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A SIMPLE PROOF OF THE BOREL EXTENSION THEOREM AND WEAK COMPACTNESS OF OPERATORS

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Abstract. Let T be a locally compact Hausdorff space and let $C_0(T)$ be the Banach space of all complex valued continuous functions vanishing at infinity in T, provided with the supremum norm. Let X be a quasicomplete locally convex Hausdorff space. A simple proof of the theorem on regular Borel extension of X-valued σ-additive Baire measures on T is given, which is more natural and direct than the existing ones. Using this result the integral representation and weak compactness of a continuous linear map $u\colon C_0(T) \to X$ when $c_0 \not\subset X$ are obtained. The proof of the latter result is independent of the use of powerful results such as Theorem 6 of [6] or Theorem 3 (vii) of [13].

Keywords: weakly compact operator on $C_0(T)$, representing measure, lcHs-valued σ-additive Baire (or regular Borel, or regular σ-Borel) measures

MSC 2000: 28B05, 28C05, 28C15

1. Introduction

Let T be a locally compact Hausdorff space and $C_0(T)$ the Banach space of all complex valued continuous functions vanishing at infinity in T, endowed with the supremum norm.

If X is a Banach space with $c_0 \not\subset X$ and S is a compact Hausdorff space, then Pelczyński [14] proved that each continuous linear map $u\colon C(S) \to X$ admits an integral representation with respect to a σ-additive X-valued Borel measure on T

* The research was done before I. Dobrakov died.

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and that u is weakly compact. His proof is a modification of the proof of Theorem VI.7.6 of Dunford and Schwartz [4], where the argument of reduction to the compact metrizable case plays a key role.

Later, in 1970, this result was extended in Theorem 5.3 of Thomas [17] to continuous linear maps $u\colon C_0(T) \to X$, where X is a locally convex Hausdorff space (briefly, an lcHs) which is quasicomplete and Σ-complete in the sense of [17], and this includes the converse, too. Thomas also used the technique of reduction to the compact metrizable case. While Pelczyński [14] used the Bartle-Dunford-Schwartz representation theorem (Theorem VI.7.2 of [4]), Thomas [17] used the Grothendieck characterizations of weakly compact operators on $C(K)$, K a compact Hausdorff space, as given in Theorem 6 of [6]. We also note that by Theorem 4 of Tumarkin [18] the Σ-completeness of X as given in [17] is equivalent to the condition that $c_0 \not\subset X$. The proof of Thomas is highly technical as it uses not only his theory of Radon vector measures but also Theorem 6 of [6] whose proof depends on some deep results such as Theorems 2 and 3 and Proposition 11 of [6].

The aim of the present note is to give a simple direct proof of the Borel extension theorem for quasicomplete lcHs valued Baire measures on T and then, as an application, to deduce Theorem 5.3 of Thomas [17]. For the latter we just use Lemma 1 and Theorem 2 of Grothendieck [6] (no other result of Grothendieck [6] is used—even in the Banach space situation its analogue has been used in the proof of [14]) and the first part of Theorem 1 of [13] (which is the locally convex space analogue of the Bartle-Dunford-Schwartz representation theorem for continuous linear maps on $C_0(T)$). The present proof dispenses with the argument of reduction to the compact metrizable case unlike the above mentioned proofs of [4], [14], [17].

The present proof emphasizes the fact that the weak compactness of the operators in question is due to the existence of a regular Borel extension of X-valued Baire measures on T. We would like to observe that this fact is not at all brought out explicitly both in the earlier proofs (based on the technique of reduction to the compact metrizable case) of [4], [14], [17] and in the recent proof of Theorem 13 of [13].

2. Preliminaries

In this section we fix the notation and terminology. For the convenience of the reader we also give some definitions and results from literature.

In the sequel T will denote a locally compact Hausdorff space and $C_0(T)$ the Banach space of all complex valued continuous functions vanishing at infinity in T, endowed with a norm $\|\cdot\|_T$ given by $\|f\|_T = \sup\limits_{t\in T} |f(t)|$.

Let $\mathcal{K}$ (or $\mathcal{K}_0$) be the family of all compacts (compact $G_\delta s$, respectively) in T. $\mathcal{B}_0(T)$, $\mathcal{B}_c(T)$ and $\mathcal{B}(T)$ are the σ-rings generated by $\mathcal{K}_0$, $\mathcal{K}$ and the class of all open sets in T, respectively. The members of $\mathcal{B}_0(T)$ are called *Baire sets* of T and those of $\mathcal{B}_c(T)$ are called *σ-Borel sets* of T. The members of $\mathcal{B}(T)$ are called *Borel sets* of T. Since a subset E of T belongs to $\mathcal{B}_c(T)$ if and only if E is a σ-bounded Borel set, the members of $\mathcal{B}_c(T)$ are called σ-Borel sets.

