac{1}{q} \geqslant \frac{1}{p} - \frac{\gamma}{6}\right\}$.

We consider the Borel measure ν on $\mathbb{R}^4$ given by

$$\nu(E) = \int \chi_E(x_1, x_2, x_1^2, x_1x_2)\Psi(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

where $\Psi(x_1,x_2)$ is a function in $C_c^\infty(\mathbb{R}^2)$ satisfying $0 \leqslant \Psi \leqslant 1$ and $\Psi(x) = 1$ for $|x| \leqslant 2$. We will check now that $(\frac{5}{8}, \frac{3}{8})$ belongs to E_ν.

A direct application of Corollary to Proposition 5, p. 342 in [7] gives, for $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$,

$$|\widehat{\nu}(\xi)| \leqslant \frac{c}{|\xi_3|^{1/2}}. \tag{4.2}$$

On the other hand, let $U_{\xi_3,\xi_4} \in S'(\mathbb{R}^2)$ be given by

$$\langle U_{\xi_3,\xi_4}, f\rangle = \int \mathrm{e}^{-\mathrm{i}(\xi_3 x_1^2+\xi_4 x_1 x_2)} f(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

Now, $\xi_3 x_1^2 + \xi_4 x_1 x_2$ is a quadratic form in (x_1, x_2), so $\widehat{U}_{\xi_3,\xi_4}$ is a locally integrable and explicitly computable function (see e.g. [5], p. 349). Moreover,

$$|\widehat{U}_{\xi_3,\xi_4}(\xi_1,\xi_2)| \leqslant \frac{c}{|\det(A)|^{1/2}} = \frac{c}{|\xi_4|}$$

with c independent of ξ, where A is the symmetric matrix defining the quadratic form $\xi_3 x_1^2 + \xi_4 x_1 x_2$. Now

$$\begin{aligned}|\widehat{\nu}(\xi)| &= |(\Psi U_{\xi_3,\xi_4})^\wedge(\xi_1,\xi_2)| = |(\widehat{\Psi} * \widehat{U}_{\xi_3,\xi_4})(\xi_1,\xi_2)| \\ &\leqslant \|\widehat{\Psi} * \widehat{U}_{\xi_3,\xi_4}\|_\infty \leqslant \|\widehat{\Psi}\|_1 \|\widehat{U}_{\xi_3,\xi_4}\|_\infty \leqslant \frac{c}{|\xi_4|}.\end{aligned}$$

From this inequality and (4.2) we obtain

$$|\widehat{\nu}(\xi)| \leqslant \frac{c}{|\xi_3|^{1/3}|\xi_4|^{1/3}}. \tag{4.3}$$

Now, for $z \in C$, we consider the analytic family of distributions I_z which for $\mathrm{Re}(z) > 0$ are given by $I_z(t) = \frac{2^{-z/2}}{\Gamma(z/2)}|t|^{z-1}$, $t \in \mathbb{R}$. Let $J_z = \delta \otimes \delta \otimes I_z \otimes I_z$, hence $\widehat{J}_z = 1 \otimes 1 \otimes I_{1-z} \otimes I_{1-z}$. We define the analytic family of operators given by $T_z f = \nu * J_z * f$, $f \in S(\mathbb{R}^4)$. It is easy to show that if $\mathrm{Re}(z) = 1$ then $\|T_z\|_{1,\infty} = \|\nu * J_z\|_\infty \leqslant c_z$. Also, for $\mathrm{Re}(z) = -\frac{1}{3}$, (4.3) implies that $\|T_z\|_{2,2} \leqslant \|\widehat{\nu}\widehat{J}_z\|_\infty \leqslant c'_z$. Now we apply the complex interpolation theorem (see [S-W], p. 205) in the strip $-\frac{1}{3} \leqslant \mathrm{Re}(z) \leqslant 1$. Since $T_0 = cT_\nu$ it follows that $(\frac{5}{8}, \frac{3}{8})$ belongs to E_ν.

