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ON PRODUCT MV-ALGEBRAS

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Abstract. In this paper we apply the notion of the product MV-algebra in accordance with the definition given by B. Riečan. We investigate the convex embeddability of an MV-algebra into a product MV-algebra. We found sufficient conditions under which any two direct product decompositions of a product MV-algebra have isomorphic refinements.

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In this paper we apply the notion of the product MV-algebra in accordance with the article [11]; it is defined to be an MV-algebra with a further binary operation (called product) satisfying certain axioms. The definition is recalled in Section 1 below.

Different definitions of the operation of product in MV-algebras have been used in [4] and [5]. If a binary operation satisfies the postulates from [5] then it will be called a DR-product.

If $\mathcal{A}$ is an MV-algebra, then its underlying set will be denoted by A.

Let $\mathcal{M}_1$ be the class of all MV-algebras $\mathcal{A}$ such that there exists a binary operation $\cdot$ on A having the property that the algebraic system $(\mathcal{A}, \cdot\,)$ turns out to be a product MV-algebra. Further, let $\mathcal{M}_2$ be the class of all product MV-algebras.

For $\mathcal{A} \in \mathcal{M}_1$ we denote by $P(\mathcal{A})$ the set of all binary operations op on A such that $(\mathcal{A}, \mathrm{op})$ belongs to $\mathcal{M}_2$. Put

$$\mathcal{P} = \{\operatorname{card} P(\mathcal{A})\colon\ \mathcal{A} \in \mathcal{M}_1\}.$$

In the present paper we show that $\mathcal{P}$ is a proper class. We prove that each MV-algebra can be convexly embedded into an MV-algebra which is an element of $\mathcal{M}_1$.

Let $\mathcal{A}$ be an MV-algebra and let $\cdot$ be a DR-product defined on A. We investigate some relations between the direct product decompositions of the MV-algebra $\mathcal{A}$ and the properties of the operation of the DR-product under consideration. In particular, we find sufficient conditions under which any two direct product decompositions of $(\mathcal{A}, \cdot)$ have isomorphic refinements.

1. Preliminaries

For MV-algebras several different (but equivalent) systems of axioms have been applied (cf., e.g., [1], [7], [11]).

In this paper the system from [7] will be used; cf. also the articles [8] and [9].

Hence an MV-algebra $\mathcal{A}$ is defined to be a nonempty set A with binary operations $\oplus, *$, a unary operation $\neg$ and nulary operations $0, 1$ on A such that the conditions (M_1)–(M_8) from [7] are satisfied.

For lattice ordered groups we apply the notation and the terminology from [3].

Let G be an abelian lattice ordered group with a strong unit u. Let A be the interval $[0, u]$ of G. For $a, b \in A$ we put

$$a \oplus b = (a + b) \wedge u, \quad \neg a = u - a,$$
$$1 = u, \quad a * b = \neg(\neg a \oplus \neg b).$$

Then (cf. Mundici [10]) the algebraic system $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$ is an MV-algebra. In accordance with [2] we denote this MV-algebra by $\Gamma(G, u)$. (In [8] and [9] the notation $\mathcal{A}_0(G, u)$ has been used.)

Further, for each MV-algebra $\mathcal{A}$ there exists an abelian lattice ordered group G with a strong unit u such that $\mathcal{A} = \Gamma(G, u)$. (Again, cf. Mundici [10].)

Let X be a partially ordered set. A sequence (x_n) in X is called decreasing if $x_n \geqslant x_{n+1}$ for each $n \in \mathbb{N}$. For $x \in X$, the symbol $x_n \searrow x$ has the usual meaning.

Let $\mathcal{A}$ and G be as above.

1.1. Definition (cf. [11]). Assume that a binary operation $\cdot$ is defined on the set A such that the following conditions are satisfied:

(i) $u \cdot u = u$.
(ii) The operation $\cdot$ is commutative and associative.
(iii) If $a, b \in A$ and $a + b \leqslant u$, then $c \cdot (a + b) = c \cdot a + c \cdot b$ for any $c \in A$.
(iv) If $a_n \searrow 0$ and $b_n \searrow 0$, then $a_n \cdot b_n \searrow 0$.