Definition 1. Let $\mathcal{S}$ be a σ-ring of sets in T such that $\mathcal{K} \subset \mathcal{S}$ or $\mathcal{K}_0 \subset \mathcal{S}$. A complex ($\sigma$-additive) measure μ on $\mathcal{S}$ is said to be *$\mathcal{S}$-regular* if, given $E \in \mathcal{S}$ and $\varepsilon > 0$, there exist a compact $K \in \mathcal{S}$ and an open set $U \in \mathcal{S}$ with $K \subset E \subset U$ such that $|\mu(B)| < \varepsilon$ for every $B \in \mathcal{S}$ with $B \subset U \backslash K$. When $\mathcal{S} = \mathcal{B}(T)$ ($\mathcal{S} = \mathcal{B}_c(T)$, $\mathcal{S} = \mathcal{B}_0(T)$), we use the terminology *Borel* (*σ-Borel, Baire*, respectively) *regularity* in place of $\mathcal{S}$-regularity.

The following proposition is well known. See, for example, Theorem 3.7 of [10] and Theorem 2.4 of [11].

Proposition 1. *Every complex Baire measure μ_0 on T is regular and has a unique extension μ on $\mathcal{B}(T)$ (μ_c on $\mathcal{B}_c(T)$) such that μ is a Borel (σ-Borel, respectively) regular complex measure. Moreover, $\mu|_{\mathcal{B}_c(T)} = \mu_c$. Besides, μ and μ_c are positive and finite if μ_0 is so.*

$M(T)$ is the Banach dual of $C_0(T)$ and hence it is identified with the space of all bounded complex Radon measures on T with their domain restricted to $\mathcal{B}(T)$ so that each $\mu \in M(T)$ is a regular (bounded) complex Borel measure on T and has a norm $\|\cdot\|$ given by $\|\mu\| = \operatorname{var}(\mu, T)$ where the variation of μ is taken with respect to $\mathcal{B}(T)$. We denote $\operatorname{var}(\mu, E)$ by $|\mu|(E)$, for $E \in \mathcal{B}(T)$.

A vector measure is an additive set function defined on a ring of sets with values in an lcHs. In the sequel X denotes an lcHs with a topology τ. Γ is the set of all τ-continuous seminorms on X. The dual of X is denoted by X^*.

The strong topology $\beta(X^*, X)$ of X^* is the locally convex topology induced by the seminorms $\{p_B\colon B \text{ bounded in } X\}$, where $p_B(x^*) = \sup_{x \in B} |x^*(x)|$. X^{**} denotes the dual of $(X^*, \beta(X^*, X))$ and is endowed with the locally convex toplogy τ_e of uniform convergence on equicontinuous subsets of X^*. Note that $(X^*, \beta(X^*, X))$ and (X^{**}, τ_e) are lcHs.

It is well known that the canonical injection $J\colon X \to X^{**}$ given by $\langle Jx, x^* \rangle = \langle x, x^* \rangle$ for all $x \in X$ and $x^* \in X^*$, is linear. On identifying X with $JX \subset X^{**}$, one has $\tau_e|_{JX} = \tau_e|_X = \tau$.

Let $\mathcal{E} = \{A \subset X^*\colon A$ is equicontinuous$\}$. Then the family of seminorms $\Gamma_{\mathcal{E}} = \{p_A\colon A \in \mathcal{E}\}$ induces the topology τ of X and the topology τ_e of X^{**}, where $p_A(x) = \sup\limits_{x^* \in A} |x^*(x)|$ for $x \in X$ and $p_A(x^{**}) = \sup\limits_{x^* \in A} |x^{**}(x^*)|$ for $x^{**} \in X^{**}$.

Definition 2. A linear map $u\colon C_0(T) \to X$ is called *a weakly compact operator* on $C_0(T)$ if $\{uf\colon \|f\|_T \leqslant 1\}$ is relatively weakly compact in X.

The following result is the same as Lemma 2 of [13], where the hypothesis of quasicompleteness of X is redundant.

Proposition 2. *Let X be an* lcHs *and let $u\colon C_0(T) \to X$ be a continuous linear map. Then u^*A is bounded in $M(T)$ for each $A \in \mathcal{E}$.*

The following result (Corollary 9.3.2 of Edwards [5] which is essentially due to Lemma 1 of Grothendieck [6]) plays a key role in Section 4.

