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ON UNSTABLE NEUTRAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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Abstract. The aim of this paper is to present sufficient conditions for all bounded solutions of the second order neutral differential equation

$$\big(x(t)-px(t-\tau)\big)''-q(t)x\big(\sigma(t)\big)=0$$

to be oscillatory and to improve some existing results. The main results are based on the comparison principles.

Keywords: neutral equation, delayed argument

MSC 2000: 34C10

We consider the second order neutral differential equation of the form

$$\big(x(t)-px(t-\tau)\big)''-q(t)x\big(\sigma(t)\big)=0. \tag{1}$$

In the sequel we will assume that

(i) $0<p<1$ and $\tau>0$ are constants;
(ii) q, $\sigma\in C(\mathbb{R}_+,\mathbb{R}_+)$, $\lim\limits_{t\to\infty}\sigma(t)=\infty$, $\sigma(t)<t$;
(iii) σ is nondecreasing.

We put $z(t)=x(t)-px(t-\tau)$. By a proper solution of Eq. (1) we mean a function $x\colon[T_x,\infty)\to\mathbb{R}$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)|: t\geqslant T\}>0$ for any $T\geqslant T_x$ so that $z(t)$ is twice continuously differentiable. Such a solution is called oscillatory if it has a sequence of zeros tending to infinity; otherwise it is called nonoscillatory. Eq. (1) is said to be oscillatory if all its solutions are oscillatory.

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Recently, research on the oscillation theory of functional differential equations of neutral type has been very active and fruitful and many papers devoted to differential equations and systems with neutral terms have appeared. Many good results known for differential equations without neutral terms have been extended to neutral equations. The recent books by D. D. Bainov and D. P. Mishev [1], by I. Győri and G. Ladas [4], and by L. H. Erbe, Q. Kong and B. G. Zhang [3], gather some important work in this area and reflect the overall new developments in the theory of neutral equations.

We recall the following result presented in [3, Theorem 4.6.1]:

Theorem A. *Assume that* (i)–(iii) *hold and*

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}\big(s-\sigma(t)\big)q(s)\,\mathrm{d}s>1. \tag{2}$$

Then every bounded solution of Eq. (1) *is oscillatory.*

The first objective of this paper is to present several bounded oscillation criteria for the second order neutral differential equation of unstable type. We are interested in such criteria which include the coefficient p explicitly. It is known that Eq. (1) always has an unbounded nonoscillatory solution (see e.g. [3]). Therefore we only need to find conditions for all bounded solutions of (1) to be oscillatory.

Theorem 1. *Assume that* (i)–(iii) *hold. Let there exist an integer* $n\geqslant 0$ *such that*

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}\big(s-\sigma(t)\big)q(s)\,\mathrm{d}s>\frac{1-p}{1-p^{n+1}}. \tag{3}$$

Then every bounded solution of Eq. (1) *is oscillatory.*

P r o o f. Assume the contrary and let $x(t)$ be an eventually positive bounded solution of Eq. (1). Define

$$z(t)=x(t)-px(t-\tau). \tag{4}$$

We have $z''(t)>0$ for all large t, say $t\geqslant t_0$. If $z'(t)>0$ eventually, then $\lim\limits_{t\to\infty}z(t)=\infty$, which contradicts the boundedness of x. Therefore $z'(t)<0$. There are two possibilities for $z(t)$:

(a) $z(t)>0$ for $t\geqslant t_1\geqslant t_0$,
(b) $z(t)<0$ for $t\geqslant t_1$.

In the case (a), Eq. (1) can be written in the form

$$z''(t) = q(t)x\big(\sigma(t)\big).$$

Using (4) we get

$$z''(t) = q(t)z\big(\sigma(t)\big) + pq(t)x\big(\sigma(t) - \tau\big).$$

Repeating this procedure we arrive at

$$z''(t) = q(t)\sum_{i=0}^{n} p^i z\big(\sigma(t) - i\tau\big) + p^{n+1}q(t)x\big(\sigma(t) - (n+1)\tau\big).$$