The MV-algebra $\mathcal{A}$ with the operation $\cdot$ is called a product MV-algebra.

Let $a, b \in A$. If $a + b \leqslant u$, then we say that $a + b$ exists in A or that $a + b$ is defined in A.

1.2. Definition (cf. [5]). A binary operation $\cdot$ on the set A will be called a DR-product if the following condition is satisfied for each $a, b, c \in A$:

Whenever $a + b$ is defined in A, then $a \cdot c + b \cdot c$ and $c \cdot a + c \cdot b$ exist in A and

$$(a + b) \cdot c = a \cdot c + b \cdot c,$$
$$c \cdot (a + b) = c \cdot a + c \cdot b.$$

1.3. Definition. Let $\mathcal{A}$ be an MV-algebra. Assume that a binary operation $\cdot$ is defined on the set A such that

a) the conditions (i), (ii) and (iv) from 1.1 are satisfied;

b) if $a, b \in A$ and $a + b \leqslant u$, then $c \cdot (a + b) = c \cdot a \oplus c \cdot b$ for any $c \in A$.

Under these assumptions $(\mathcal{A}, \cdot)$ is called a weak product MV-algebra.

Recall that whenever $x, y \in A$ and $x + y \leqslant u$, then $x \oplus y = x + y$. Hence each product MV-algebra is a weak product MV-algebra.

2. Examples

2.1. Example. Let R be the additive group of all reals with the natural linear order and let $u = 1$. Put $\mathcal{A} = \Gamma(R, u)$. Let $\cdot$ be the usual multiplication of reals. Then $(\mathcal{A}, \cdot)$ is a product MV-algebra.

2.2. Example. Let $\mathcal{A}$ be a finite MV-algebra with $\operatorname{card} A \geqslant 2$. For $x, y \in A$ the product $x \cdot y$ is defined as follows:

$$x \cdot y = \begin{cases} 0 & \text{if either } x = 0 \text{ or } y = 0, \\ u & \text{otherwise.} \end{cases}$$

Then $(\mathcal{A}, \cdot)$ is a weak product MV-algebra.

The following example shows that there exist infinite weak product MV-algebras with the operation $\cdot$ defined as in 2.2.

2.3. Example. Let Z be the additive group of all integers with the natural linear order and $G_1 = G_2 = Z$. Consider the lexicographic product $G = G_1 \circ G_2$ (we apply the notation as in Fuchs [6]). Denote $u = (1, 0)$, $\overline{0} = (0, 0)$ and let $\mathcal{A} = \Gamma(G, u)$. Then A is the interval $[\overline{0}, u]$ of G. If (x_n) is a sequence in A with $x_n \searrow \overline{0}$, then there is a positive integer m such that $x_n = \overline{0}$ whenever $n \geqslant m$. Let us define the product $x \cdot y$ analogously as in 2.2. Then $\mathcal{A}$ is an infinite weak product MV-algebra.

For an MV-algebra $\mathcal{A}$ we denote by $\ell(\mathcal{A})$ the underlying lattice. We remark that the validity of the condition (iv) from 1.1 in the previous example is due to the fact that the linearly ordered set $\ell(\mathcal{A})$ has an atom.

The following example shows that an analogous situation can occur also in the case when $\ell(\mathcal{A})$ has no atom.

Let J be a linearly ordered set such that

(i) J has the greatest element j_0;

(ii) if (j_n) is a decreasing sequence in J, then there exists $\overline{j} \in J$ such that $\overline{j} < j_n$ for each $n \in \mathbb{N}$.

There exists a proper class of linearly ordered sets satisfying (i) and (ii); in fact, for each cardinal α there exists J with the above properties such that $\operatorname{card} J \geqslant \alpha$.

2.4. Example. Let J be as above and for each $j \in J$ let $G_j = R$. If $r_1, r_2 \in R$, then the multiplication $r_1 r_2$ has the usual meaning. Let

$$G = \Gamma_{j \in J} G_j$$

(the lexicographic product of linearly ordered groups G_j; cf. [6]). For $g \in G$ and $j \in J$ let g_j be the component of g in G_j.