Proposition 3. *Let E and F be* lcHs *with F quasicomplete and let $u\colon E \to F$ be linear and continuous. Then the following assertions are equivalent:*

(i) *$u^{**}(E^{**}) \subset F$.*

(ii) *u maps bounded subsets of E into relatively weakly compact subsets of F.*

(iii) *$u^*(A)$ is relatively $\sigma(E^*, E^{**})$-compact for each equicontinuous subset A of F^*.*

The following result is due to Theorem 2 of Grothendieck [6], and is needed in Section 4.

Proposition 4. *A bounded set A in $M(T)$ is relatively weakly compact if and only if, for each disjoint sequence $\{U_n\}_1^\infty$ of open sets in T,*

$$\sup_{\mu \in A} |\mu(U_n)| \to 0$$

as $n \to \infty$.

For each τ-continuous seminorm p on X, let $p(x) = \|x\|_p$, $x \in X$, and let $X_p = (X, \|\cdot\|_p)$ be the associated seminormed space. The completion of the quotient normed space $X_p/p^{-1}(0)$ is denoted by $\tilde{X}_p$. Let $\Pi_p\colon X_p \to X_p/p^{-1}(0) \subset \tilde{X}_p$ be the canonical quotient map.

Let $\mathcal{S}$ be a σ-ring of subsets of a non empty set Ω. Given a vector measure $m\colon \mathcal{S} \to X$, for each τ-continuous seminorm p on X let $m_p\colon \mathcal{S} \to \tilde{X}_p$ be given by $m_p(E) = \Pi_p \circ m(E)$ for $E \in \mathcal{S}$. Then m_p is a Banach space valued vector measure on $\mathcal{S}$. We define the p-semivariation $\|m\|_p$ of m by

$$\|m\|_p(E) = \|m_p\|(E) \ \text{ for } \ E \in \mathcal{S}$$

and

$$\|m\|_p(\Omega) = \|m_p\|(\Omega) = \sup_{E\in\mathcal{S}} \|m_p\|(E)$$

where $\|m_p\|$ is the semivariation of the vector measure m_p. When m is σ-additive, m_p is a Banach space valued σ-additive vector measure and hence, by a well known theorem on vector measures, $\|m\|_p(\Omega) = \|m_p\|(\Omega) \leqslant 4 \sup_{E\in\mathcal{S}} \|m(E)\|_p < \infty$.

An X-valued vector measure m on a σ-ring $\mathcal{S}$ of subsets of Ω is said to be *bounded* if $\{m(E)\colon E \in \mathcal{S}\}$ is bounded in X or equivalently, if $\|m\|_p(\Omega) < \infty$ for each τ-continuous seminorm p on X.

For the theory of integration of bounded $\mathcal{S}$-measurable scalar functions with respect to a bounded quasicomplete lcHs-valued vector measure defined on the σ-ring $\mathcal{S}$, the reader may refer to [12] or [13]. We need the following results from Lemma 6 of [12] and Proposition 7 of [13].

Proposition 5. *Let X be a quasicomplete* lcHs. *Then:*

(i) *If f is a bounded $\mathcal{S}$-measurable scalar function and m is an X-valued bounded vector measure on $\mathcal{S}$, then f is m-integrable and*

$$x^*\left(\int_\Omega f\,\mathrm{d}m\right) = \int_\Omega f\,\mathrm{d}(x^*m)$$

for each $x^* \in X^*$.

(ii) (Lebesgue bounded convergence theorem). *If m is an X-valued σ-additive vector measure on $\mathcal{S}$ and (f_n) is a bounded sequence of $\mathcal{S}$-measurable scalar functions with $\lim_n f_n(w) = f(w)$ for each $w \in \Omega$, then f is m-integrable and*

$$\int_E f\,\mathrm{d}m = \lim_n \int_E f_n\,\mathrm{d}m$$

for each $E \in \mathcal{S}$.

The following result is due to the first part of Theorem 1 of [13] and is analogous to Theorem VI.2.1 of [2] for lcHs-valued continuous linear maps on $C_0(T)$. It plays a key role in Section 4.

Proposition 6 (Generalized Bartle-Dunford-Schwartz representation theorem)**.** *Let X be an lcHs and let $u\colon C_0(T) \to X$ be a continuous linear map. Then there* *exists a unique X^{**}-valued vector measure m on $\mathcal{B}(T)$ possessing the following properties:*

(i) *$x^*(m) \in M(T)$ for each $x^* \in X^*$ and consequently, $m\colon \mathcal{B}(T) \to X^{**}$ is σ-additive in the $\sigma(X^{**}, X^*)$-topology.*

(ii) *The mapping $x^* \to x^*m$ of X^* into $M(T)$ is weak*-weak* continuous. Moreover, $u^*x^* = x^*m$, $x^* \in X^*$.*

(iii) *$x^*uf = \int_T f \,\mathrm{d}x^*m$ for each $f \in C_0(T)$ and $x^* \in X^*$.*

(iv) *The range of m is τ_e-bounded in X^{**}.*

(v) *$m(E) = u^{**}(\chi_E)$ for $E \in \mathcal{B}(T)$.*

Definition 3. Let $u\colon C_0(T) \to X$ be a continuous linear map. Then the vector measure m given in Proposition 6 is called the *representing measure* of u.

