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TOPOLOGICAL CHARACTERIZATIONS OF ORDERED GROUPS WITH QUASI-DIVISOR THEORY

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Abstract. For an order embedding $G \xrightarrow{h} \Gamma$ of a partly ordered group G into an l-group Γ a topology $\mathcal{T}_{\widehat{W}}$ is introduced on Γ which is defined by a family of valuations W on G. Some density properties of sets $h(G)$, $h(X_t)$ and $(h(X_t) \setminus \{h(g_1), \ldots, h(g_n)\})$ (X_t being t-ideals in G) in the topological space $(\Gamma, \mathcal{T}_{\widehat{W}})$ are then investigated, each of them being equivalent to the statement that h is a strong theory of quasi-divisors.

Keywords: quasi-divisor theory, ordered group, valuations, t-ideal

MSC 2000: 13F05, 06F15

1. Introduction

L. Skula [22] introduced the notion of a theory of divisors for a partly ordered group (*po*-group) (or, equivalently, for a semigroup with a cancellation law) as a very natural generalization of a theory of divisors for rings and derived an extensive theory of these *po*-groups.

A step towards further generalization of a divisor theory was done by K. E. Aubert in [3], where for the first time the notion of a quasi-divisors theory was introduced. Recall that a directed *po*-group $(G, \cdot)$ has a *theory of quasi-divisors* if there exists an l-group $(\Gamma, \cdot)$ and a map $h\colon\ G \to \Gamma$ such that

(i) h is an order isomorphism from G into Γ,

(ii) $(\forall \alpha \in \Gamma_+)(\exists g_1, \ldots, g_n \in G_+)\alpha = h(g_1) \wedge \ldots \wedge h(g_n)$.

The principal tool for an investigation of these properties in *po*-groups seems to be the notion of an r-ideal. We recall here that by an r-system of ideals in a directed *po*-group G we mean a map $X \mapsto X_r$ (X_r is called an r-ideal) from the set of all

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lower bounded subsets X of G into the power set of G which satisfies the following conditions:

(1) $X \subseteq X_r$,

(2) $X \subseteq Y_r \Longrightarrow X_r \subseteq Y_r$,

(3) $\{a\}_r = a \cdot G^+ = (a)$ for all $a \in G$,

(4) $a \cdot X_r = (a \cdot X)_r$ for all $a \in G$.

One of the first characterizations of *po*-groups with a theory of quasi-divisors was established by P. Jaffard [12]. He proved that a directed *po*-group G has a theory of quasi-divisors if and only if the semigroup $(\mathcal{I}_t^{(f)}, \times)$ of finitely generated t-ideals is a group, i.e. if and only if G is a t-Prüfer group. (For comprehensive description see e.g. [3].)

In [14] we introduced a stronger version of *po*-groups with a theory of quasi-divisors. Recall that a theory of quasi-divisors $h\colon G \to \Gamma$ is called a *strong theory of quasi-divisors*, if

$$(\forall \alpha, \beta \in \Gamma_+)(\exists \gamma \in \Gamma_+) \alpha \cdot \gamma \in h(G), \ \ \beta \wedge \gamma = 1.$$

It may be proved that any strong theory of quasi-divisors is also a theory of quasi-divisors.

It was again L. Skula [22] who proved for the first time that a theory of divisors can be characterized by some *density* property. For an *o*-embedding $G \xrightarrow{h} \Gamma$ of a *po*-group G into an l-group Γ ($\Gamma = \mathbb{Z}^{(P)}$ in his approach) he introduced a short exact sequence

$$0 \to G \xrightarrow{h} \Gamma \xrightarrow{\varphi_h} \mathcal{C}_h \to 0$$

and proved that h is a strong divisor theory if and only if a map φ_h has some algebraic density property. Namely, he proved the following theorem.

Theorem ([22]). *Let G be a po-group and let $h\colon G \to \mathbb{Z}^{(P)}$ be an o-isomorphism into. Then the following conditions are equivalent.*

(1) *h is a strong theory of divisors.*

(2) *For $p_1, \ldots, p_n \in P$ ($n \geqslant 1$), the set $\varphi_h(P \setminus \{p_1, \ldots, p_n\})$ is a semigroup generator of a divisor class group $\mathcal{C}_h$.*

In this paper we want to investigate some density properties of *po*-groups with a strong theory of quasi-divisors which can be expressed not by using a map φ_h from the above short exact sequence but directly by a map h. To do it we have to change this notion of density used by Skula—instead of the *density in an algebraic sense* (i.e. $X \subseteq \Gamma$ is dense in $\mathcal{C}_h$ if $\varphi_h(X)$ is a semigroup generator of $\mathcal{C}_h$) we will use the *density in a topological sense*, i.e. we will define a topology $\mathcal{T}_{\widehat{W}}$ on Γ and investigate conditions under which for a set $X \subseteq G$, $h(X)$ is topologically dense in $(\Gamma, \mathcal{T}_{\widehat{W}})$. The principal result of this paper will be then Theorem 2.9 which introduces nine new density conditions, each of them being equivalent to the statement that h is a strong theory of quasi-divisors.

