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GENERALIZED INDICES OF BOOLEAN MATRICES

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Abstract. We obtain upper bounds for generalized indices of matrices in the class of nearly reducible Boolean matrices and in the class of critically reducible Boolean matrices, and prove that these bounds are the best possible.

Keywords: Boolean matrix, index of convergence, digraph

MSC 2000: 15A33, 05C20

1. Introduction

A Boolean matrix is a matrix whose entries are 0 and 1; the arithmetic underlying the matrix multiplication and addition is Boolean, that is, it is the usual integer arithmetic except that $1+1=1$. Let B_n be the set of all $n\times n$ Boolean matrices. For a matrix $A\in B_n$, the sequence of powers $A^0=I$, $A, A^2,\ldots$ is a finite subsemigroup of B_n. Thus there is a minimum nonnegative integer $k=k(A)$ such that $A^k=A^{k+t}$ for some $t\geqslant 1$, and a minimum positive integer $p=p(A)$ such that $A^k=A^{k+p}$. The integers $k=k(A)$ and $p=p(A)$ are called the index of convergence of A and the period of A, respectively.

For a matrix $A\in B_n$, the digraph $D(A)$ of A is the digraph on vertices $1,2,\ldots,n$ such that (i,j) is an arc if and only if $a_{ij}=1$. The girth of a digraph D is the length of a shortest cycle of D.

A Boolean matrix A is primitive if there is a positive integer m such that $A^m=J$, the all-ones matrix. Note that $p(A)=1$ if $A\in B_n$ is primitive.

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In 1990, Brualdi and Liu [2] introduced the generalized exponents for primitive matrices. Recently, some of the concepts in [2] have been further extended in [3] to matrices which are not necessarily primitive.

Let A be an $n \times n$ Boolean matrix and let i be an integer with $1 \leqslant i \leqslant n$. The i-th generalized index $k(A, i)$ of A is the minimum nonnegative k such that i rows of A^k and A^{k+t} are mutually equal for some $t \geqslant 1$. Clearly we have

$$k(A,1) \leqslant k(A,2) \leqslant \ldots \leqslant k(A,n) = k(A).$$

As remarked in [3], the above integer t can be chosen as $p(A)$.

Note that if A is primitive, then $k(A, i)$ is just the parameter $\exp_{D(A)}(i)$ introduced in [2].

Hence the concept of the generalized index is a generalization of both the concept of the classical index of convergence for a Boolean matrix and the concept of the generalized exponent for a primitive matrix.

A matrix $A \in B_n$ is reducible if there is a permutation matrix P such that

$$PAP^{-1} = \begin{pmatrix} A_1 & 0 \\ C & A_2 \end{pmatrix}$$

where A_1 and A_2 are square, and A is irreducible if it is not reducible.

It is well known that if a matrix $A \in B_n$ is reducible, then there is a permutation matrix P such that

$$PAP^{-1} = \begin{pmatrix} A_{11} & 0 & \ldots & 0 \\ A_{21} & A_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{t1} & A_{t2} & \ldots & A_{tt} \end{pmatrix},$$

where $A_{11}, A_{22}, \ldots, A_{tt}$ $(t \geqslant 2)$ are irreducible matrices which we call the components of A. Clearly, $D(A_{11}), \ldots, D(A_{tt})$ are the strong components of $D(A)$.

The irreducible matrix $A \in B_n$ is nearly reducible if the matrix obtained from A by replacing any non-zero entry with 0 is reducible. In particular, the 1×1 zero matrix is regarded as a nearly reducible matrix. $A \in B_n$ is critically reducible if A is reducible and all its components are nearly reducible.

In this paper, we obtain upper bounds for generalized indices of matrices in the class of nearly reducible Boolean matrices and in the class of critically reducible Boolean matrices, and prove that these bounds are the best possible.

2. Preliminaries

In this section we establish some lemmas that will be used later.

A strongly connected digraph D is called minimally strong provided no digraph obtained from D by the removal of an arc is strongly connected. We observe that $A \in B_n$ is irreducible if and only if $D(A)$ is strongly connected, while A is nearly reducible if and only if $D(A)$ is minimally strong.