3. Regular Borel (σ-Borel) extension of quasicomplete lcHs-valued Baire measures

The regular Borel extension theorem for Banach space- and complete lcHs-valued Baire measures on T are well known since the publication of [3], [8] and has also been generalized to group-valued measures by Sion [16] and to semigroup-valued measures by Weber [19]. Using Lemma 2 and Theorem 2 of [3] and the lemma in § 68 of Berberian [1] we give here a simple direct proof of the theorem on regular Borel and σ-Borel extensions of a quasicomplete lcHs-valued Baire measure on T. We would like to remark that even for the case of Banach space-valued Baire measures, the proof given in [8] is quite involved, presupposing several results from the earlier papers of the author. Unlike [8], the present proof further dispenses with the technique of one-point compactification.

Let us begin with the following definitions.

Definition 4. A σ-additive vector measure $m\colon \mathcal{B}_0(T) \to X$ ($\mathcal{B}_c(T) \to X$, $\mathcal{B}(T) \to X$) is called an X-valued *Baire* (*σ-Borel, Borel* respectively), *measure* on T.

Definition 5. Let $\mathcal{S}$ be one of $\mathcal{B}_0(T)$, $\mathcal{B}_c(T)$ or $\mathcal{B}(T)$. An X-valued vector measure m on $\mathcal{S}$ is said to be *$\mathcal{S}$-regular* if, given $E \in \mathcal{S}$, a seminorm $p \in \Gamma$ and $\varepsilon > 0$, there exist a compact $K \in \mathcal{S}$ and an open set $U \in \mathcal{S}$ with $K \subset E \subset U$ such that $\|m(B)\|_p < \varepsilon$ for every $B \in \mathcal{S}$ with $B \subset U \backslash K$. When $\mathcal{S} = \mathcal{B}_0(T)$ (or $\mathcal{B}_c(T)$, $\mathcal{B}(T)$) we use the terminology *Baire* (or *σ-Borel, Borel*, respectively) *regular.*

Theorem 1. *Let m be an X-valued Baire measure on T and let X be a quasicomplete* lcHs*. Then there exists a unique X-valued Borel (or σ-Borel) regular σ-additive extension $\hat{m}$ (or m_c) of m on $\mathcal{B}(T)$ ($\mathcal{B}_c(T)$, respectively). Moreover, $\hat{m}|_{\mathcal{B}_c(T)} = m_c$.*

P r o o f. For each $p \in \Gamma$, $m_p\colon \mathcal{B}_0(T) \to \tilde{X}_p$ is σ-additive. Since the proof of Theorem I.2.4 of [2] holds for σ-rings too, for each $p \in \Gamma$ there exists a finite positive measure μ_p on $\mathcal{B}_0(T)$ such that

$$\lim_{\mu_p(A)\to 0} \|m_p(A)\|_p = 0, \quad A \in \mathcal{B}_0(T).$$

By Proposition 1 μ_p has a unique extension $\hat{\mu}_p$ (or μ_p^c) on $\mathcal{B}(T)$ (or $\mathcal{B}_c(T)$) such that $\hat{\mu}_p$ (or μ_p^c) is a (σ-additive) regular Borel (σ-Borel, respectively) finite positive measure. Moreover, $\hat{\mu}_p|_{\mathcal{B}_c(T)} = \mu_p^c$.

For $p \in \Gamma$, let $\varrho_p(E,F) = \hat{\mu}_p(E\Delta F)$, for $E, F \in \mathcal{B}(T)$. Then $\varrho_p(E,F) = \mu_p^c(E\Delta F)$ for $E, F \in \mathcal{B}_c(T)$. Let $s(\Gamma)$ be the uniform structure defined by the family $\{\varrho_p\}_{p\in\Gamma}$ of semidistances on $\mathcal{B}(T)$ (or $\mathcal{B}_c(T)$) and let Θ (or Θ_c) be the topology induced by $s(\Gamma)$ on $\mathcal{B}(T)$ (on $\mathcal{B}_c(T)$, respectively). Then clearly, $\Theta|_{\mathcal{B}_c(T)} = \Theta_c$.