In this paper all *po*-groups are assumed to be abelian and directed. As we have mentioned in the introduction, ideal systems are the principal tools for an investigation of *po*-groups with various divisors theory. Among these ideal systems, t-ideals play the principal role. Recall that an r-system is called a v-system, if

$$X_v = \bigcap_{X \subseteq (y),\, y \in G} (y),$$

and it is called a t-system, if

$$X_t = \bigcup_{Y \subseteq X,\, Y \text{ finite}} Y_v.$$

An r-system r is said to be of *a finite character*, if

$$X_r = \bigcup_{Y \subseteq X,\, Y \text{ finite}} Y_r.$$

An r-ideal X_r is *finitely generated* if $X_r = Y_r$ for some finite subset Y. Clearly, any t-system is of a finite character and for any r-system r of a finite character on G, $X_r \subseteq X_t$ ($r \leqslant t$, in symbols). An o-homomorphism φ from a *po*-group G_1 with an r-system r_1 into a *po*-group G_2 with an r-system r_2 is an (r_1, r_2)-*morphism* if $\varphi(X_{r_1}) \subseteq (\varphi(X))_{r_2}$ for any lower bounded subset X. If G_2 is totally ordered (i.e. an *o-group*) and φ is surjective, then φ is called an r_1-*valuation* if it is an (r_1, t)-morphism. Sometimes t-valuations will be simply called valuations. Moreover, an o-homomorphism $\varphi\colon G_1 \to G_2$ is called *essential* if it is an o-epimorphism and $\ker\varphi$ is a directed convex subgroup of G_1 (i.e. an *o-ideal* of G_1). In [8, Theorem 3.8], it is proved that the existence of a theory of quasi-divisors of a finite character is equivalent to the existence of a family W of essential t-valuations such that

(1) $\forall g \in G,\ g \geqslant 1 \Leftrightarrow (\forall w \in W) w(g) \geqslant 1$,

(2) $\forall g \in G,\ g \neq 1,\ \{w \in W\colon\ w(g) \neq 1\}$ is finite.

In this case W is called a *defining family of a finite character.* If $G \xrightarrow{h} \Gamma$ is a theory of quasi-divisors then any t-valuation $G \xrightarrow{w} G_w$ from a defining family W can be uniquely extended onto a t-valuation $\Gamma \xrightarrow{\widehat{w}} G_w$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ h\ } & \Gamma \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle \widehat{w}} \\ G_w & = & G_w \end{array}$$

commutes. The set of these extended t-valuations will be then denoted by $\widehat{W}$. It is clear that in this case $\widehat{W}$ is a defining family of t-valuations for Γ.

Let $(G, \cdot, \leqslant)$ be a *po*-group and let W be a defining family of valuations $w\colon G \to G_w$ for G. Let r be an ideal system in G. Recall that r is then said to be defined by W, if for any finite subset $X \subseteq G$,

$$(\forall g \in G) g \in X_r \Leftrightarrow (\forall w \in W) w(g) \in (w(X))_t$$

holds. Moreover, it was proved by Jaffard [12] that, conversely, any defining family of valuations W defines in this way an ideal system. In both these cases, any valuation $w \in W$ is then an r-valuation. Now, if r is defined by W and for any lower directed subset $X \subseteq G$ we set $X_r = \bigcup_{K \subseteq X, X \text{ finite}} K_r$, then we obtain an ideal system of a finite character in G and any $w \in W$ is also r-valuation.

For our purposes various types of approximation theorems for valuations are of principal importance. Let w, v be valuations of G with value groups G_w, G_v, respectively. Then the canonical o-homomorphism $G \to G/[\ker w, \ker v]$, where $[\ker w, \ker v]$ is the smallest o-ideal generated by the corresponding kernels, is a valuation and there are o-homomorphisms d_{vw}, d_{wv} such that $d_{vw} \cdot v = d_{wv} \cdot w$. This common valuation will be denoted by $v \wedge w$. Now, elements $(g_1, g_2) \in G_w \times G_v$ are called *compatible*, if $d_{wv}(g_1) = d_{vw}(g_2)$. Moreover, if W is a set of valuations, an element $(g_w)_w \in \prod\limits_{w \in W'} G_w$ (where $W' \subseteq W$) is called *compatible* if any pair (g_w, g_v) from this element is compatible. Finally, we say that an element $(g_w)_w \in \prod\limits_{w \in W} G_w$ is W'-*complete* for $W' \subseteq W$, if $\bigcup\limits_{w \in W'} W(g_w) \subseteq W'$, where $W(g_w) = \{v \in W\colon\ d_{wv}(g_w) \neq 1\}$. We set $W(1) = \emptyset$.

Then we say that G with a defining family W of valuations satisfies *Positive Weak Approximation Theorem* (P.W.A.T.) if for any finite subset $F \subseteq W$ and any compatible system $(\alpha_w)_w \in \prod\limits_{w \in F} G_w^+$ there exists $g \in G_+$ such that $w(g) = \alpha_w$ for all $w \in F$. Further, we say that G with W satisfies the *Weak Approximation Theorem* (W.A.T.) if for any finite subset $F \subseteq W$ and any compatible system $(\alpha_w)_w \in \prod\limits_{w \in F} G_w$ there exists $g \in G$ such that $w(g) = \alpha_w$ for all $w \in F$. Finally, we say that G with W satisfies the *Approximation Theorem* (A.T.), if for any finite subset $F \subseteq W$ and any compatible and F-complete system $(\alpha_w)_w \in \prod\limits_{w \in F} G_w^+$ there exists $g \in G_+$ such that $w(g) = \alpha_w$ for all $w \in F$ and $w(g) \geqslant 1$ for all $w \in W \setminus F$.