Therefore

$$z''(t) \geqslant q(t)\sum_{i=0}^{n} p^i z\big(\sigma(t) - i\tau\big).$$

For simplicity denote $\sum\limits_{i=0}^{n} p^i = k$. Then in view of the monotonicity of $z(t)$ one gets

$$z''(t) \geqslant kq(t)z(\sigma(t)). \tag{5}$$

Integration of (5) from s to t yields

$$z'(t) - z'(s) \geqslant \int_s^t kq(u)z(\sigma(u))\,\mathrm{d}u.$$

Then integrating with respect to s from $\sigma(t)$ to t we see that

$$\begin{aligned} z'(t)(t - \sigma(t)) - z(t) + z(\sigma(t)) &\geqslant \int_{\sigma(t)}^t \int_s^t kq(u)z(\sigma(u))\,\mathrm{d}u\,\mathrm{d}s \\ &\geqslant \int_{\sigma(t)}^t kq(s)\big(s - \sigma(t)\big)z(\sigma(s))\,\mathrm{d}s \\ &\geqslant z(\sigma(t))\int_{\sigma(t)}^t kq(s)\big(s - \sigma(t)\big)\,\mathrm{d}s. \end{aligned}$$

Hence for $t \geqslant t_1$ we obtain

$$0 \geqslant z'(t)(t - \sigma(t)) \geqslant z(\sigma(t))\left(k\int_{\sigma(t)}^t q(s)\big(s - \sigma(t)\big)\,\mathrm{d}s - 1\right), \tag{6}$$

which contradicts the positiveness of $z(t)$ and (3).

In the case (b) we have

$$x(t) < px(t - \tau) < p^2x(t - 2\tau) < \ldots < p^n x(t - n\tau)$$

for $t \geqslant t_1 + n\tau$ and we conclude that $\lim\limits_{t\to\infty} x(t) = 0$. Consequently, $\lim\limits_{t\to\infty} z(t) = 0$. This is a contradiction. □

The conclusion of Theorem 1 can be strengthened as follows:

Theorem 2. *Assume that* (i)–(iii) *hold. Further assume that*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} \big(s-\sigma(t)\big) q(s)\,\mathrm{d}s > 1-p. \tag{7}$$

Then every bounded solution of Eq. (1) *is oscillatory.*

P r o o f. Denote $a = \limsup\limits_{t\to\infty} \int_{\sigma(t)}^{t} \big(s-\sigma(t)\big) q(s)\,\mathrm{d}s$. Let an integer n be chosen so that

$$a > \frac{1-p}{1-p^{n+1}}.$$

Then by Theorem 1 the solutions of Eq. (1) have the claimed property. □

Remark 1. Theorem 1 improves the result of Theorem A, since there is the coefficient p included in our criterion.

Remark 2. Theorem 2 is also true for the "singular case" when $p = 0$. This result is due to Koplatadze and Čanturia [2] (see also [6, Theorem 4.3.1]).

Example 1. Consider the neutral differential equation

$$\big(x(t) - px(t-\tau)\big)'' - \frac{1}{t^2} x\big(\lambda t\big) = 0, \tag{8}$$

where $p \in (0,1)$, $\lambda \in (0,1)$ and $\tau > 0$. Condition (7) for Eq. (8) reduces to

$$\ln\Big(\frac{1}{\lambda}\Big) + \lambda \geqslant 2 - p \tag{9}$$

and so for example for $p = 1/2$ and $\lambda = 1/4$ condition (9) is fulfilled and therefore all bounded solutions of Eq. (7) are oscillatory. On the other hand, the criterion (2) fails.

The following comparison theorem is intended to cover also the case when the condition (7) is violated.

For our forthcoming consideration we need functions $a(t)$ and $\beta(t)$ satisfying

$$a \in C^1((t_0,\infty)), \quad a(t) > 0 \quad \text{and} \quad a'(t) \leqslant 0, \tag{10}$$

$$\beta \in C^1((t_0,\infty)), \quad \beta(t) < t \quad \text{and} \quad \beta'(t) > 0. \tag{11}$$

Moreover, denote by $k = \frac{1-p^{n+1}}{1-p}$, where $n \geqslant 0$ is an integer.