Let $u \in G$ be such that $u_{j_0} = 1$ and $u_j = 0$ whenever $j \neq j_0$. Hence u is a strong unit of G. Consider the MV-algebra $\mathcal{A} = \Gamma(G, u)$. Then $\ell(\mathcal{A})$ is a linearly ordered set and it has no atom. Similarly as in 2.3, if $x_n \searrow 0$ in $\ell(\mathcal{A})$, then there is $m \in \mathbb{N}$ such that $x_n = 0$ for each $n \geqslant m$. Let us define the operation $\cdot$ in A analogously as in 2.2. Then $(\mathcal{A}, \cdot)$ is a weak product MV-algebra.

2.5. Example. Let G be an abelian lattice ordered group, $G \neq \{0\}$, and let Z be the additive group of all integers with the natural linear order. Put $H = Z \circ G$. For $h \in H$ let $h(Z)$ and $h(G)$ be the components of h in Z and in G, respectively. Let $u \in H$ be such that $u(Z) = 1$ and $u(G) = 0$. Then u is a strong unit of H. Consider the MV-algebra $\mathcal{A} = \Gamma(H, u)$. Denote

$$X = \{a \in A\colon\ a(Z) = 1\}, \quad Y = \{a \in A\colon\ a(Z) = 0\}.$$

Then $X \cup Z = A$ and $X \cap Y = \emptyset$.

If $x_1, x_2 \in X$, then $x_1 + x_2 > u$, hence $x_1 + x_2 \notin A$.

We define a binary operation $\cdot$ on A as follows. Let $z_1, z_2 \in A$. We put $z_1 \cdot z_2 = u$ if both z_1 and z_2 belong to X. Otherwise we set $z_1 \cdot z_2 = 0$.

It is obvious that the conditions (i), (ii) and (iv) from 1.1 are satisfied. Let us verify that (iii) holds as well.

Let $a, b, c \in A$, $a+b \leqslant u$. Hence we cannot have $a, b \in X$. If $c \in Y$, then

$$(a+b) \cdot c = 0 = a \cdot c + b \cdot c.$$

The same holds if both a and b belong to Y. In the remaining case we can suppose that $c \in X$, $a \in X$ and $b \in Y$. Thus $a+b \in X$ and $a \cdot c = u$, $b \cdot c = 0$, whence

$$(a+b) \cdot c = u = a \cdot c + b \cdot c.$$

2.6. Example. Let G, H and $\mathcal{A}$ be as in 2.5; we use also X and Y in the same sense as in 2.5.

Consider the binary operation $\cdot(1)$ on A which is defined as follows. Let $z_1, z_2 \in A$.

a) If $z_1, z_2 \in X$, then we put $z_1 \cdot (1) z_2 = (1, z_1(G) + z_2(G))$.

b) In the case $z_1 \in X$, $z_2 \in Y$ we set $z_1 \cdot (1) z_2 = z_2 \cdot (1) z_1 = z_2$.

c) For $z_1, z_2 \in Y$ we put $z_1 \cdot (1) z_2 = 0$.

Let us remark that in the case a) we have $z_1(G) \leqslant 0$, $z_2(G) \leqslant 0$, whence $(1, z_1(G) + z_2(G)) \in A$; therefore the operation $\cdot$ is correctly defined.

Then we have $u \cdot (1) u = u$. The commutativity of the operation $\cdot(1)$ is obvious. Let $z_1, z_2, z_3 \in A$.