Assertion 1. *$\mathcal{B}_0(T)$ is Θ-dense (Θ_c-dense) in $\mathcal{B}(T)$ ($\mathcal{B}_c(T)$, respectively).*

In fact, given $A \in \mathcal{B}(T)$ (or $\mathcal{B}_c(T)$), $p \in \Gamma$ and $\varepsilon > 0$, it suffices to show that there exists $E \in \mathcal{B}_0(T)$ such that $\varrho_p(A,E) < \varepsilon$. Since $\hat{\mu}_p$ is Borel regular (μ_p^c is σ-Borel regular), there exist a compact K and an open set U (an open set $U \in \mathcal{B}_c(T)$) such that $K \subset A \subset U$ and $\hat{\mu}_p(U\backslash K) < \varepsilon$ (and $\mu_p^c(U\backslash K) < \varepsilon$, respectively). As $K \in \mathcal{B}_c(T)$ and $\hat{\mu}_p|_{\mathcal{B}_c(T)} = \mu_p^c$ is σ-Borel regular, by the lemma in § 68 of Berberian [1] there exists $E \in \mathcal{B}_0(T)$ such that $\hat{\mu}_p(K\Delta E) = \mu_p^c(K\Delta E) = 0$. Then $\varrho_p(A,E) \leqslant \hat{\mu}_p(A\Delta K) + \hat{\mu}_p(K\Delta E) \leqslant \hat{\mu}_p(U\backslash K) < \varepsilon$ ($\varrho_p(A,E) \leqslant \mu_p^c(U\backslash K) < \varepsilon$, respectively). Hence the assertion holds.

Let $\tilde{X}$ be the completion of X. Then by Assertion 1 and by Theorem 2 of Dinculeanu and Kluvánek [3] there exists an additive set function $\hat{m}\colon \mathcal{B}(T) \to \tilde{X}$ (or $m_c\colon \mathcal{B}_c(T) \to \tilde{X}$) such that $\hat{m}|_{\mathcal{B}_0(T)} = m$ (or $m_c|_{\mathcal{B}_0(T)} = m$) and for every $p \in \Gamma$ we have

$$(1) \qquad \lim_{\hat{\mu}_p(A)\to 0} \|\hat{m}(A)\|_p = 0, \quad A \in \mathcal{B}(T)$$

$$(1') \qquad \Big(\lim_{\mu_p^c(A)\to 0} \|m_c(A)\|_p = 0, \quad A \in \mathcal{B}_c(T), \text{ respectively}\Big).$$

Moreover, given $A \in \mathcal{B}(T)$ (or $A \in \mathcal{B}_c(T)$), by Assertion 1 there exists a net $\{E_\alpha\} \subset \mathcal{B}_0(T)$ such that $E_\alpha \to A$ in Θ and hence by Lemma 2 and Theorem 2 of [3] we have

$$(2) \qquad \hat{m}(A) = \lim_\alpha m(E_\alpha)$$

$$(2') \qquad (m_c(A) = \lim_\alpha m(E_\alpha), \text{ respectively}).$$

Since m is σ-additive on $\mathcal{B}_0(T)$, m is bounded and hence there exists a τ-bounded closed set H in X such that $m(\mathcal{B}_0(T)) \subset H$. Since $(m(E_\alpha))$ is τ-Cauchy in X by (2) (or (2$'$)) and is contained in the τ-bounded closed set H, it follows from the quasicompleteness of X that $\hat{m}(A)$ (or $m_c(A)$, respectively) belongs to H. Hence the range of $\hat{m}$ (or m_c) is contained in X. Moreover, by (2) and (2$'$) we also have that $\hat{m}(A) = m_c(A)$ for $A \in \mathcal{B}_c(T)$. Thus $\hat{m}|_{\mathcal{B}_c(T)} = m_c$.

From (1) (or (1$'$)) and the fact that $\hat{\mu}_p$ (or μ_p^c) is a finite Borel (or σ-Borel) regular positive measure, it follows that $\hat{m}$ (or m_c) is a σ-additive (X-valued) regular Borel (σ-Borel, respectively) vector measure.

If $\hat{m}'$ (or m_c') is another X-valued σ-additive regular Borel (or σ-Borel) extension of m, then for each $x^* \in X^*$, $x^*\hat{m}'$ and $x^*\hat{m}$ (or x^*m_c and x^*m_c') are regular Borel (σ-Borel, respectively) complex measures extending x^*m. Then by the uniqueness part of Proposition 1 and by the Hahn-Banach theorem we conclude that $\hat{m} = \hat{m}'$ ($m_c = m_c'$, respectively). Thus the extension is unique.

This completes the proof of the theorem. □

Remark 1. The above proof is much simpler than that given by Kluvánek [8] for Banach spaces. A sophisticated operator theoretic proof of the above theorem is found in [13].

