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THE SINGLE-VALUED EXTENSION PROPERTY FOR SUMS AND PRODUCTS OF COMMUTING OPERATORS

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Abstract. It is shown that the sum and the product of two commuting Banach space operators with Dunford's property (C) have the single-valued extension property.

Keywords: single-valued extension property, Dunford's property (C), decomposable operators

MSC 2000: 47A11

Introduction

One of the most challenging open problems in the local spectral theory of operators on Banach spaces is the question of the extent to which the sum and product of two commuting decomposable operators remain decomposable. Since the corresponding permanence property holds, by Fuglede's classical theorem, for normal operators and, by Corollary 4.3.4 of [5], for regular generalized scalar operators, one might expect a positive answer also in the decomposable case, but so far the problem has been settled only in certain special cases; see [3], [6], [9] and [13].

Similarly, it is not known if spectral conditions such as the single-valued extension property (SVEP), Dunford's property (C), Bishop's property (β), or the decomposition property (δ) are preserved under sums and products of commuting operators. In fact, in light of the duality between the properties (β) and (δ) from [1], the questions for (β) and (δ) are equivalent. Thus a positive answer in either case would lead to a positive solution in the case of decomposability, since, as noted in [2], an operator is decomposable precisely when it has both properties (β) and (δ).

Partial results for commuting operators with SVEP have recently been obtained in [4], based on the theory of subharmonic functions. Here we pursue the line of argument initiated by Sun [13] to establish SVEP for sums and products of commuting operators with property (C). Sun [13] developed a beautiful approach to handle the case of sums, and then applied a result due to Apostol, Lemma 3.1 of [3], to reduce the case of products to that of sums. We feel that this reduction step is on shaky ground, since the proof of Apostol's lemma contains a serious gap. In fact, this proof seems to be easily reparable only if SVEP were known to hold for sums of arbitrary commuting operators with SVEP.

In the present paper, we establish directly SVEP for products of commuting operators with property (C), and then reduce the case of sums to that of products by a very simple argument. As a by-product of our approach, we obtain that the product of a semi-shift and an arbitrary commuting operator always has SVEP. This leads to new insight into the problem of the preservation of SVEP for sums of commuting operators. The necessary tools from local spectral theory are collected in the next section.

1. Preliminaries

Throughout this note, let X be a complex Banach space, and let $L(X)$ denote the Banach algebra of all bounded linear operators on X. An operator $T \in L(X)$ is said to have the *single-valued extension property* (SVEP) if, for every open subset U of $\mathbb{C}$, the only analytic solution $f\colon U \to X$ of the equation $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function $f \equiv 0$ on U. For each $x \in X$, let $\varrho_T(x)$ denote the set of all $\lambda \in \mathbb{C}$ for which there exists an analytic function $f\colon U \to X$ on some open neighborhood U of λ such that $(T-\mu)f(\mu) = x$ for all $\mu \in U$. The set $\sigma_T(x) := \mathbb{C} \setminus \varrho_T(x)$ is called the *local spectrum* of T at x. The operator $T \in L(X)$ is said to have *Dunford's property* (C) if, for each closed subset F of $\mathbb{C}$, the corresponding local spectral subspace $X_T(F) := \{x \in X\colon \sigma_T(x) \subseteq F\}$ is closed. These notions date back to the early days of the theory of spectral operators, and are of fundamental importance in local theory, as witnessed by the monographs [5], [9] and [14].

By Proposition 1.2 of [10], property (C) implies SVEP, but the converse is far from being true in general. For instance, all multipliers on a semi-simple commutative Banach algebra have SVEP, while, as illustrated in [7], property (C) plays quite a distinguished role in this context.

If an operator $T \in L(X)$ has property (C), then, by Proposition 1.3.8 of [5], the inclusion $\sigma(T|X_T(F)) \subseteq F$ holds for every closed set $F \subseteq \mathbb{C}$. Moreover, it is not hard to see that, for every T-invariant closed linear subspace Y of X, the restriction $T|Y$ inherits property (C) from T.

Both SVEP and property (C) are preserved under the Riesz functional calculus. More precisely, given an operator $T \in L(X)$, let f be an analytic complex-valued function on an open neighborhood U of the spectrum $\sigma(T)$, and suppose that f is non-constant on each connected component of U. Then, by Theorem 1.1.5 of [5], and also by Proposition 1.6 of [15], T has SVEP precisely when $f(T)$ has SVEP. Moreover, it follows from Proposition 1.2 of [10] and Theorem 1.1.6 of [5] that property (C) is transferred from T to $f(T)$, but the converse seems to be an open problem.