At the beginning of this section we introduce the notion of a topology defined by a defining family of valuations in a *po*-group.

Definition 2.1. Let G be a *po*-group with an ideal system r and let W be a defining family of r- valuations for G. By $\mathcal{T}_W$ we denote a topology on G such that $\{\ker w\colon\ w \in W\}$ is a subbase of neighbourhoods of 1_G.

Clearly $(G, \mathcal{T}_W)$ is a topological group and if G_w is considered to be a discrete space, then w is a continuous map. By $\overline{X}$ we denote the closure of a set $X \subseteq G$ in this topology $\mathcal{T}_W$. It is clear that for any $X \subseteq G$ we have

$$\overline{X} = \{g \in G\colon\ (\forall F \subseteq W,\ F \text{ finite})(\exists a \in X)(\forall w \in F)w(a) = w(g)\}.$$

First we summarize some simple relationships between this topology and the ideal systems in G.

Lemma 2.2. *Let G be a po-group with a defining family of r-valuations W, where r is an ideal system defined on G. Further, let s be the ideal system in G defined by W. Let X be a lower bounded subset in G.*

(1) *For any $w \in W$ we have $w(\overline{X_r}) \subseteq (w(X))_t$.*

(2) *For any $w \in W$ we have $\overline{X} \subseteq X_s = (\overline{X})_s = \overline{X_s}$.*

P r o o f. (1) Let $g \in \overline{X_r}$. Then for $F = \{w\}$ there exists $a \in X_r$ such that $w(g) = w(a)$. Since w is a (r,t)-morphism, we have $w(g) = w(a) \in w(X_r) \subseteq (w(X))_t$.

(2) Let $g \in \overline{X}$ and let $w \in W$. Then there exists $a \in X$ such that $w(g) = w(a)$ and it follows that $w(g) \in w(X) \subseteq w(X_s) \subseteq (w(X))_t$. Since s is defined by W, we have $g \in X_s$. Further, let $g \in \overline{X_s}$ and let us suppose that $g \notin X_s$. Then there exists $w \in W$ such that $w(g) \notin (w(X))_t$. On the other hand, there exists $a \in X_s$ such that $w(g) = w(a)$ and it follows that $w(g) = w(a) \in w(X_s) \subseteq (w(X))_t$, a contradiction. Hence, $\overline{X_s} = X_s$. Finally, since $\overline{X} \subseteq X_s$, it follows that $\overline{X}$ is lower bounded. Then we have $\overline{X}_s \subseteq X_{ss} = X_s \subseteq \overline{X_s}$. □

It is clear that any topology $\mathcal{T}_W$ defined on G by a defining family of valuations is a T_1-topology.

Let G and G' be *po*-groups, $h\colon\ G \to G'$ an *o*-homomorphism, and let W and W', respectively, be defining families of valuations of G and G'. Then W' is said to be *coarser than* W (*with respect to* h), in symbols $W' \leqslant_h W$, if there exists an injective map $\sigma\colon\ W' \to W$ such that for each $w' \in W'$ there exists an *o*-homomorphism $h_{w'}$

such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\ h\ } & G' \\ {\scriptstyle \sigma(w')}\big\downarrow & & \big\downarrow{\scriptstyle w'} \\ G_{\sigma(w')} & \xrightarrow[h_{w'}]{} & G_{w'} \end{array}$$

In the next proposition we investigate some relationship between topologies (defined by families of valuations) on a *po*-group G and its factor *po*-group G/H, respectively.

Proposition 2.3. *Let W be a defininig family for a po-group G, let H be a convex subgroup in G, $h\colon G \to G/H$ a canonical o-epimorphism and W_1 a defining family of G/H such that $W_1 \leqslant_h W$. Then $(G/H, \mathcal{T}_{W_1})$ is a factor topological group of the topological group $(G, \mathcal{T}_W)$.*

Proof. It is clear that $\mathcal{T}_{W_1}$ is a factor topology if and only if for any $\mathbf{A} \subseteq G/H$, $h^{-1}(\overline{\mathbf{A}})$ is closed in $\mathcal{T}_{W_1}$. Hence, let $g \in \overline{h^{-1}(\overline{\mathbf{A}})}$ and let us assume that $g \notin h^{-1}(\overline{\mathbf{A}})$. Then there exists a finite subset F in W_1 such that for any $\mathbf{x} \in \mathbf{A}$ there exists $w_{\mathbf{x}} \in F$ such that $w_{\mathbf{x}}(\mathbf{x}) \neq w_{\mathbf{x}}(h(g))$. Since $W_1 \leqslant_h W$, there exists an injective map $\sigma\colon W_1 \to W$ such that for any $w \in W_1$ there exists an *o*-homomorphism h_w such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\ h\ } & G/H \\ {\scriptstyle \sigma(w)}\big\downarrow & & \big\downarrow{\scriptstyle w} \\ G_{\sigma(w)} & \xrightarrow[h_{w}]{} & G_{w}. \end{array}$$