4. Proof of Theorem 5.3 of Thomas [17] by the method of Borel extension

In this section we employ the Borel extension theorem to give (see Theorem 2) a direct simple proof of Theorem 5.3 of Thomas [17] for which he employed his theory of Radon vector measures, the Grothendieck characterizations of weakly compact operators on $C(K)$, K a compact Hausdorff space (as given in Theorem 6 of [6]) and the technique of reduction to the compact metrizable case. This result was also recently obtained in Theorem 13 of [13] as an application of some deep results of the earlier sections of [13], without employing the technique of reduction to the compact metrizable case. The present proof is based just on Propositions 3 and 4 (namely, Lemma 1 and Theorem 2 of Grothendieck [6]), Proposition 6 (namely, the first part of Theorem 1 of [13]) and Theorem 1. As mentioned in Introduction, the reader can note that the present proof is much simpler than the proofs in [13], [17].

Lemma 1. *Let $u\colon C_0(T) \to X$ be a continuous linear map where X is a quasicomplete* lcHs. *Let m be the representing measure of u and let $m_0 = m|_{\mathcal{B}_0(T)}$. If the range of m_0 is contained in X, then the following assertions hold.*

(i) *m_0 is σ-additive in τ.*

(ii) *m is an X-valued σ-additive (in τ) regular Borel measure.*
(iii) $uf = \int_T f \,\mathrm{d}m, \quad f \in C_0(T)$.
(iv) *m is uniquely determined by* (ii) *and* (iii).
(v) *u is a weakly compact operator.*

Proof. As m_0 is X-valued and $x^* \circ m_0$ is σ-additive by Proposition 6 (i), it follows by the Orlicz-Pettis theorem for lcHs (see [9]) that m_0 is σ-addive in τ. Thus (i) holds.

As m_0 is an X-valued Baire measure on T and X is quasicomplete, by Theorem 1 there exists a unique X-valued σ-additive regular Borel measure $\hat{m}_0$ on T such that $\hat{m}_0|_{\mathcal{B}_0(T)} = m_0$. By Theorem 51.B of Halmos [7], each $f \in C_0(T)$ is $\mathcal{B}_0(T)$-measurable and clearly also bounded. Consequently, f is m_0-integrable in the sense of Definition 1 of [12] and

$$\int_T f \,\mathrm{d}m_0 \in X, \quad f \in C_0(T) \tag{3}$$

since m_0 is an X-valued bounded vector measure on $\mathcal{B}_0(T)$. Then by (3), Proposition 5 (i) and Proposition 6 (iii), we have

$$x^*\left(\int_T f \,\mathrm{d}m_0\right) = \int_T f \,\mathrm{d}(x^*m_0) = \int_T f \,\mathrm{d}(x^*m) = x^*uf$$

and

$$\int_T f \,\mathrm{d}(x^*m_0) = \int_T f \,\mathrm{d}(x^*\hat{m}_0)$$

for $x^* \in X^*$ and $f \in C_0(T)$. Thus the bounded linear functional x^*u on $C_0(T)$ is represented by the regular complex Borel measures x^*m and $x^*\hat{m}_0$ and consequently, by the uniqueness part of the Riesz representation theorem we conclude that $x^*m = x^*\hat{m}_0$. Since this holds for all $x^* \in X^*$, $\hat{m}_0$ is X-valued and m is X^{**}-valued, it follows that $m = \hat{m}_0$. Thus m is X-valued, σ-additive in τ and Borel regular. Hence (ii) holds.

(iii) By Proposition 6 (vi) and by (ii) above, m is a bounded X-valued vector measure as $\tau_e|_X = \tau$. Let $f \in C_0(T)$. Then f is bounded and by Theorem 51.B of [7], it is Borel measurable so that f is the uniform limit of a sequence of Borel simple functions. Hence by Lemma 7 of [12], f is m-integrable in the sense of Definition 2 of [12] and $\int_T f \,\mathrm{d}m \in X$. Then by Proposition 5(i) and by Proposition 6(iii) we have

$$x^*\left(\int_T f \,\mathrm{d}m\right) = \int_T f \,\mathrm{d}(x^*m) = x^*uf$$

for each $x^* \in X$. Then by the Hahn-Banach theorem (iii) holds.

If $\tilde{m}\colon \mathcal{B}(T) \to X$ satisfies (ii) and (iii), then x^*m and $x^*\tilde{m} \in M(T)$ and by Proposition 5 (i), they represent the bounded linear functional x^*u on $C_0(T)$. Hence $x^*m = x^*\tilde{m}$ for each $x^* \in X^*$. Then by the Hahn-Banach theorem we conclude that $m = \tilde{m}$. Thus (iv) holds.