2. The results

As usual, the spectral radius of an operator $T \in L(X)$ will be denoted by $r(T)$. For arbitrary $s, t \geqslant 0$, we introduce the annulus $\mathcal{A}(s,t) := \{\lambda \in \mathbb{C}\colon\ s \leqslant |\lambda| \leqslant t\}$, and define $\mathcal{A}_0(s,t) := \mathcal{A}(s,t) \cup \{0\}$. Finally, for $F, G \subseteq \mathbb{C}$, let F/G consist of all fractions λ/μ, where $\lambda \in F$ and $\mu \in G \setminus \{0\}$. The following result will be our main tool.

Lemma 1. *Let $S, T \in L(X)$ be two commuting operators on a complex Banach space X, suppose that U is a non-empty, bounded, open, and connected subset of $\mathbb{C}$, and let $m := \inf\{|\lambda|\colon\ \lambda \in U\}$ and $M := \sup\{|\lambda|\colon\ \lambda \in U\}$. Then, for every analytic function $f\colon\ U \to X$ for which*

$$(ST - \lambda) f(\lambda) = 0 \qquad \text{for all } \lambda \in U,$$

the following assertions hold:

(a) *if S has Dunford's property* (C)*, then*

$$f(\lambda) \in X_S(\mathcal{A}_0(M/r(T), r(S))) \qquad \text{for all } \lambda \in U;$$

(b) *if both S and T have Dunford's property* (C)*, then*

$$f(\lambda) \in X_T(\mathcal{A}_0(M/r(S), (m/M)\, r(T))) \qquad \text{for all } \lambda \in U.$$

Proof. Without loss of generality, we may assume that U does not contain the origin, since otherwise U may be replaced by $U \setminus \{0\}$, and the assertion for $f(0)$ then follows by continuity.

To establish assertion (a), suppose that S has property (C), and consider an arbitrary point $\lambda \in U$. Evidently, the set $V_\lambda := \{\lambda\}/\varrho(T)$ is an open subset of $\mathbb{C}$, and the definition

$$g_\lambda(\mu) := \frac{1}{\mu}\Big(\frac{\lambda}{\mu} - T\Big)^{-1} T f(\lambda) \qquad \text{for all } \mu \in V_\lambda$$

yields an analytic function $g_\lambda\colon V_\lambda \to X$. For each $\mu \in V_\lambda$, we obtain from the identity $(ST-\lambda)f(\lambda)=0$ that

$$(S-\mu)g_\lambda(\mu) = \frac{1}{\mu}\Big(\frac{\lambda}{\mu}-T\Big)^{-1}(ST-\mu T)f(\lambda) = \frac{1}{\mu}\Big(\frac{\lambda}{\mu}-T\Big)^{-1}(\lambda-\mu T)f(\lambda),$$

and hence $(S-\mu)g_\lambda(\mu) = f(\lambda)$. Since $\mathbb{C}\setminus V_\lambda = \{0\}\cup(\{\lambda\}/\sigma(T))$, we conclude that

$$f(\lambda) \in X_S\big(\{0\}\cup(\{\lambda\}/\sigma(T))\big) \qquad \text{for all } \lambda \in U.$$

Now let $\mu \in U$ be given, and let $\varepsilon > 0$ be small enough so that the closed disc $\nabla(\mu,\varepsilon)$ with center μ and radius ε is contained in U. For every $\lambda \in \nabla(\mu,\varepsilon)$, we obtain that

$$f(\lambda) \in X_S\big(\{0\}\cup(\{\lambda\}/\sigma(T))\big) \subseteq X_S\big(\{0\}\cup(\nabla(\mu,\varepsilon)/\sigma(T))\big).$$

Since the latter space is closed and U is connected, the identity principle for analytic functions, in connection with the Hahn-Banach theorem, then ensures that $f(\lambda) \in X_S\big(\{0\}\cup(\nabla(\mu,\varepsilon)/\sigma(T))\big)$ for arbitrary $\lambda \in U$. Taking the intersection over all sufficiently small $\varepsilon > 0$, we conclude that

$$f(\lambda) \in X_S\big(\{0\}\cup(\{\mu\}/\sigma(T))\big) \qquad \text{for all } \lambda,\mu \in U,$$

and consequently $f(\lambda) \in X_S(F)$ for all $\lambda \in U$, where

$$F := \sigma(S) \cap \bigcap_{\mu\in U} \big(\{0\}\cup(\{\mu\}/\sigma(T))\big).$$

It is easily seen that F is contained in $\mathcal{A}_0(M/r(T), r(S))$. Indeed, given an arbitrary $\lambda \in F$, we have $|\lambda| \leqslant r(S)$, since $F \subseteq \sigma(S)$. Moreover, if $\lambda \in F$ is non-zero, then, for every $\mu \in U$, there exists some non-zero $\zeta \in \sigma(T)$ for which $\mu = \lambda\zeta$. This implies that $|\mu| \leqslant |\lambda|\, r(T)$ for all $\mu \in U$, and hence $M \leqslant |\lambda|\, r(T)$. If $r(T) > 0$, we thus obtain that $F \subseteq \mathcal{A}_0(M/r(T), r(S))$. Finally, if $r(T) = 0$, then the definition of F shows that $F \subseteq \{0\}$, again as desired. Assertion (a) is now immediate.

