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ON INTERVALS AND ISOMETRIES OF MV-ALGEBRAS

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Abstract. Let $\operatorname{Int}\mathcal{A}$ be the lattice of all intervals of an MV-algebra $\mathcal{A}$. In the present paper we investigate the relations between direct product decompositions of $\mathcal{A}$ and (i) the lattice $\operatorname{Int}\mathcal{A}$, or (ii) 2-periodic isometries on $\mathcal{A}$, respectively.

Keywords: MV-algebra, duality, interval, autometrization, 2-periodic isometry

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1. Introduction

The system $\operatorname{Int} L$ of intervals of a lattice L has been investigated in several papers; for detailed references cf. [11].

Let $\mathcal{A}$ be an MV-algebra with the underlying set A. In view of [13], $\mathcal{A}$ can be constructed by means of an abelian lattice ordered group having a strong unit. This yields that without loss of generality we can suppose that on the set A lattice operations $\vee$ and $\wedge$ (implying a partial order $\leqslant$ on A) are defined and that for each $x, y \in A$ with $x \leqslant y$ the difference $y - x$ is defined in A.

Let $\ell(\mathcal{A})$ be the lattice $(A; \vee, \wedge)$; we put $\operatorname{Int}\ell(\mathcal{A}) = \operatorname{Int}\mathcal{A}$.

We denote by $\mathcal{A}^{\mathrm{dual}}$ the MV-algebra dual to $\mathcal{A}$ (for the terminology, cf. Section 2 below).

Further, we denote by $M_1(\mathcal{A})$, $M_2(\mathcal{A})$ and $M_3(\mathcal{A})$ the systems of all MV-algebras $\mathcal{A}_1$ such that

$$\operatorname{Int}\mathcal{A}_1 = \operatorname{Int}\mathcal{A}, \quad \ell(\mathcal{A}_1) = \ell(\mathcal{A}), \quad \text{or} \quad \ell(\mathcal{A}_1) = \ell(\mathcal{A}^{\mathrm{dual}}),$$

respectively.

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We always have

$$M_2(\mathcal{A}) \cup M_3(\mathcal{A}) \subseteq M_1(\mathcal{A}).$$

In the present paper we prove:

(∗) Let $\mathcal{A}$ be an MV-algebra. The following conditions are equivalent:

(i) $M_2(\mathcal{A}) \cup M_3(\mathcal{A}) = M_1(\mathcal{A})$.

(ii) The MV-algebra A is directly indecomposable.

The basic papers on isometries in autometrized lattice ordered groups are the articles [16] and [17]; cf. also [6], [7], [14], [15]. For more detailed references concerning isometries in some other types of autometrized partially ordered algebraic structures cf. [10].

Let $\mathcal{A}$ and A be as above. For $a, b \in A$ we put

$$\varrho(a,b) = (a \vee b) - (a \wedge b).$$

The mapping $\varrho\colon A \times A \to A$ will be called the autometrization of $\mathcal{A}$.

A bijection $f\colon A \to A$ is said to be an isometry of A if the relation

$$\varrho(f(a), f(b)) = \varrho(a,b)$$

identically holds.

An isometry f is called 2-periodic if $f(f(a)) = a$ for each $a \in A$. Let F be the set of all 2-periodic isometries on $\mathcal{A}$.

We show that a 2-periodic isometry f is uniquely determined by the element $f(0)$.

Namely, let us denote $f(0) = b$. Then b has a (uniquely determined) complement c in $\ell(\mathcal{A})$. We prove that for each $t \in A$ the following formula is valid:

$$f(t) = (b - (t \wedge b)) \vee (t \wedge c).$$

For $f_1, f_2 \in F$ we put $f_1 \leqslant f_2$ if $f_1(0) \leqslant f_2(0)$. We show that the structure $(F; \leqslant)$ is a Boolean algebra.

When dealing with isometries on $\mathcal{A}$ we shall apply direct product decompositions of $\mathcal{A}$.

2. Preliminaries

For defining MV-algebras several equivalent systems of axioms have been applied.

Let us recall the system from [3] (cf. also [2]); this system will be useful for defining the dual of an MV-algebra.