Theorem 3. *Assume that* (i)–(iii) *hold. Let there exist functions* $a(t)$ *and* $\beta(t)$ *satisfying* (10) *and* (11), *respectively. Assume that*

$$\beta(\beta(t)) \geqslant \sigma(t). \tag{12}$$

Further assume that there exists an integer $n \geqslant 0$ *such that*

$$kq(t) \geqslant -\frac{a'(t)}{a(t)} \frac{1}{t-\sigma(t)} \left\{ 1 - k \int_{\sigma(t)}^{t} [s-\sigma(t)] q(s) \, \mathrm{d}s \right\} + a(t) a(\beta(t)) \beta'(t). \tag{13}$$

If the first order differential inequality

$$v'(t) + a(\beta(t)) \beta'(t) v(\beta(t)) \geqslant 0 \tag{14}$$

has no eventually negative solutions, then all bounded solutions of (1) *are oscillatory.*

P r o o f. Let $x(t)$ be a positive solution of (1). Let $z(t)$ be defined by (4). Then proceeding exactly as in the case (a) of the proof of Theorem 1 we arrive at (6). The case when (3) holds is covered by Theorem 1 and so we may assume that (3) is violated. Set

$$y(t) = z'(t) - a(t) z(\beta(t)). \tag{15}$$

Then $y(t)$ is negative for sufficiently large t. Differentiation of both sides of (15) yields

$$y'(t) = z''(t) - a'(t) z(\beta(t)) - a(t) \beta'(t) z'(\beta(t)).$$

Hence

$$y'(t) + a(t) \beta'(t) y(\beta(t)) = z''(t) - a'(t) z(\beta(t)) - a(t) a(\beta(t)) \beta'(t) z(\beta(\beta(t))),$$

which together with (15), (10) and the monotonicity of z implies

$$\begin{aligned} y'(t) - \frac{a'(t)}{a(t)} y(t) &+ a(t) \beta'(t) y(\beta(t)) \\ &\geqslant z''(t) - \frac{a'(t)}{a(t)} z'(t) - a(t) a(\beta(t)) \beta'(t) z(\sigma(t)). \end{aligned} \tag{16}$$

Combining the last inequality with (6) and (13) one gets

$$y'(t) - \frac{a'(t)}{a(t)} y(t) + a(t) \beta'(t) y(\beta(t)) \geqslant z''(t) - kq(t) z(\sigma(t)) \geqslant 0.$$

Consequently, the differential inequality

$$y'(t) - \frac{a'(t)}{a(t)}y(t) + a(t)\beta'(t)y(\beta(t)) \geqslant 0 \tag{17}$$

has an eventually negative solution. Put

$$y(t) = a(t)v(t), \tag{18}$$

then (17) becomes (14). Noting that the transformation (18) preserves the existence of negative solutions we have a contradiction with the hypothesis. The case (b) can be led to contradiction exactly as in the proof of Theorem 1. □

Corollary 1. *Assume that* (i)–(iii) *are satisfied. Let* (10)–(13) *hold. If*

$$\liminf_{t\to\infty} \int_{\beta(t)}^{t} a(\beta(s))\beta'(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{19}$$

then (1) *does not allow bounded nonoscillatory solutions.*

Proof. It is known (see [6]) that (19) is sufficient for (14) to have no eventually negative solutions. The assertion of this corollary follows from Theorem 1. □

Remark 3. As a matter of fact we can use any sufficient condition for (14) to have no eventually negative solutions and Theorem 1 guarantees bounded oscillation of (1).

The following illustrative example is intended to show that Theorem 3 together with Corollary 1 extends the result of Theorem 2.