To prove assertion (b), suppose that both S and T have property (C). Since the space $Y := X_S(\mathcal{A}_0(M/r(T), r(S)))$ is closed and invariant under S and T, we may consider the restrictions $\widehat{S} := T|Y$ and $\widehat{T} := S|Y$ as operators on Y. Evidently, $\widehat{S}$ and $\widehat{T}$ commute, and property (C) for S entails that $\sigma(\widehat{T}) \subseteq \mathcal{A}_0(M/r(T), r(S))$. Moreover, $\widehat{S}$ has property (C), since this property is inherited by restrictions to arbitrary closed invariant subspaces. By the preceding part of the proof, we obtain that $f(\lambda) \in Y_{\widehat{S}}(\widehat{F}) \subseteq X_T(\widehat{F})$ for all $\lambda \in U$, where

$$\widehat{F} := \sigma(\widehat{S}) \cap \bigcap_{\mu\in U} \big(\{0\}\cup(\{\mu\}/\sigma(\widehat{T}))\big).$$

This establishes assertion (b), since $\widehat{F}$ is contained in $\mathcal{A}_0(M/r(S),(m/M)\,r(T))$. Indeed, given an arbitrary non-zero $\lambda \in \widehat{F}$, we obtain, for every $\mu \in U$, a non-zero element $\zeta \in \sigma(\widehat{T})$ for which $\mu = \lambda\zeta$. Because $\sigma(\widehat{T}) \subseteq \mathcal{A}_0(M/r(T), r(S))$, this implies that $|\lambda| M/r(T) \leqslant |\mu| \leqslant |\lambda|\,r(S)$ for all $\mu \in U$, and therefore

$$M/r(S) \leqslant |\lambda| \leqslant (m/M)r(T).$$

Thus $\lambda \in \mathcal{A}_0(M/r(S),(m/M)r(T))$, as desired. □

As a first application, we shall put the validity of Sun's result, Theorem 5 of [13], beyond doubt. We mention that this result is quite useful, for instance, in the local spectral theory of multipliers. In fact, as shown in Theorem 7 of [8], Sun's theorem leads to an elementary proof of the fact that the decomposable multipliers on a commutative Banach algebra with a bounded approximate identity form a subalgebra of the multiplier algebra.

Theorem 2. *Suppose that $S, T \in L(X)$ are two commuting operators with Dunford's property* (C) *on a Banach space X. Then both ST and $S+T$ have SVEP.*

P r o o f. To establish SVEP for ST, it suffices to show that, for every open disc $U \subseteq \mathbb{C}$ and every analytic function $f\colon\, U \to X$ for which $(ST-\lambda)f(\lambda) = 0$ for all $\lambda \in U$, it follows that $f \equiv 0$ on U. With the notation of Lemma 1, let

$$Y := X_T(\mathcal{A}_0(M/r(S),(m/M)r(T))).$$

Then part (b) ensures that $f(\lambda) \in Y$ for all $\lambda \in U$. Moreover, the restrictions $S|Y$ and $T|Y$ are commuting operators with property (C) on Y and we have $r(S|Y) \leqslant r(S)$ and $r(T|Y) \leqslant (m/M)r(T)$. Thus, again by part (b) of Lemma 1, we obtain that

$$f(\lambda) \in X_T\big(\mathcal{A}_0(M/r(S),(m/M)^2 r(T))\big) \qquad \text{for all } \lambda \in U.$$

Because $m/M < 1$, a finite number of repetitions of this argument then shows that $f(\lambda) \in X_T(\{0\})$ for all $\lambda \in U$. Let $\widehat{S}$ and $\widehat{T}$ denote the restrictions of S and T, respectively, to the closed invariant subspace $X_T(\{0\})$. Since property (C) for T ensures that $\sigma(\widehat{T}) \subseteq \{0\}$, the operator $\widehat{T}$ is quasi-nilpotent. Moreover, since S and T commute, we obtain that $r(\widehat{S}\widehat{T}) \leqslant r(\widehat{S})r(\widehat{T})$, and hence that $\widehat{S}\widehat{T}$ is quasi-nilpotent. Consequently, from

$$(\widehat{S}\widehat{T}-\lambda)f(\lambda) = (ST-\lambda)f(\lambda) = 0 \qquad \text{for all } \lambda \in U$$

we infer that $f(\lambda) = 0$ for all non-zero $\lambda \in U$, and therefore that $f \equiv 0$ on U. This shows that ST has SVEP.