Suppose that A is a nonempty set, $\oplus$ and $\odot$ are binary operations, $\neg$ is a unary operation, and 0, 1 are nullary operations (i.e., constants) on A. By means of these operations we define binary operations $\vee$ and $\wedge$ on A by putting

$$x \vee y = (x \odot \neg y) \oplus y, \quad x \wedge y = (x \oplus \neg y) \odot y.$$

2.1. Definition. The algebraic structure $\mathcal{A} = (A; \oplus, \odot, \neg, 0, 1)$ is an MV-algebra if it satisfies the following axioms:

Ax. 1. $x \oplus y = y \oplus x$	Ax. 1′. $x \odot y = y \odot x$,
Ax. 2. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,	Ax. 2′. $x \odot (y \odot z) = (x \odot y) \odot z$,
Ax. 3. $x \oplus \neg x = 1$,	Ax. 3′. $x \odot \neg x = 0$,
Ax. 4. $x \oplus 1 = 1$,	Ax. 4′. $x \odot 0 = 0$,
Ax. 5. $x \oplus 0 = x$,	Ax. 5′. $x \odot 1 = x$,
Ax. 6. $\neg(x \oplus y) = \neg x \odot \neg y$,	Ax. 6′. $\neg(x \odot y) = \neg x \oplus \neg y$,
Ax. 7. $x = \neg(\neg x)$,	Ax. 8. $\neg 0 = 1$,
Ax. 9. $x \vee y = y \vee x$,	Ax. 9′. $x \wedge y = y \wedge x$,
Ax. 10. $x \vee (y \vee z) = (x \vee y) \vee z$,	Ax. 10′. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
Ax. 11. $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$,	Ax. 11′. $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.

Further, let us consider the following system of axioms for an algebraic structure $\mathcal{A} = (a, \oplus, \odot, \neg, 0, 1)$ (cf. [5]):

(M1) $(x \oplus y) \oplus z = z \oplus (y \oplus z)$,
(M2) $x \oplus 0 = x$,
(M3) $x \oplus y = y \oplus x$,
(M4) $x \oplus 1 = 1$,
(M5) $\neg\neg x = x$,
(M6) $\neg 0 = 1$,
(M7) $x \oplus \neg x = 1$,
(M8) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$,
(M9) $x \odot y = \neg(\neg x \oplus \neg y)$.

2.2. Proposition (cf. [12]). *Assume that the algebraic structure* $\mathcal{A} = (A; \oplus, \odot, \neg, 0, 1)$ *satisfies the axioms* (M1)–(M9). *Then* $\mathcal{A}$ *is an* MV*-algebra.*

In some papers (cf., e.g., [5], [8]) the axioms (M1)–(M9) are applied under a slightly modified notation (instead of $\odot$ the symbol $*$ is used).

A simplified system of axioms for an MV-algebra was given in [2]; moreover, it was shown that the axioms of this system are independent.

If $\mathcal{A}_1$ is another MV-algebra then we sometimes use the notation

$$\mathcal{A}_1 = (A_1; \oplus_1, \odot_1, \neg_1, 0_1, 1_1) \tag{1}$$

(e.g., in the case when $A_1 = A$ and when the operations from $\mathcal{A}_1$ need not coincide with those of $\mathcal{A}$).

2.3. Lemma. *Let $\mathcal{A}$ be as in 2.1 and let*

$$A_1 = A, \quad \oplus_1 = \odot, \quad \odot_1 = \oplus, \quad \neg_1 = \neg, \quad 0_1 = 1, \quad 1_1 = 0.$$

Then the algebraic structure $\mathcal{A}_1$ from (1) is an MV-algebra. Moreover, if $\vee_1$ and $\wedge_1$ are defined analogously as $\vee$ and $\wedge$ above, then

$$\vee_1 = \wedge, \quad \wedge_1 = \vee.$$

Proof. This is an immediate consequence of Definition 2.1. $\square$

We say that the MV-algebra $\mathcal{A}_1$ from 2.3 is dual to the MV-algebra $\mathcal{A}$ and write

$$\mathcal{A}_1 = \mathcal{A}^{\text{dual}}.$$

3. The lattice $\ell(\mathcal{A})$

For lattice ordered groups we apply the notation and the terminology as in [1] and [4].

For the following results $(*_1)$ and $(**)$ cf. [13].