For $1 \leqslant i \leqslant n$ and $n \geqslant 4$, denote

$$f(n,i) = \begin{cases} n^2 - 5n + 7 + i & \text{for } 1 \leqslant i \leqslant n-2, \\ n^2 - 5n + 6 + i & \text{for } i = n-1 \quad \text{or } n. \end{cases}$$

Lemma 1 [4]**.** *Suppose $A \in B_n$ is irreducible with period p and the girth of $D(A)$ is s. Then*

$$k(A) \leqslant n + s\Big(\frac{n}{p} - 2\Big).$$

Lemma 2 [5]**.** *Let*

$$G_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \text{ for } n \geqslant 5, \qquad \text{and } G_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then for $n \geqslant 4$ and $1 \leqslant i \leqslant n$ we have $k(G_n, i) = \exp_{D(G_n)}(i) = f(n,i)$.

Let $A \in B_n$ with $p(A) = p$. For all i, j, $k_A(i,j)$ is defined to be the minimum nonnegative integer k such that $(A^{l+p})_{ij} = (A^l)_{ij}$ for every integer $l \geqslant k$, and $m_A(i,j)$ is defined to be the minimum nonnegative integer m such that $(A^{a+mp})_{ij} = 1$ for every integer $a \geqslant 0$. It is easy to verify that $k(A) = \max\{k_A(i,j)\colon\ 1 \leqslant i,j \leqslant n\}$ and $k_A(i,j) = \max\{m_A(i,j) - p + 1, 1\}$.

Lemma 3. *Suppose $A \in B_n$ with $n \geqslant 4$ is critically reducible, and A has no component of 1×1. Then $k(A) \leqslant n^2 - 7n + 16$.*

P r o o f. Let s_0 and n_0 be respectively the maximum of all the girths and the maximum of all the orders of the strong components of $D(A)$, let f_0 be the greatest common divisor of all cycle lengths of $D(A)$. It follows from Theorem A and Corollary 2.1 in [6] that $k(A) \leqslant n + s_0(n_0/f_0 - 2)$. Note that $s_0 \leqslant n_0 \leqslant n-2$ and the components of A are all nearly reducible.

Case 1. $s_0 \leqslant n-4$. In this case $n \geqslant 6$. Hence $k(A) \leqslant n + s_0(n_0-2) \leqslant n + (n-4)(n-2-2) = n^2 - 7n + 16$.

Case 2. $s_0 = n-2$. Then $n_0 = n-2$. Corresponding to the components C and F of A, the strong components of $D(A)$ are cycles of length 2 and $n-2$, respectively. Suppose without loss of generality that $A = \begin{pmatrix} C & 0 \\ E & F \end{pmatrix}$. If $E = 0$, then $k(A) = 0$. Suppose $E \neq 0$. For $1 \leqslant i \leqslant 2$ and $3 \leqslant j \leqslant n$ we have $(A^m)_{ij} = 1$ for nonnegative integer $m \leqslant n-1$. Then $m_A(i,j) \leqslant n-1+\tilde{\varphi}(2,n-2) \leqslant n-1+(2-1)(n-2-1) = 2n-4$ by Lemma 3.2 in [7] (where for positive integers a and b with greatest common divisor p, the generalized Frobenius number $\tilde{\varphi}(a,b)$ is the least multiple $\tilde{\varphi}$ of p such that for all multiples $r \geqslant \tilde{\varphi}$ of p, r can be expressed as a nonnegative integral combination of a and b), and hence $k_A(i,j) = \max\{m_A(i,j) - p(A) + 1, 1\} \leqslant 2n-4$. It follows that

$$\begin{aligned} k(A) &= \max\{k_A(i,j)\colon\ 1 \leqslant i,j \leqslant n\} \\ &= \max\{k_A(i,j)\colon\ 1 \leqslant i \leqslant 2,\ \ 3 \leqslant j \leqslant n\} \\ &\leqslant 2n-4 \leqslant n^2 - 7n + 16. \end{aligned}$$

Case 3. $s_0 = n-3$. Then $n \geqslant 5$.

If $n = 5$, then $n_0 = 3$ and the cycle length set of $D(A)$ is $\{2\}$. Hence we have $f_0 = 2$ and $k(A) \leqslant n + s_0(n_0/f_0 - 2) \leqslant 5 + 2(3/2 - 2) = 4 \leqslant n^2 - 7n + 16$ as in Case 1.