Since $g \in \overline{h^{-1}(\overline{\mathbf{A}})}$, for a finite set $\sigma(F) \subseteq W$ there exists $b \in h^{-1}(\overline{\mathbf{A}})$ such that $\sigma(w)(g) = \sigma(w)(b)$ for all $w \in F$. Further, since $h(b) \in \overline{\mathbf{A}}$, there exists $\mathbf{c} \in \mathbf{A}$ such that $w(\mathbf{c}) = w(h(b))$ for all $w \in F$. Especially, for any $w_{\mathbf{c}} \in F$, $\mathbf{c} \in \mathbf{A}$, we have $w_{\mathbf{c}}(h(g)) \neq w_{\mathbf{c}}(\mathbf{c}) = w_{\mathbf{c}}(h(b))$. From the commutativity of the above diagram we then obtain

$$w_{\mathbf{c}}(\mathbf{c}) = w_{\mathbf{c}}(h(b)) = h_{w_{\mathbf{c}}}(\sigma(w_{\mathbf{c}})(b) = h_{w_{\mathbf{c}}}\sigma(w_{\mathbf{c}}(g) = w_{\mathbf{c}}(h(g)),$$

a contradiction. Therefore, $g \in h^{-1}(\overline{\mathbf{A}})$ and $\mathcal{T}_{W_1}$ is a factor topology. □

Corollary. *Let W be a defining family of valuations for G of a finite character and let H be an o-ideal of G. Then there exists a defining family W_1 of valuations for G/H such that $(G/H, \mathcal{T}_{W_1})$ is the factor topological group of $(G, \mathcal{T}_W)$.*

P r o o f. According to [18, 2.7], there exists a defining family W_1 for G/H such that $W_1 \leqslant_h W$, where $h\colon G \to G/H$ is the canonical o-epimorphism. The rest follows from 2.3. $\square$

The following proposition is also corollary of the above proposition.

Proposition 2.4. *Let G be a po-group and let W be a defining family of valuations of G of a finite character. Let H be an o-ideal in G. Then H is closed in the topology $\mathcal{T}_W$.*

P r o o f. According to [17, Proposition 2.7], there exists a defining family W_H of G/H such that $W_H \leqslant_h W$, where h is a canonical o-epimorphism. Hence, according to 2.3, the topological group $(G/H, \mathcal{T}_{W_H})$ is a factor topological group of the topological group $(G, \mathcal{T}_W)$. Since any topology defined by a family of valuations is a T_1-topology, we obtain that H has to be closed in $\mathcal{T}_W$. $\square$

Let w_1, w_2 be valuations of G. Then we set $w_1 \geqslant w_2$ if there exists an o-epimorphism d such that $w_2 = d \cdot w_1$.

Lemma 2.5. *Let W be a defining family of a po-group G and let W_1 be such that for any $w \in W$ there exists $w' \in W_1$ with $w' \geqslant w$. Then W_1 is a defininig family of G and $\mathcal{T}_W = \mathcal{T}_{W_1}$.*

P r o o f. The lemma follows directly from the fact that for any $w \in W$ there exists an o-homomorphism $h_w\colon G_{w'} \to G_w$ such that $w = h_w \cdot w'$. $\square$

Lemma 2.6. *Let W be a system of valuations in a po-group G and let $(\beta_w)_{w\in W} \in \prod\limits_{w\in W} G_w$ be a compatible system. Let $W_1 = \{w \in W\colon \beta_w \neq 1\}$. Then $(\beta_w)_{w\in W'}$ is W'-complete for any W' such that $W_1 \subseteq W' \subseteq W$.*

P r o o f. Let $w \in W'$. If $\beta_w = 1$, then $W(\beta_w) = \{w\} \subseteq W'$. Let $\beta_w \neq 1$, and let $v \in W(\beta_w)$. Since $1 \neq d_{wv}(\beta_w) = d_{vw}(\beta_v)$, we have $\beta_v \neq 1$ and it follows that $v \in W_1 \subseteq W'$. Hence $\bigcup\limits_{w\in W'} W(\beta_w) = W'$ and $(\beta_w)_{w\in W'}$ is W'-complete. $\square$

The next theorem is the first example of topological density properties of po-groups with a strong theory of quasi-divisors. In some aspect it represents a topological analogue of Skula's algebraic density property.