$(*_1)$ Let G be an abelian lattice ordered group with a strong unit u. Let A be the interval $[0, u]$ of G. For $a, b \in A$ we put

$$a \oplus b = (a + b) \wedge u, \quad \neg a = u - a,$$
$$1 = u, \quad a \odot b = \neg(\neg a \oplus \neg b).$$

Then the algebraic system $\mathcal{A} = (A; \oplus, \odot, \neg, 0, 1)$ is an MV-algebra.

The MV-algebra from $(*)$ will be denoted by $\Gamma(G,u)$ (in [14], the notation $G_0(G,u)$ was applied).

$(**)$ For each MV-algebra $\mathcal{A}$ there exists an abelian lattice ordered group G with a strong unit u such that $\mathcal{A} = \Gamma(G,u)$.

In what follows we assume that $\mathcal{A}$ is an MV-algebra and that G is as in $(**)$. Then the operation $\vee$ on the set A (induced from G) coincides with the operation $\vee$ from 2.1; the situation for the operation $\wedge$ is analogous. The partial order $\leqslant$ on A is defined by means of the operations $\vee$ and $\wedge$. We have $0 \leqslant x \leqslant u$ for each $x \in A$. Further, if x and y are elements of A with $x \leqslant y$, then $y - x \in A$; hence we can consider—to be a partial binary operation on A. We denote

$$(A; \vee, \wedge) = \ell(\mathcal{A}).$$

We remark that if $\mathcal{A}$ and $\mathcal{A}'$ are MV-algebras such that

$$\ell(\mathcal{A}) = \ell(\mathcal{A}'),$$

then neither $\mathcal{A} = \mathcal{A}'$ nor $\mathcal{A}^{\mathrm{dual}} = \mathcal{A}'$ need be valid.

Let L be a lattice. The corresponding dual lattice will be denoted by L^d.

The direct product of lattices L_1 and L_2 is defined in the usual way and we denote it by $L_1 \times L_2$.

A lattice L is called directly indecomposable if, whenever L is isomorphic to a direct product $L_1 \times L_2$, then either L_1 or L_2 is a one-element set.

An analogous notation and terminology will be applied for direct products of MV-algebras.

The meaning of $\operatorname{Int} L$ is as in Section 1. Further, let $\operatorname{Csub} L$ be the set of all convex sublattices of L. We obviously have

3.1. Lemma. *Let L be a lattice. Then* $\operatorname{Int} L^d = \operatorname{Int} L$.

As a corollary we obtain

3.1.1. Corollary. *Let L_1 and L_2 be lattices. Then*

$$\operatorname{Int}(L_1 \times L_2) = \operatorname{Int}(L_1^d \times L_2).$$

The proof of the following lemma is simple; it will be omitted.

3.2. Lemma. *Let L and L' be lattices defined on the same underlying set M. Then the following conditions are equivalent:*

(i) $\operatorname{Int} L = \operatorname{Int} L'$;
(ii) $\operatorname{Csub} L = \operatorname{Csub} L'$.

3.3. Lemma. *Let L and L' be distributive lattices defined on the same underlying set M. Then the following conditions are equivalent:*
(i) $\operatorname{Int} L = \operatorname{Int} L'$;
(ii) *There exist lattices L_1, L_2 and a bijection*

$$\varphi\colon\ M \to L_1 \times L_2$$

such that φ is an isomorphism of L onto $L_1 \times L_2$ and, at the same time, φ is an isomorphism of L' onto $L_1^d \times L_2$.

Proof. This is a consequence of 3.2 and of the results of [9]. □

3.4. Lemma. *Let $\mathcal{A}$ be an MV-algebra. Then*

$$M_2(\mathcal{A}) \cup M_3(\mathcal{A}) \subseteq M_1(\mathcal{A}).$$

Proof. In view of the definition of the MV-algebra $\mathcal{A}^{\text{dual}}$ we have

$$\ell(\mathcal{A}^{\text{dual}}) = (\ell(\mathcal{A}))^d. \tag{1}$$

Now it suffices to apply 3.1. □

Now suppose that L_1 and L_2 are lattices with $\operatorname{card} L_1 \neq 1 \neq \operatorname{card} L_2$. Put $L = L_1 \times L_2$ and $L' = L_1^d \times L_2$. The partial orders on L, L^d and L' will be denoted by $\leqslant_1, \leqslant_2$ or $\leqslant_3$, respectively.