To settle the case of sums, we first observe that $\exp(S)$ and $\exp(T)$ have property (C), since, as noted earlier, this property is invariant under the analytic functional calculus. Because S and T commute, the first part of the proof then confirms that $\exp(S+T)$ has SVEP. By Theorem 1.1.5 of [5] or Proposition 1.6 of [5], this implies that $S+T$ has SVEP. $\square$

It would be interesting to know if, in the preceding result, the hypothesis can be weakened from property (C) to SVEP. Similarly, it remains open if the conclusion can be strengthened from SVEP to property (C). Since Bishop's property (β) implies property (C), the latter question may be viewed as a weakened version of the decomposability problem for sums and products of commuting decomposable operators.

We conclude with another application of Lemma 1. Recall from [12] that an operator $T \in L(X)$ on a Banach space X is said to have *fat local spectra* provided that $\sigma_T(x) = \sigma(T)$ for all non-zero $x \in X$. Trivial examples are given by operators for which the spectrum is a singleton, but, as noted in [12], there are also many more substantial examples.

For instance, by Proposition 4.2 of [12], an isometry $T \in L(X)$ has fat local spectra if and only if $\bigcap\{T^n(X)\colon\ n \in \mathbb{N}\} = \{0\}$. Isometries with the latter property are known as *semi-shifts*. Natural examples include, for arbitrary $1 \leqslant p \leqslant \infty$, the unilateral right shifts of arbitrary multiplicity on $\ell^p(\mathbb{N})$, and the right translation operators on $L^p([0,\infty))$. Moreover, it follows easily from the von Neumann-Wold decomposition that, on Hilbert spaces, the semi-shifts are precisely the pure isometries.

Theorem 3. *Let $S \in L(X)$ be a non-invertible operator with fat local spectra. Then, for every operator $T \in L(X)$ commuting with S, the product ST has* SVEP.

P r o o f. Since S has fat local spectra, its local spectral subspaces are trivial, in the sense that $X_S(F) = \{0\}$ for every closed set $F \subseteq \mathbb{C}$ that does not contain $\sigma(S)$, while $X_S(F) = X$ otherwise. In particular, it follows that S has property (C). Moreover, a simple application of the Riesz projections corresponding to the clopen subsets of the spectrum shows that $\sigma(S)$ is connected.

Now let $T \in L(X)$ commute with S and let $f\colon U \to X$ be an analytic function on an open disc U in $\mathbb{C}$ for which $(ST-\lambda)f(\lambda) = 0$ for all $\lambda \in U$. Then part (a) of Lemma 1 ensures that $f(\lambda) \in X_S(\mathcal{A}_0(M/r(T), r(S)))$ for all $\lambda \in U$. If $\sigma(S)$ is not contained in $\mathcal{A}_0(M/r(T), r(S))$, then it follows that $f \equiv 0$ on U, as desired. But if $\sigma(S) \subseteq \mathcal{A}_0(M/r(T), r(S))$, then 0 is an isolated point of $\sigma(S)$. Since $\sigma(S)$ is connected, this implies that S is quasi-nilpotent. Since S and T commute, we infer that ST is quasi-nilpotent, and hence that $f \equiv 0$ on U, again as desired. $\square$

Evidently, Theorem 3 is bound to fail without the condition of non-invertibility. The following simple consequence suggests the investigation of the commutant for specific operators with fat local spectra. Natural test cases arise in the theory of unilateral weighted right shifts on $\ell^2(\mathbb{N})$. However, at least in the examples considered in [11] and [12], it turns out that every operator in the commutant has SVEP. Thus it remains open if the following result leads to a counter-example to the preservation of SVEP under sums.

Corollary 4. *Let $S \in L(X)$ be an operator with fat local spectra on a Banach space X, and suppose that $\sigma(S)$ is not a singleton. If there exists an operator $T \in L(X)$ without SVEP that commutes with S, then there exist two commuting operators with SVEP for which the sum fails to have SVEP.*

Proof. We choose two distinct points $\lambda, \mu \in \sigma(S)$, and observe that each of the operators $\lambda - S$ and $S - \mu$ is non-invertible and has fat local spectra. By Theorem 3, it follows that $(\lambda - S)T$ and $(S - \mu)T$ both have SVEP. But, by the condition on T, the sum of these operators does not have SVEP. □

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