Theorem 2.7. *Let G be a po-group with a strong theory of quasi-divisors of a finite character and let W be its infinite defining family of t-valuations of finite character. Let X be a lower bounded subset in G. Then for any $g_1, \ldots, g_n \in X_t$, the set $X_t \setminus \{g_1, \ldots, g_n\}$ is dense in X_t in the topology $\mathcal{T}_W$, i.e.*

$$\overline{X_t \setminus \{g_1, \ldots, g_n\}} = X_t.$$

P r o o f. We suppose first that X is a finite subset in G. According to 2.2, $\overline{X_t} = X_t$ since the t-system is defined by any defining family of valuations. To prove the theorem it suffices to show that $g_1 \in \overline{X_t \setminus \{g_1, \ldots, g_n\}}$. Hence, let $F \subset W$ be a finite subset. Since $g_1 \neq g_j$, $j = 2, \ldots, n$, for any $j \geqslant 2$ there exists $v_j \in W$ such that $v_j(g_1) \neq v_j(g_j)$. Let $X = \{x_1, \ldots, x_m\}$ and let $\beta_w^X = w(x_1) \wedge \ldots \wedge w(x_m)$ for any $w \in W$. Since a t-system in G is defined by any defining family of valuations in G, we obtain

$$(\forall a \in G) a \in X_t \Leftrightarrow (\forall w \in W) w(a) \in (w(X))_t = \{\gamma \in G_w \colon \gamma \geqslant \beta_w^X\}.$$

Moreover, according to [18, Lemma 2.9 and Lemma 2.6], $(\beta_w^X)_w$ is a compatible and $W_{\mathbf{b}}$-complete system, where $W_{\mathbf{b}} = \{w \in W \colon \beta_w^X \neq 1\}$. We put

$$W_1 = F \cup \{v_2, \ldots, v_n\} \cup \{w \in W \colon w(g_1) \neq 1\} \cup W_{\mathbf{b}}.$$

Since W is of a finite character, W_1 is a finite set.

Now, let $w_0 \in W \setminus W_1$ be an arbitrary valuation. Then for any $w \in W_1$ and the o-homomorphism $d_{w_0,w} \colon G_{w_0} \to G_{w_0 \wedge w}$ we have $w_0 \wedge w = d_{w_0,w} \cdot w_0$ in the $\wedge$-semilattice of valuations over G. Without any loss of generality we can require that elements from W are pairwise incomparable. Hence, for any $w \in W_1$ there exists $1 < \delta_w \in (\ker d_{w_0,w})_+ \subseteq G_{w_0}^+$. Let $\delta = \min\{\delta_w \colon w \in W_1\} > 1$. Since $1 < \delta \leqslant \delta_w \in \ker d_{w_0,w}$, we have $\delta \in \bigcap\limits_{w \in W_1} \ker d_{w_0,w}$ and it follows that $(1, \delta) \in G_w^+ \times G_{w_0}^+$ is a compatible system for all $w \in W_1$. Since G has a strong theory of quasi-divisors of a finite character, a defining family W of valuations satisfies the Positive Weak Approximation Theorem (see [17, Theorem 3.3]). Then for a compatible system $\mathbf{c}' = (1, \ldots, 1, \delta) \in \prod\limits_{w \in W_1} G_w \times G_{w_0}$ there exists an element $e \in G_+$ such that

$$w(e) = \mathbf{c}_w; \quad w \in W_1 \cup \{w_0\}.$$

Now we set $W_{\mathbf{c}} = \{w \in W \colon w(e) \neq 1\} \cup W_1$. Further, let us denote

$$\mathbf{a} = (w(g_1))_{w \in W_{\mathbf{c}}}, \quad \mathbf{b} = ((\beta_w^X)_{w \in W_{\mathbf{c}}}, \quad \mathbf{c} = (w(e))_{w \in W_{\mathbf{c}}}.$$

Then $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are compatible systems and according to Lemma 2.6, these systems are $W_{\mathbf{c}}$-complete, since $W_{\mathbf{b}} \subseteq W_1 \subseteq W_{\mathbf{c}}$. Hence according to [18, 2.9], it follows that $(\mathbf{a} \vee \mathbf{b}) \cdot \mathbf{c}$ is a compatible and $W_{\mathbf{c}}$-complete system as well, where the operations are done pointwise. Then according to the Approximation Theorem which holds for any po-group with a strong theory of quasi-divisors of a finite character (see [17, 3.5]), there exists $a \in G$ such that

$$\begin{aligned} w(a) &= ((\mathbf{a} \vee \mathbf{b}) \cdot \mathbf{c})_w; \quad && w \in W_{\mathbf{c}}, \\ w(a) &\geqslant 1, && w \in W \setminus W_{\mathbf{c}}. \end{aligned}$$

Hence $a \in X_t$. In fact, let $w \in W$. If $\beta_w^X = 1$, then in the case $w \in W_{\mathbf{c}}$ we have

$$w(a) = ((\mathbf{a} \vee \mathbf{b}) \cdot \mathbf{c})_w = (1 \vee w(g_1)) \cdot w(e) \geqslant 1 = \beta_w^X$$

and in the case $w \in W \setminus W_{\mathbf{c}}$ we have $w(a) \geqslant 1 = \beta_w^X$. If $\beta_w^X > 1$, then $w \in W_1 \subseteq W_{\mathbf{c}}$ and we have $w(a) = (\mathbf{a}_w \vee \mathbf{b}_w) \cdot \mathbf{c}_w = (w(g_1) \vee \beta_w^X) \cdot w(e) \geqslant \beta_w^X$. Hence $w(a) \geqslant \beta_w^X$ for any $w \in W$ and it follows that $a \in X_t$.

Further, for any $w \in F \subseteq W_1$ we have $w(a) = (\mathbf{a}_w \vee \mathbf{b}_w) \cdot \mathbf{c}_w = w(g_1) \vee \beta_w^X = w(g_1)$, since $g_1 \in X_t$ and $w(e) = 1$ for any $w \in W_1$.