3.5. Lemma. *The partial order $\leqslant_3$ coincides neither with $\leqslant_1$ nor with $\leqslant_2$.*

Proof. There exist $u_1, v_1 \in L_1$ and $u_2, v_2 \in L_2$ such that the relation $u_i < v_i$ is valid in L_i $(i = 1, 2)$. Then we have

$$(v_1, u_2) <_3 (u_1, v_2),$$

but the analogous relation fails to hold for both $<_1$ and $<_2$. □

If $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ are MV-algebras such that $\mathcal{A}$ is isomorphic to $\mathcal{A}_1 \times \mathcal{A}_2$, then $\ell(\mathcal{A})$ is isomorphic to $\ell(\mathcal{A}_1) \times \ell(\mathcal{A}_2)$. Thus 3.5 and (1) yield

3.6. Lemma. *Assume that $\mathcal{A}$ is a directly decomposable MV-algebra. Then*

$$M_2(\mathcal{A}) \cup M_3(\mathcal{A}) \neq M_1(\mathcal{A}).$$

Now suppose that $\mathcal{A}$ and $\mathcal{A}'$ are MV-algebras such that
(i) $\mathcal{A}$ and $\mathcal{A}'$ have the same underlying set A;
(ii) $\operatorname{Int} \mathcal{A} = \operatorname{Int} \mathcal{A}'$.
Denote

$$\ell(\mathcal{A}) = L, \quad \ell(\mathcal{A}') = L'.$$

Then both L and L' have the same underlying set A and

$$\operatorname{Int} L = \operatorname{Int} L'.$$

Hence the condition (ii) from 3.3 is satisfied. We denote by A_1 and A_2 the underlying sets of the lattices L_1 and L_2, respectively.

In view of [8] there exist MV-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ such that
a) $\ell(\mathcal{A}_i) = L_i$ for $i = 1, 2$;
b) the mapping φ is an isomorphism of $\mathcal{A}$ onto $\mathcal{A}_1 \times \mathcal{A}_2$.

Similarly we obtain that there exist MV-algebras $\mathcal{A}_1'$ and $\mathcal{A}_2'$ such that
a) $\ell(\mathcal{A}_1') = L_1^d$, $\ell(\mathcal{A}_2') = L_2$;
b) the mapping φ is an isomorphism of $\mathcal{A}'$ onto $\mathcal{A}_1' \times \mathcal{A}_2'$.

Summarizing, we conclude

3.7. Proposition. *Let $\mathcal{A}$ and $\mathcal{A}'$ be MV-algebras such that $\mathcal{A}' \in M_1(\mathcal{A})$. Then there exist direct product decompositions*

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2, \quad \mathcal{A}' = \mathcal{A}_1' \times \mathcal{A}_2'$$

such that

$$\mathcal{A}_1' \in M_3(\mathcal{A}_1), \quad \mathcal{A}_2' \in M_2(\mathcal{A}_2).$$

P r o o f of $(*)$ from Section 1. Let the condition (i) from $(*)$ be valid. Then in view of 3.6 the MV-algebra $\mathcal{A}$ is directly indecomposable.

Conversely, assume that the condition (ii) from $(*)$ holds. Let $\mathcal{A}' \in M_1(\mathcal{A})$. We apply 3.7. Since $\mathcal{A}$ is directly indecomposable we infer that either $\mathcal{A}_1$ or $\mathcal{A}_2$ has a one-element underlying set. Hence either $\mathcal{A} = \mathcal{A}_1$ or $\mathcal{A} = \mathcal{A}_2$. Therefore (i) holds.

□

4. Autometrization and isometries

Assume that $\mathcal{A}$ and G are as above.

Let $a, b \in A$. From the definition of $\varrho(a,b)$ in Section 1 we get

$$\varrho(a,b) = |a-b|.$$

Since the autometrization ϱ_G on G considered in [16] was given by

$$\varrho_G(x,y) = |x-y|$$

for each $x, y \in G$, we conclude that the autometrization ϱ on A is induced from that studied in [6] on the whole G.

This immediately yields

1) $\varrho(a,b) = 0$ if and only if $a = b$.
2) $\varrho(a,b) = \varrho(b,a)$.