To prove a) it remains to check that $(\frac{5}{8}, \frac{3}{8})$ belongs to E_σ. Now, if $\gamma \geqslant 2$ and $f \geqslant 0$, then $T_\sigma f(x) \leqslant T_\nu f(x)$ and so in this case a) follows. For $3/2 \leqslant \gamma < 2$, we use Christ's argument as in Section 2. In fact, we observe that $T_{\mu_0} f(x) \leqslant cT_\nu f(x)$ and then $(\frac{5}{8}, \frac{3}{8})$ belongs to E_{μ_0}. Lemma 2.1 implies that $\{T_{\mu_j}\}_{j\in\mathbb{Z}}$ are uniformly bounded operators from $L^{8/5}$ into $L^{8/3}$.

To prove b) we proceed as in the case $\frac{3}{2} \leqslant \gamma < 2$. Since $\gamma < \frac{3}{2}$ we interpolate between $(\frac{5}{8}, \frac{3}{8})$ and $(1, 1)$. The Riesz Thorin theorem implies that $\bigl(1 - \frac{1}{4}\gamma, 1 - \frac{5}{12}\gamma\bigr) \in E_{\mu_0}$. We invoke again Lemma 2.1 to obtain that $\{T_{\mu_j}\}_{j\in\mathbb{Z}}$ are uniformly bounded operators from L^p into L^q if $\frac{1}{p} = 1 - \frac{1}{4}\gamma$ and $\frac{1}{q} = 1 - \frac{5}{12}\gamma$. So we obtain that $\bigl(1 - \frac{1}{4}\gamma, 1 - \frac{5}{12}\gamma\bigr) \in E_\sigma$. □

Now, we return to the global case IV). We have

Theorem 4.4. *Assume* $\varphi(x) = (x_1^2, x_1x_2)$ *and* $\gamma > 0$. *Then* $E_\mu = \emptyset$ *for* $\gamma > \frac{3}{2}$ *and, for* $\gamma \leqslant \frac{3}{2}$, E_μ *is a segment that contains the closed segment with endpoints* $\left(1 - \frac{1}{4}\gamma, 1 - \frac{5}{12}\gamma\right)$ *and* $\left(\frac{5}{12}\gamma, \frac{1}{4}\gamma\right)$.

Proof. $E_\mu \subset E_\sigma$, and $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\sigma$ implies $\frac{1}{q} \geqslant \frac{1}{p} - \frac{1}{4}$ (see the proof of Theorem 4.1), and by Lemma 2.1, $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\mu$ implies $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{6}$, so the case $\gamma > 3/2$ follows. If $\gamma \leqslant 3/2$, then, as before, $\{T_{\mu_j}\}_{j\in\mathbb{Z}}$ are uniformly bounded operators from L^p into L^q if $\frac{1}{p} = 1 - \frac{1}{4}\gamma$ and $\frac{1}{q} = 1 - \frac{5}{12}\gamma$. Now we can proceed as in the proof of Theorem 3.5 in order to see that $\left(1 - \frac{1}{4}\gamma, 1 - \frac{5}{12}\gamma\right) \in E_\mu$. Finally, the proof of the theorem follows by the convexity and symmetry of E_μ and by Lemma 2.1. □

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References

[1] *M. Christ*: Endpoint bounds for singular fractional integral operators. UCLA Preprint (1988).
[2] *S. W. Drury and K. Guo*: Convolution estimates related to surfaces of half the ambient dimension. Math. Proc. Camb. Phil. Soc. *110* (1991), 151–159.
[3] *E. Ferreyra, T. Godoy and M. Urciuolo*: Convolution operators with fractional measures associated to holomorphic functions. Acta Math. Hungar *92* (2001), 27–38.
[4] *D. Oberlin*: Convolution estimates for some measures on curves. Proc. Amer. Math. Soc. *99* (1987), 56–60.
[5] *F. Ricci*: Limitatezza L^p-L^q per operatori di convoluzione definiti da misure singolari in $\mathbb{R}^n$. Bollettino U.M.I. *7 11-A* (1997), 237–252.
[6] *S. Secco*: Fractional integration along homogeneous curves in $\mathbb{R}^3$. Math. Scand. *85* (1999), 259–270.
[7] *E. M. Stein*: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, 1970.
[8] *E. M. Stein*: Harmonic Analysis. Real Variable Methods, Orthogonality and Oscillatory Integrals. Princeton University Press, 1993.

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