Finally, for any $j \geqslant 2$ we have $v_j(a) = ((\mathbf{a} \vee \mathbf{b}) \cdot \mathbf{c})_{v_j} = v_j(g_1) \vee \beta_{v_j}^X = v_j(g_1) \neq v_j(g_j)$, since again $g_1 \in X_t$ and $v_j \in W_1$. Hence $a \neq g_j$, $j \geqslant 2$. Moreover, we have $w_0(a) = ((\mathbf{a} \vee \mathbf{b}) \cdot \mathbf{c})_{w_0} = \delta \neq 1 = w_0(g_1)$. Hence $a \neq g_1$.

Therefore, we conclude that $g_1 \in \overline{X_t \setminus \{g_1, \ldots, g_n\}}$.

Now, suppose that X is any lower bounded subset in G and let $g_1, \ldots, g_n \in X_t$. For any i, $i = 1, \ldots, n$, there exists a finite subset $K^i \subseteq X$ such that $g_i \in K_t^i$. Let $K = \bigcup\limits_i K^i$. Then $g_1, \ldots, g_n \in K_t$ and according to the first part of this proof we have $g_i \in \overline{K_t \setminus \{g_1, \ldots, g_n\}} \subseteq \overline{X_t \setminus \{g_1, \ldots, g_n\}}$. Therefore, $\overline{X_t \setminus \{g_1, \ldots, g_n\}} = X_t$. □

Corollary. *Let $g_1, \ldots, g_n \in G_+$. Then $\overline{G_+ \setminus \{g_1, \ldots, g_n\}} = G_+$.*

The proof follows directly from 2.7, since $\{1\}_t = G_+$.

Let G be a *po*-group with an ideal system r of a finite character and let H be an o- ideal of G, $h\colon G \to G/H$ a canonical o-homomorphism. Then for any lower bounded subset $\mathbf{A} \subseteq G/H$ we can find a lower bounded subset $A \subset G$ such that $A/H = \mathbf{A}$. Then we set $\mathbf{A}_{r_H} = A_r/H$. In [17] it was proved that r_H is an ideal system in G/H.

Lemma 2.8. *Let G be a po-group with a defining family W of valuations, let r be an ideal system of a finite character defined by W and let H be an o-ideal in G, $h\colon G \to G/H$ a canonical o- homomorphism. Let W_H be any defining family of G/H such that $W_H \leqslant_h W$. Then any valuation in W_H is an r_H-valuation.*

Proof. Let $\sigma\colon W_H \to W$ be an injective map such that $h \cdot w = h_w \cdot \sigma(w)$ for any $w \in W_H$, where $h_w\colon G_{\sigma(w)} \to G_w$ is an o-homomorphism. Let $\mathbf{A} \subseteq G/H$ be a lower bounded set and let A be a lower bounded set in G such that $\mathbf{A}_{r_H} = A_r/H$. Let $a \in A_r$. Then we have $w(h(a)) = h_w\sigma(w)(a) \in h_w((\sigma(w)A))_t) \subseteq (h_w\sigma(w)(A))_t = (wh(A))_t = (w(\mathbf{A}))_t$. Hence $w(\mathbf{A}_{r_H}) \subseteq (w(\mathbf{A}))_{\mathbf{t}}$. □

Now, let $h\colon G \to D$ be an o-embedding of a *po*-group G into another *po*-group D and let W $(\widehat{W})$ be a defining family of valuations for G (D, respectively). Then we say that $\widehat{W}$ is an *extension of* W, if there is a bijection $\sigma\colon W \to \widehat{W}$ such that

(1) $(\forall w \in W) G_w = G_{\sigma(w)}$,
(2) $(\forall w \in W) \sigma(w) \cdot h = w$.

The next theorem is the principal result of the paper and it presents some topological characterizations of *po*-groups with a strong theory of quasi-divisors.

Theorem 2.9. *Let G be a po-group with an infinite defining family of valuations W of a finite character and let $h\colon G \to \Gamma$ be an o-embedding of G into an l-group. Let r be an ideal system in G defined by W. Finally, let $\widehat{W}$ be a defining family of valuations for Γ such that $\widehat{W}$ is an extension of W. Then the following statements are equivalent.*