Let (U_n) be a disjoint sequence of open sets in T and let A be an equicontinuous subset of X^*. Recall that the topology τ is the same as the topology $\tau_e|_X$ of uniform convergence on equicontinuous subsets of X^*. Thus, if $U = \bigcup_1^\infty U_n$, then (ii) implies that $\|m(U_n)\|_{p_A} \to 0$ as $n \to \infty$, where $p_A(x) = \sup_{x^* \in A} |x^*(x)|$ for $x \in X$. Then by Proposition 6 (ii) we have $\lim_n \sup_{x^* \in A} |x^* \circ m(U_n)| = \lim_n \sup_{\mu \in u^* A} |\mu(U_n)| = 0$. As $u^*(A)$ is bounded in $M(T)$ by Proposition 2, it follows by Proposition 4 that u^*A is relatively weakly compact in $M(T)$. Consequently, by Proposition 3 we conclude that u is a weakly compact operator. Thus (v) holds.

This completes the proof of the lemma. □

Theorem 2 (Theorem 5.3 of Thomas [17]). *Let $u\colon C_0(T) \to X$ be a continuous linear map and let X be a quasicomplete* lcHs *with $c_0 \not\subset X$. If m is the representing measure of u and $m_0 = m|_{\mathcal{B}_0(T)}$, then m_0 has range in X. Consequently, assertions* (i)–(v) *of Lemma* 1 *hold and in particular, u is weakly compact.*

Conversely, if X is a quasicomplete lcHs *such that each continuous linear map $u\colon C_0(T) \to X$ is weakly compact for every locally compact Hausdorff space T, then $c_0 \not\subset X$.*

In other words, a quasicomplete lcHs *X contains no copy of c_0 (or equivalently, is Σ-complete in the sense of Definition* 5.2 *of Thomas* [17] *by Theorem* 4 *of Tumarkin* [18]*) if and only if each continuous linear map $u\colon C_0(T) \to X$ is weakly compact for every locally compact Hausdorff space T.*

P r o o f. Let $c_0 \not\subset X$ and let $u\colon C_0(T) \to X$ be a continuous linear map. By Proposition 6 there exists a unique X^{**}-valued vector measure m on $\mathcal{B}(T)$ such that

$$x^*uf = \int_T f \,\mathrm{d}(x^*m) \quad f \in C_0(T) \tag{4}$$

and $x^*m \in M(T)$ for each $x^* \in X^*$.

Let $C \in \mathcal{K}_0$. Then by Theorem 55.B of Halmos [7] there exists a decreasing sequence (f_n) in $C_0(T)$ such that $f_n \searrow \chi_C$ pointwise in T. Then by (4) and by the Lebesgue dominated convergence theorem we have

$$x^*m(C) = \lim_n \int_T f_n \,\mathrm{d}(x^*m) = \lim_n x^*uf_n \tag{5}$$

for each $x^* \in X^*$.

Let $uf_n = x_n$. For $x^* \in X^*$ we have $x^*m \in M(T)$ and hence there exist finite positive measures $\mu_{x^*,j}$ on $\mathcal{B}(T)$, $j = 1, 2, 3, 4$, such that

$$x^*m = (\mu_{x^*,1} - \mu_{x^*,2}) + i(\mu_{x^*,3} - \mu_{x^*,4}).$$

Again by (4) and by the Lebesgue dominated convergence theorem we have

$$\begin{aligned}\sum_{n=1}^{\infty} |(x^*(x_n - x_{n+1})| &= \sum_{n=1}^{\infty} \left| \int_T (f_n - f_{n+1}) \, \mathrm{d}(x^*m) \right| \\ &\leqslant \sum_{j=1}^{4} \left(\sum_{n=1}^{\infty} \int_T (f_n - f_{n+1}) \, \mathrm{d}\mu_{x^*,j} \right) \\ &= \sum_{j=1}^{4} \left(\int_T f_1 \, \mathrm{d}\mu_{x^*,j} + \mu_{x^*,j}(C) \right) < \infty.\end{aligned}$$

Hence

$$|x^*(x_1)| + \sum_{n=1}^{\infty} |x^*(x_n - x_{n+1})| < \infty$$

for each $x^* \in X^*$. Since $c_0 \not\subset X$, by Theorem 4 of Tumarkin [18] the formal series $x_1 + \sum_{n=1}^{\infty} (x_{n+1} - x_n)$ converges unconditionally in the topology τ to some vector $x_0 \in X$. In other words, $\lim_n x_n = x_0$. Then by (5) we have

$$x^*(x_0) = \lim_n x^*(x_n) = \lim_n x^*uf_n = x^*m(C)$$

for each $x^* \in X^*$. Since $m(C) \in X^{**}$, it follows that $m(C) = x_0 \in X$. Thus we have proved that $m(\mathcal{K}_0) \subset X$.