If $n \geqslant 6$, then since A is critically reducible, we have $n_0 = n-3$. By an argument similar to that in Case 2, we have $k(A) \leqslant n - 1 + 2(n-4) \leqslant n^2 - 7n + 16$.

The proof is complete. □

3. Results

First we consider the nearly reducible matrices; the results obtained will be needed to prove an upper bound for the generalized indices for critically reducible matrices.

Suppose $A \in B_n$ where $n = 2$ or 3 is nearly reducible. If $n = 2$, then $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $k(A,1) = k(A,2) = 0$. If $n = 3$, then there is a permutation matrix P such that $PAP^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, and $k(A,i) = 0$ for $1 \leqslant i \leqslant 3$ or $k(A,1) = 0$ and $k(A,2) = k(A,3) = 1$.

Theorem 1. *Suppose $A \in B_n$ with $n \geqslant 4$ is nearly reducible. Then*

$$k(A,i) \leqslant f(n,i),$$

and this bound is the best possible.

P r o o f. Let s be the girth of $D = D(A)$.

Case 1. A is not primitive. Then $p = p(A) \geqslant 2$.

If $s = n$ or $n-1$, we have $p = n$ or $n-1$, and hence $k(A) = 0$ or $k(A) \leqslant 2$ by Lemma 1. Hence we have $k(A,i) < f(n,i)$ for all i.

If $s \leqslant n-2$, by Lemma 1 again

$$\begin{aligned} k(A,i) &\leqslant k(A,n) = k(A) \\ &\leqslant n + s\Big(\frac{n}{2} - 2\Big) \\ &\leqslant n + (n-2)\Big(\frac{n}{2} - 2\Big) \\ &= \frac{n^2 - 4n + 8}{2} < f(n,i) \ \text{ for all } \ i. \end{aligned}$$

Case 2. A is primitive. Then by Theorem 1 in [5]

$$k(A,i) = \exp_D(i) \leqslant f(n,i).$$

Combining Cases 1 and 2, we have $k(A,i) \leqslant f(n,i)$. Note that G_n is nearly reducible. By Lemma 2, this bound is the best possible. □

Now we turn to the critically reducible matrices.

Theorem 2. *Suppose $A \in B_n$ with $n \geqslant 2$ is critically reducible. Then $k(A,i) \leqslant g(n,i)$ where*

$$g(n,i) = \begin{cases} n^2 - 7n + 13 + i & \text{for } 1 \leqslant i \leqslant n-3, \\ n^2 - 7n + 12 + i & \text{for } n-2 \leqslant i \leqslant n \end{cases}$$

if $n \geqslant 5$ and $g(n,i) = i$ for $1 \leqslant i \leqslant n$ if $2 \leqslant n \leqslant 4$, and this bound is the best possible.

P r o o f. Let $f(n,i) = i$ for $1 \leqslant i \leqslant n$ and $1 \leqslant n \leqslant 3$. Then it is easy to verify that

$$g(n,i) = \begin{cases} f(n-1,i) & \text{for } 1 \leqslant i \leqslant n-1, \\ f(n-1,n-1) + 1 & \text{for } i = n, \end{cases}$$

where $n \geqslant 2$.

First we use induction on n to prove $k(A,i) \leqslant g(n,i)$ for all $n \geqslant 2$.

If $n = 2$, this can be easily verified. Suppose it is true for all critically reducible matrices of order less than n. Denote by $|X|$ the order of the square matrix X.

Claim. *$k(X, i) \leqslant f(|X|, i)$ for $2 \leqslant |X| \leqslant n-1$ if X is nearly reducible or critically reducible. This follows from Theorem 1 and the remark before it if X is nearly reducible, and from the induction hypothesis if X is critically reducible.*

The proof is now divided into the following three cases.

Case 1. A has no component of order 1. We have $n \geqslant 4$. There is a permutation matrix P such that $PAP^{-1} = \begin{pmatrix} C & 0 \\ E & F \end{pmatrix}$ where C and F are of orders at most $n-2$. By the above Claim and the definition of the generalized index, we have

$$\begin{aligned} k(A, i) &\leqslant k(C, i) \leqslant f(|C|, i) \leqslant f(n-2, i) \\ &= \begin{cases} i & \text{for } \ n = 4, 5 \\ n^2 - 9n + 21 + i & \text{for } \ n \geqslant 6 \end{cases} \\ &\leqslant g(n, i) \end{aligned}$$

for $1 \leqslant i \leqslant 2$. Suppose $3 \leqslant i \leqslant n$. By Lemma 3, $k(A, i) \leqslant n^2 - 7n + 16 \leqslant g(n, i)$.