(1) *h is a strong theory of quasi-divisors of a finite character.*
(2) *$h(G)$ is a dense set in Γ in the topology $\mathcal{T}_{\widehat{W}}$.*
(3) *$h(G_+)$ is a dense set in Γ_+ in the topology $\mathcal{T}_{\widehat{W}}$.*
(4) *For any finite set $X \subset G_+$, $h(X_r)$ is a dense set in $(h(X))_t$ in the topology $\mathcal{T}_{\widehat{W}}$.*
(5) *For any finite set $X \subset G$, $h(X_r)$ is a dense set in $(h(X))_t$ in the topology $\mathcal{T}_{\widehat{W}}$.*
(6) *For any lower bounded set $X \subseteq G$, $h(X_r)$ is a dense set in $(h(X))_t$ in the topology $\mathcal{T}_{\widehat{W}}$.*
(7) *For any finite set $X \subseteq G$ and any elements $g_1, \dots, g_n \in X_r$, $h(X_r \setminus \{g_1, \dots, g_n\})$ is a dense set in $(h(X))_t$ in the topology $\mathcal{T}_{\widehat{W}}$.*
(8) *For any elements $g_1, \dots, g_n \in G_+$, $h(G_+ \setminus \{g_1, \dots, g_n\})$ is a dense set in Γ_+ in the topology $\mathcal{T}_{\widehat{W}}$.*
(9) *For any elements $g_1, \dots, g_n \in G$, $h(G \setminus \{g_1, \dots, g_n\})$ is a dense set in Γ in the topology $\mathcal{T}_{\widehat{W}}$.*
(10) *For any lower bounded set $X \subseteq G$ and any elements $g_1, \dots, g_n \in X_r$, $h(X_r \setminus \{g_1, \dots, g_n\})$ is a dense set in $(h(X))_t$ in the topology $\mathcal{T}_{\widehat{W}}$.*

Proof. The proof will be done according to the following scheme.

$$\begin{array}{ccccccccccccc}
 & & & & & & (8) & & \longleftarrow & & (10) & & \\
 & & & & & & \downarrow & & & & \uparrow & & \\
(6) & \longrightarrow & (5) & \longrightarrow & (4) & \longrightarrow & (3) & \longrightarrow & (2) & \longrightarrow & (1) & \longrightarrow & (6) \\
 & & & & & & & & \uparrow & & \downarrow & & \uparrow \\
 & & & & & & & & (9) & \longleftarrow & (10) & \longrightarrow & (7)
\end{array}$$

(2) $\Longrightarrow$ (1): We prove that W satisfies the Weak Approximation Theorem (W.A.T.). Let $F \subseteq W$ be a finite set and let $(\alpha_w)_w \in \prod\limits_{w \in F} G_w$ be a compatible system. Since $1_\Gamma\colon \Gamma \to \Gamma$ is a strong theory of quasi-divisors of a finite character (Γ is defined by $\widehat{W}$), according to the W.A.T. (see [17, Theorem 3.3 and Theorem 3.4])

applied to this system there exists $\mathbf{a} \in \Gamma$ such that $\widehat{w}(\mathbf{a}) = \alpha_w$ for all $w \in F$. Since $h(G)$ is dense in Γ, there exists $g \in G$ such that $w(g) = \widehat{w}(\mathbf{a})$ for $w \in F$. Hence W satisfies the W.A.T. and it follows that (1) holds (see [17, Theorem 3.5]).

(3) $\Longrightarrow$ (2): Let $\mathbf{a} \in \Gamma$, $\mathbf{a} = \mathbf{a}_1 \cdot \mathbf{a}_2^{-1}$, where $\mathbf{a}_i \geqslant 1$. Let $F \subseteq W$ be a finite set. Then there exist $g_1, g_2 \in G_+$ such that $w(g_i) = \widehat{w}(\mathbf{a}_i)$ for all $w \in F$. Hence $w(g_1 \cdot g_2^{-1}) = \widehat{w}(\mathbf{a})$ for all $w \in F$ and (2) holds.

Implications (4) $\Longrightarrow$ (3), (5) $\Longrightarrow$ (4) and (6) $\Longrightarrow$ (5) are clearly trivial.

(1) $\Longrightarrow$ (6): Let $X \subseteq G$ be a lower bounded set. Since h is an (r,t)-morphism, we have $h(X_r) \subseteq (h(X))_t$. Let $\mathbf{a} \in (h(X))_t$. Then there exists a finite subset $K \subseteq X$ such that $\mathbf{a} \in (h(K))_t$ and it follows that $\mathbf{a} \geqslant \wedge_{k \in K} h(k)$. For any $w \in W$ we set $\beta_w = \wedge_{k \in K} w(k)$. Let $F \subseteq W$ be a finite set. Then we put

$$W_1 = \{w \in W\colon\ \beta_w \neq 1\} \cup F \cup \{w \in W\colon\ \widehat{w}(\mathbf{a}) \neq 1\}.$$

Since W is of finite character, W_1 is a finite set. Further, we put $\alpha = (\widehat{w}(\mathbf{a}))_{w \in W_1}$. According to Lemma 2.6, α is a compatible and W_1-complete system. Since G satisfies the Approximation Theorem ([17, Theorem 3.5]), there exists $g \in G$ such that

$$\begin{aligned} w(g) &= \widehat{w}(\mathbf{a}), \quad && w \in W_1, \\ w(g) &\geqslant 1, && w \in W \setminus W_1. \end{aligned}$$

Then $g \in K_r$. In fact, let $w \in W$. If $\beta_w \neq 1$, then $w \in W_1$ and it follows that $w(g) = \widehat{(\mathbf{a})} \geqslant \beta_w$. If $\beta_w = 1$ then in the case $w \in W_1$ we have $w(g) = \widehat{w}(\mathbf{a}) \geqslant \beta_w$ and in the case $w \notin W_1$ we obtain $w(g) \geqslant 1 = \beta_w$. Hence, for any $w \in W$ we have $w(g) \geqslant \beta_w$ and it follows that $g \in K_r$. Further, since $F \subseteq W_1$, we have

$$\widehat{w}(h(g)) = w(g) = \widehat{w}(\mathbf{a}), \quad w \in F,$$

and it follows that $\mathbf{a} \in \overline{h(K_r)} \subseteq \overline{h(X_r)}$.