Now let $\Sigma = \{E \in \mathcal{B}_0(T) \colon\ m(E) \in X\}$. As $\mathcal{K}_0$ is contained in Σ, it follows that the ring $\mathcal{R}(\mathcal{K}_0)$ generated by $\mathcal{K}_0$ is also contained in Σ. Let (E_n) be a monotone sequence in Σ with $E = \lim_n E_n$. When $E_n \nearrow$, put $F_n = E_n - E_{n-1}$ with $E_0 = \emptyset$ and $n \in \mathbb{N}$. When $E_n \searrow$, put $F_n = E_n - E_{n+1}$ for $n \in \mathbb{N}$. Clearly, $m(F_n) \in X$ for all n. Then $E = \bigcup_1^{\infty} F_n$ when $E_n \nearrow$ and $E_1 \backslash E = \bigcup_1^{\infty} F_n$ when $E_n \searrow$. Since x^*m is σ-additive on $\mathcal{B}(T)$, we have

$$x^*m(E) = \sum_1^{\infty} x^*m(F_n) \quad \text{if} \quad E_n \nearrow$$

and

$$x^*m(E_1) - x^*m(E) = \sum_1^{\infty} x^*m(F_n) \quad \text{if} \quad E_n \searrow .$$

Then in both the cases we have $\sum\limits_1^\infty |x^*m(F_n)| < \infty$ for each $x^* \in X^*$. As $c_0 \not\subset X$, then by Theorem 4 of Tumarkin [18] the formal series $\sum\limits_1^\infty m(F_n)$ is unconditionally convergent to some vector in X in the topology τ. Then it follows in both the cases that there exists a vector $w_0 \in X$ such that $\lim\limits_n m(E_n) = w_0$ (in the topology τ). Since x^*m is σ-additive and complex valued, we have

$$x^*m(E) = \lim_n x^*m(E_n) = x^*w_0$$

for all $x^* \in X^*$. As $m(E) \in X^{**}$, we conclude that $m(E) = w_0$. This shows that $E \in \Sigma$ and hence Σ is a monotone class. Now by Theorem 6.B of Halmos [7] it follows that $\Sigma = \mathcal{B}_0(T)$ and so $m(\mathcal{B}_0(T)) \subset X$. Consequently, the assertions (i)–(v) of Lemma 1 hold and thus, in particular, u is weakly compact.

To prove the converse, let ω be the set $\mathbb{N}$ endowed with the discrete topology. Then ω is a locally compact Hausdorff space. Let (x_n) be a sequence in X such that $\sum\limits_1^\infty |x^*(x_n)| < \infty$ for each $x^* \in X^*$. For each $n \in \mathbb{N}$, let $u(\chi_{\{n\}}) = x_n$ and let u be extended linearly onto the set S of all $\mathcal{P}(\mathbb{N})$-simple functions. By the hypothesis on (x_n), the set $\{uf\colon\ f \in S,\ \ \|f\|_{\mathbb{N}} \leqslant 1\}$ is weakly bounded and hence τ-bounded. Then by Theorem 1.32 of Rudin [15], u is continuous. Since X is sequentially complete and S is norm dense in $C_0(\omega)$, u has a unique continuous linear extension to the whole of $C_0(\omega)$; let us denote the extension again by u. Let m be the representing measure of u. By hypothesis, u is weakly compact and hence by Proposition 3, u^{**} has range in X so that by Proposition 6 (v) we have $m(E) = u^{**}(\chi_E) \in X$ for all $E \subset \mathbb{N}$. Then by Proposition 6 (i) and by the Orlicz-Pettis theorem for lcHs we conclude that m is σ-additive in the toplogy τ of X and hence $\sum\limits_1^\infty x_n = \sum\limits_1^\infty u(\chi_{\{n\}}) = \sum\limits_1^\infty u^{**}(\chi_{\{n\}}) = \sum\limits_1^\infty m(\{n\}) = m(\mathbb{N}) \in X$. Thus the series $\sum\limits_1^\infty x_n$ is unconditonally convergent in X. Now Theorem 4 of Tumarkin [18] implies that $c_0 \not\subset X$.

This completes the proof of the theorem. □

Remark 2. The reader can note that the proof of the first part of the above theorem is much simpler than those of Thomas [17] and Panchapagesan [13]. Moreover, the argument given in the last part is also much simpler than the corresponding one in the proof of Theorem 13 of [13].

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