Example 2. We consider the differential equation

$$\Big(x(t) - \frac{1}{2}x(t-\tau)\Big)'' - \frac{1}{t^2}x(\lambda t) = 0. \tag{20}$$

From Example 1 we know that all bounded solutions of (20) are oscillatory provided that

$$\ln\Big(\frac{1}{\lambda}\Big) + \lambda \geqslant 1.5 \qquad (\text{i.e. } \lambda < 0.3017).$$

Theorem 3 enables us to dilate the set of values of λ for which all bounded solutions of (20) are oscillatory. Put $\beta(t) = \sqrt{\lambda}t$ and $a(t) = c/t$, where c is a positive constant which will be given later. Then (13) for $n \to \infty$ reduces to

$$2 \geqslant \frac{1}{1-\lambda}(3 + 2\ln\lambda - 2\lambda) + c^2.$$

We let $c = \sqrt{\frac{-1-2\ln\lambda}{1-\lambda}}$ (provided that $\lambda < \mathrm{e}^{0.5}$). Hence, by Corollary 1 all bounded solutions of (20) are oscillatory provided

$$-\frac{c}{2}\ln\lambda > \frac{1}{\mathrm{e}} \qquad \text{(i.e. } \lambda \leqslant 0.4711\text{)}.$$

And so, indeed, Theorem 3 (and Corollary 1) conveniently supplements Theorem 2.

In the assumptions of Theorem 3 the function $q(t)$ is required to satisfy condition (13). That means roughly speaking that function $q(t)$ should be greater than the square of a positive decreasing function $a(t)$. If $q(t)$ is greater than the square of a positive nondecreasing function then the conclusion of Theorem 3 can be reformulated as follows.

Theorem 4. *Assume that* (i)–(iii) *are satisfied. Let* (11)–(12) *hold. Assume that*

$$a \in C^1((t_0,\infty)), \quad a(t) > 0 \quad \textit{and} \quad a'(t) \geqslant 0. \tag{21}$$

Further assume that

$$\frac{1}{1-p}q(t) > a(t)a(\beta(t))\beta'(t). \tag{22}$$

If the differential inequality (14) *has no eventually negative solutions, then all nonoscillatory solutions* (1) *are unbounded.*

P r o o f. To obtain contradiction assume that (1) has an eventually positive solution $x(t)$. Let an integer $n \geqslant 0$ be chosen so that

$$\frac{1-p^{n+1}}{1-p}q(t) > a(t)a(\beta(t))\beta'(t).$$

Then proceeding exactly as in the proof of Theorem 3 we arrive at (16), which in view of (21) and (22) implies

$$\begin{aligned} y'(t) - \frac{a'(t)}{a(t)}y(t) + a(t)\beta'(t)y(\beta(t)) &\geqslant z''(t) - a(t)a(\beta(t))\beta'(t)z(\sigma(t)) \\ &\geqslant z''(t) - kq(t)z(\sigma(t)) \geqslant 0. \end{aligned}$$

Taking the transformation (18) into account, one gets that (14) has an eventually negative solution, which contradicts the hypothesis. The proof is complete. □

Corollary 2. *Assume that* (i)–(iii) *are satisfied. Let* (11), (12), (21), (22) *and* (19) *hold. Then* (1) *does not allow bounded nonoscillatory solutions.*

Corollary 3. *Assume that* (i)–(iii) *hold. If* $\sigma > 0$ *and there exists a constant* q *such that*

$$q(t) \geqslant q > (1-p)\frac{4}{\sigma^2 \mathrm{e}^2}, \tag{23}$$

then all bounded solutions of

$$\bigl(x(t) - px(t-\tau)\bigr)'' - q(t)x\bigl(t-\sigma\bigr) = 0 \tag{24}$$

are oscillatory.

Proof. Let $\varepsilon > 0$ be such that $q(t) \geqslant (1-p)\frac{(2+\varepsilon)^2}{\sigma^2 \mathrm{e}^2}$. Set $a(t) = \frac{2+\varepsilon}{\sigma\,\mathrm{e}}$ and $\beta(t) = t - \frac{\sigma}{2}$. Then the proof immediately follows from Corollary 2. □

Remark 4. In the case when $q(t) \equiv q$ is a constant, (23) is also a necessary condition for the bounded oscillation of (24).

Remark 5. The conclusion of Corollary 3 holds also for $p = 0$ and Corollary 3 improves Corollary 4.3.1 in [6].

Remark 6. The results which are known for ordinary differential equations are often extended to neutral differential equations. In this paper we have obtained a new result for neutral differential equations w ich is new even in the case when $p = 0$.

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