(1) $\Longrightarrow$ (10): Let $X \subseteq G$ be a lower bounded subset and let $g_1, \ldots, g_n \in X_r$. Then in the topology $\mathcal{T}_{\widehat{W}}$ we have

$$\overline{h(X_r \setminus \{g_1, \ldots, g_n\})} = \overline{(h(X))_t \setminus \{h(g_1), \ldots, h(g_n)\}}.$$

In fact, since h is an (r,t)-morphism, we have $h(X_r \setminus \{g_1, \ldots, g_n\}) \subseteq (h(X))_t \setminus \{h(g_1), \ldots, h(g_n)\}$ and it follows that in the above statement the inclusion $\subseteq$ holds. Conversely, let $\mathbf{x} \in \overline{(h(X))_t \setminus \{h(g_1), \ldots, h(g_n)\}}$ and let $\widehat{F} \subseteq \widehat{W}$ be a finite set. Then there exists $\mathbf{a} \in (h(X))_t \setminus \{h(g_1), \ldots, h(g_n)\}$ such that $\widehat{w}(\mathbf{a}) = \widehat{w}(\mathbf{x})$ for all $\widehat{w} \in \widehat{F}$. Since the implication (1) $\Longrightarrow$ (6) has been proved, we have $\mathbf{a} \in \overline{h(X_r)}$. Then for the same subset $\widehat{F}$ there exists $\mathbf{b} \in h(X_r)$ such that $\widehat{w}(\mathbf{b}) = \widehat{w}(\mathbf{a})$ for all

$\widehat{w} \in \widehat{F}$. We set $\mathbf{b} = h(g)$ for some $g \in X_r$. Then according to Theorem 2.7, we have $g \in X_r = \overline{X_r \setminus \{g_1, \ldots, g_n\}}$ in the topology $\mathcal{T}_W$. Let $F \subseteq W$ be a finite set such that $\widehat{F} = \{\widehat{w}\colon\ w \in F\}$. Then there exists $c \in X_r \setminus \{g_1, \ldots, g_n\}$ such that $w(c) = w(g)$ for all $w \in F$. Finally, we obtain

$$h(c) \in h(X_r) \setminus \{h(g_1), \ldots, h(g_n)\},$$
$$\widehat{w}(h(c)) = \widehat{w}((h(g)) = \widehat{w}(\mathbf{b}) = \widehat{w}(\mathbf{a}) = \widehat{w}(\mathbf{x}), \quad (\forall \widehat{w} \in \widehat{F}),$$

and the other inclusion holds in the above statement, as well. Now, since $1_\Gamma\colon\ \Gamma \to \Gamma$ is a strong theory of quasi-divisors of a finite character, as well, and $\widehat{W}$ is a defining family of Γ, according to Theorem 2.7 applied to this theory of quasi-divisors we obtain

$$\overline{(h(X))_t \setminus \{h(g_1), \ldots, h(g_n)\}} = (h(X))_t$$

in the topology $\mathcal{T}_{\widehat{W}}$. Therefore, we obtain

$$\overline{h(X_r \setminus \{g_1, \ldots, g_n\})} = \overline{(h(X))_t \setminus \{h(g_1), \ldots, h(g_n)\}} = (h(X))_t.$$

(10) $\Longrightarrow$ (7): It is trivial.

(10) $\Longrightarrow$ (8): It follows directly from $(h(\{1_G\}))_t = \Gamma_+$.

(8) $\Longrightarrow$ (3) and (10) $\Longrightarrow$ (8) are trivial.

(10) $\Longrightarrow$ (9): The following inclusion holds:

$$(h(X))_t = \overline{h(X_r \setminus \{g_1, \ldots, g_n\})} \subseteq \overline{h(G \setminus \{g_1, \ldots, g_n\})} \subseteq \Gamma.$$

Again, since $1_\Gamma\colon\ \Gamma \to \Gamma$ is a strong theory of quasi-divisors of a finite character and $\widehat{W}$ is a defining family of Γ, according to Theorem 2.7 applied to this l-group Γ we obtain that $(h(X))_t$ is a dense set in Γ in the topology $\mathcal{T}_{\widehat{W}}$. Therefore, we have

$$\Gamma = \overline{(h(X))_t} = \overline{h(X_r \setminus \{g_1, \ldots, g_n\})} \subseteq \overline{h(G \setminus \{g_1, \ldots, g_n\})} \subseteq \Gamma.$$

(9) $\Longrightarrow$ (2): It is trivial.

(7) $\Longrightarrow$ (6): Let $X \subseteq G$ be a lower bounded set and let $\mathbf{a} \in (h(X))_t$. Then there exists a finite set $K \subseteq X$ such that $\mathbf{a} \in (h(K))_t$ and according to (7), we have $\mathbf{a} \in \overline{h(K_r)} \subseteq \overline{h(X_r)}$. Hence $\overline{h(X_r)} = (h(X))_t$. □

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