Further, we have:

3) For any $a, b, c \in A$,

$$\varrho(a,b) \leqslant \varrho(a,c) \oplus \varrho(c,b).$$

P r o o f. It is well-known that

$$|a-b| \leqslant |a-c| + |c-b|.$$

Since $|a-b| \in A$ we get $|a-b| \leqslant u$ and then

$$|a-b| \leqslant (|a-c| + |c-b|) \wedge u = |a-c| \oplus |c-b|.$$

□

By checking the proofs of Lemmas 1.1–1.7′ in [7] we can verify that all assertions of these lemmas remain valid if instead of the lattice ordered group G we take the MV-algebra $\mathcal{A}$. Moreover, the duals of 1.7 and 1.7′ also hold.

Since $A = [0,u]$, we have

4.1. Lemma. *Let $t_1, t_2 \in A$, $t_2 - t_1 = u$. Then $t_1 = 0$ and $t_2 = u$.*

Let f be an isometry on $\mathcal{A}$. Denote

$$f(0) = b, \quad f(u) = c.$$

We have

$$u = |u-0| = |f(u) - f(0)| = |b-c| = (b \vee c) - (b \wedge c).$$

Hence in view of 4.1,

$$b \wedge c = 0, \quad b \vee c = u.$$

Thus we obtain

4.2. Lemma. *The element c is a complement of b.*

Now suppose that f is an element of F. Then

$$f(b) = 0, \quad f(c) = u.$$

Let us apply the terminology of Section 1, [7]. Hence we have

$$[0, b] \in M_2, \tag{1}$$

$$[b, u] \in M_1. \tag{2}$$

In view of 1.7′ from [7] and according to (1) we obtain

$$[c, u] \in M_2. \tag{3}$$

Further, in view of the dual of 1.7 from [7] and according to (2), we get

$$[0, c] \in M_1. \tag{4}$$

Remark. The assertion of 4.2 is implied also by (1)–(4) and by Lemma 1.6 of [7].

4.3. Lemma. *Let $x \in [0, b]$. Then $f(x) = b - x$.*

P r o o f. In view of (1) we have

$$f(0) \geqslant f(x) \geqslant f(b),$$

hence in view of 1.3 from [7] we get

$$0 \leqslant f(x) \leqslant b.$$

Further,

$$|x - 0| = |f(x) - f(0)|,$$

thus $x = b - f(x)$, yielding $f(x) = b - x$. □

Let $t \in A$. Denote

$$t \wedge b = t_1, \quad t \wedge c = t_2.$$

Then we easily obtain

$$t_1 \wedge t_2 = 0, \quad t_1 \vee t_2 = t.$$

In view of (4) and according to 1.3 from [7] we have $[0, t_2] \in M_1$, hence according to 1.7 of [7] we get

$$[t_1, t] \in M_1. \tag{5}$$

Further, $t - t_1 = t_2$. Thus

$$|f(t) - f(t_1)| = |t - t_1| = t_2.$$

In view of (5),

$$|f(t) - f(t_1)| = f(t) - f(t_1).$$

Hence

$$f(t) - f(t_1) = t_2.$$

Then according to 4.3,

$$f(t) = b - t_1 + t_2.$$

Since $b - t_1 \leqslant b$ and $t_2 \leqslant c$, we have

$$(t - t_1) \wedge t_2 = 0,$$

thus $(b - t_1) + t_2 = (b - t_1) \vee t_2$. Therefore

$$f(t) = (b - t_1) \vee t_2.$$

Summarizing, we have

4.4. Proposition. *Let f be a 2-periodic isometry on $\mathcal{A}$, $f(0) = b$. Then there exists a uniquely determined element $c \in A$ such that c is a complement of b in $\ell(\mathcal{A})$. For each $t \in A$ the formula*

$$f(t) = (b - (b \wedge t)) \vee (t \wedge c)$$

is valid.

Again, let $\mathcal{A}$ and G be as above.

In this section we prove that for each element $b \in A$ having a complement in $\ell(\mathcal{A})$ there exists $f \in F$ with $f(0) = b$.