Case 2. There is a permutation matrix P such that $PAP^{-1} = \begin{pmatrix} X & 0 \\ \alpha & 0 \end{pmatrix}$ where X is $(n-1) \times (n-1)$. Then $n \geqslant 3$ and

$$PA^lP^{-1} = \begin{pmatrix} X^l & 0 \\ \alpha X^{l-1} & 0 \end{pmatrix}.$$

Note that $k(A, i)$ is the minimum nonnegative k such that i rows of A^k and $A^{k+p(A)}$ are mutually equal. We have $k(A, i) \leqslant k(X, i) \leqslant f(|X|, i) = f(n-1, i)$ for all $1 \leqslant i \leqslant n-1$, and $k(A, n) \leqslant k(X, n-1) + 1 \leqslant f(|X|, n-1) + 1 = f(n-1, n-1) + 1$, i.e., $k(A, i) \leqslant g(n, i)$ for $1 \leqslant i \leqslant n$.

Case 3. There is a permutation matrix P such that $PAP^{-1} = \begin{pmatrix} X & \beta \\ 0 & 0 \end{pmatrix}$ where X is $(n-1) \times (n-1)$. Then $n \geqslant 3$ and

$$PA^lP^{-1} = \begin{pmatrix} X^l & X^{l-1}\beta \\ 0 & 0 \end{pmatrix}.$$

Note that the n-th row of PAP^{-1} is independent of l. We have $k(A, 1) \leqslant 1$. By the definition of $k(A, i)$, we have $k(A, i) \leqslant k(X, i-1) + 1 \leqslant f(|X|, i-1) + 1 = f(n-1, i-1) + 1 = f(n-1, i)$ for all $2 \leqslant i \leqslant n-1$ and $k(A, n) \leqslant f(n-1, n-1) + 1$, and hence $k(A, i) \leqslant g(n, i)$ for $1 \leqslant i \leqslant n$.

Combining the above three cases, we have $k(A, i) \leqslant g(n, i)$ for all $1 \leqslant i \leqslant n$.

In the sequel we show that the above bound can be attained for every i with $1 \leqslant i \leqslant n$ and $n \geqslant 2$.

For $n \geqslant 5$, let

$$A_0 = \begin{pmatrix} G_{n-1} & 0 \\ \gamma & 0 \end{pmatrix},$$

where $\gamma = (0, \dots, 0, 1, 0)$. Clearly $A_0 \in B_n$ is critically reducible, $p(A_0) = 1$, and $D(A_0)$ is just the digraph obtained by adding a new vertex n and an arc $(n, n-2)$ to $D(G_{n-1})$. Denote $m = k(G_{n-1}, n-1)$. Note that the $(n-2, 1)$-entry of G_{n-1}^{m-1} is zero, while G_{n-1}^{m} is the all-ones matrix, and hence the first entry of γG_{n-1}^{m-1} is zero, while each entry of γG_{n-1}^{m} is one. By the powers of A_0, we have $A_0^m \neq A_0^{m+1}$ and $A_0^{m+1} = A_0^{m+2}$. It follows that

$$\begin{aligned} k(A_0, i) &= \begin{cases} k(G_{n-1}, i) = f(n-1, i) & \text{for } 1 \leqslant i \leqslant n-1 \\ k(G_{n-1}, n-1) + 1 = f(n-1, n-1) + 1 & \text{for } i = n \end{cases} \\ &= g(n, i). \end{aligned}$$

For $n = 2$, let

$$A_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Clearly we have $k(A_0, i) = i = g(2, i)$ for all $1 \leqslant i \leqslant 2$.

For $n = 3$, let

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

We have $k(A_0, i) = i = g(3, i)$ for all $1 \leqslant i \leqslant 3$.

For $n = 4$, let

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Then $k(A_0, i) = i = g(n, i)$ for $1 \leqslant i \leqslant 4$.

The proof is now complete. $\square$

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