The main tool in this investigation are direct product decompositions (of lattices, MV-algebras and lattice ordered groups, respectively). We apply the results of [14]. Suppose that b, c are elements of A such that

$$b \wedge c = 0, \quad b \vee c = u.$$

Put $B = [0, b]$, $C = [0, c]$. For each $t \in A$ we set

$$t_1 = b \wedge t, \quad t_2 = c \wedge t, \quad \varphi(t) = (t_1, t_2).$$

Since the lattice $L = \ell(\mathcal{A})$ is distributive we obtain

5.1. Lemma. *φ is an isomorphism of L onto the direct product $B \times C$.*

From 5.1 and in view of the results of [8] we infer

5.2. Lemma. *There exist MV-algebras $\mathcal{B}$ and $\mathcal{C}$ such that*

(i) $\ell(\mathcal{B}) = B$, $\ell(\mathcal{C}) = C$,

(ii) *the mapping φ is an isomorphism of $\mathcal{A}$ onto the direct product $\mathcal{B} \times \mathcal{C}$.*

Recall that if $t \in A$ and $\varphi(t) = (t_1, t_2)$, then $t = t_1 \vee t_2$.

Again, let G be as above (i.e., $\mathcal{A} = \Gamma(G, u)$, where u is a strong unit of G).

In view of 5.2 and according to [8] we obtain that there exist abelian lattice ordered groups G_1 and G_2 having strong units b and c, respectively, such that

(i) $\mathcal{B} = \Gamma(G_1, b)$, $\mathcal{C} = \Gamma(G_2, c)$,

(ii) there exists an isomorphism φ^0 of G onto $G_1 \times G_2$ such that $\varphi^0(t) = \varphi(t)$ for each $t \in A$.

This yields that for each $t, t' \in A$ we have

$$|t - t'|_i = |t_i - t'_i| \quad (i = 1, 2).$$

For each $t \in A$ we put

$$f(t) = (b - (b \wedge t)) \vee (t \wedge c). \tag{1}$$

Since

$$b_1 = b, \quad b_2 = 0, \quad b \wedge t = t_1, \quad t \wedge c = t_2$$

we get

$$(f(t))_1 = b - t_1, \quad (f(t))_2 = t_2.$$

We want to verify that f is an isometry on $\mathcal{A}$. It suffices to verify that the relation

$$|t_i - t_i'| = |(f(t))_i - (f(t'))_i|$$

is valid for $i = 1, 2$.

The case $i = 2$ is obvious. Consider the case $i = 1$. We have

$$\begin{aligned} |t_1 - t_1'| &= (t_1 \vee t_1') - (t_1 \wedge t_1'), \\ |(f(t))_1 - (f(t')_1| &= |(b - t_1) - (b - t_1')| \\ &= ((b - t_1) \vee (b - t_1')) - ((b - t_1) \wedge (b - t_1')). \end{aligned}$$

In view of the relation between $\mathcal{A}$ and G, and since $A \subseteq G$, the last expressions can be calculated in G and we obtain

$$\begin{aligned} (b - t_1) \vee (b - t_1') &= b + ((-t_1) \vee (-t_1')) = b - (t_1 \wedge t_1'), \\ (b - t_1) \wedge (b - t_1') &= b + ((-t_1) \wedge (-t_1')) = b - (t_1 \vee t_1'), \\ |(f(t))_1 - (f(t'))_1| &= (b - (t_1 \wedge t_1')) - (b - (t_1 \vee t_1')) \\ &= (t_1 \vee t_1') - (t_1 \wedge t_1'), \end{aligned}$$

as desired. Therefore f is an isometry.

Now let us verify that f is 2-periodic. Put $f(t) = p$. Then

$$\begin{aligned} (f(p))_1 &= b - (b - t_1)_1 = b - (b - t_1) = t_1, \\ (f(p))_2 &= (f(f(t)))_2 = t_2, \\ f(p) &= f(p)_1 \vee f(p)_2 = t_1 \vee t_2 = t, \quad f(f(t)) = t. \end{aligned}$$

Hence we obtain

5.3. Proposition. *Let b and c be complementary elements of the lattice $L = \ell(\mathcal{A})$. Let f be defined by* (1)*. Then f is a* 2*-periodic isometry on $\mathcal{A}$.*

Let us now write f_b instead of f (where f is as in 5.3). Let B_0 be the set of all elements $b \in L$ which have a complement. Since the lattice L is distributive, B_0 is a Boolean algebra.

Consider the mapping $\chi\colon B_0 \to F$ defined by

$$\chi(b) = f_b$$

for each $b \in B_0$. In view of 4.4 and 5.3, χ is a bijection. Hence under the relation $\leqslant$ from Section 1, F is a Boolean algebra.

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