If $z_1, z_2, z_3 \in X$, then

$$(z_1 \cdot (1) z_2) \cdot (1) z_3 = (1, z_1(G) + z_2(G) + z_3(G) = z_1 \cdot (1)(z_2 \cdot (1) z_3).$$

If at least two indices i_1, i_2 of the set $\{1, 2, 3\}$ have the property that $z_{i_1}, z_{i_2} \in Y$, then

$$(z_1 \cdot (1) z_2) \cdot (1) z_3 = 0 = z_1 \cdot (1)(z_2 \cdot (1) z_3).$$

Let i_1, i_2 and i_3 be distinct indices belonging to the set $\{1, 2, 3\}$. Suppose that $z_{i_1}, z_{i_2} \in X$ and $z_{i_3} \in Y$. Then we have

$$(z_1 \cdot (1) z_2) \cdot (1) z_3 = z_{i_3} = z_1 \cdot (1)(z_2 \cdot (1) z_3).$$

Hence the operation $\cdot(1)$ is associative.

Again, let $z_1, z_2, z_3 \in A$ and suppose that $z_1 + z_2 \leqslant u$.

First suppose that both z_1 and z_2 belong to the set Y. Then $z_1 + z_2 \in Y$. The case $z_3 \in Y$ yields

$$z_3 \cdot (1)(z_1 + z_2) = 0 = z_3 \cdot (1) z_1 + z_3 \cdot (1) z_2;$$

if $z_3 \in X$, then

$$z_3 \cdot (1)(z_1 + z_2) = z_1 + z_2 = z_3 \cdot (1) z_1 + z_3 \cdot (1) z_2.$$

The case $z_1, z_2 \in X$ cannot occur. Suppose that $z_1 \in X$ and $z_2 \in Y$. Put $z_1 + z_2 = z_4$. Hence $z_4 \in X$, $z_4(G) = z_1(G) + z_2(G)$.

If $z_3 \in Y$, then

$$z_3 \cdot (1)(z_1 + z_2) = z_3 = z_3 \cdot (1)z_1 + z_3 cdot (1)z_2;$$

in the case $z_3 \in X$ we have

$$z_3 \cdot (1)(z_1 + z_2) = (1, z_3(G)) \cdot (1)(1, z_1(G) + z_2(G)) = (1, z_1(G) + z_2(G) + z_3(G)),$$

and at the same time

$$z_3 \cdot (1)z_1 + z_3 \cdot (1)z_2 = (1, z_3(G) + z_1(G)) + (0, z_2(G)) = (1, z_3(G) + z_1(G) + z_2(G)).$$

Hence the condition (iii) from 1.1 is satisfied.

Let (z_n) be a sequence in A such that $z_n \searrow 0$. From the construction of $\mathcal{A}$ we easily obtain that there is $m \in \mathbb{N}$ such that $z_n \in Y$ for each $n \geqslant m$. This yields that the condition (iv) from 1.1 is satisfied.

Therefore $(\mathcal{A}, \cdot(1))$ is a product MV-algebra.

Since the operations $\cdot$ and $\cdot(1)$ on A are distinct, we infer

$$\operatorname{card} P(\mathcal{A}) \geqslant 2.$$

3. The class $\mathcal{P}$ and convex embeddings

The notion of the direct product of MV-algebras is defined in the usual way. Cf., e.g., [8].

Let $(\mathcal{A}_i)_{i \in I}$ be an indexed system of MV-algebras. Consider the direct product

(1) $$\mathcal{A} = \prod_{i \in I} \mathcal{A}_i.$$

Assume that for each $i \in I$ a binary operation op_i is defined on A_i such that $(\mathcal{A}_i, \mathrm{op}_i)$ is a product MV-algebra.

For $x \in A$ and $i \in I$ let x_i be the component of x in $\mathcal{A}_i$. We define a binary operation op on $\mathcal{A}$ by putting $x \,\mathrm{op}\, y = z$, where

$$z_i = x_i \,\mathrm{op}_i\, y_i \quad \text{for each } i \in I.$$

3.1. Lemma. *$(\mathcal{A}, \mathrm{op})$ is a product MV-algebra.*

P r o o f. Let (x_n) be a sequence in A. Then $x \searrow 0$ if and only if $(x_n)_i \searrow 0$ for each $i \in I$. Hence the condition (iv) from 1.1 is valid for $(\mathcal{A}, \mathrm{op})$. It is clear that the conditions (i), (ii) and (iii) from 1.1 are valid as well. □

3.2. Corollary. *The class $\mathcal{M}_1$ is closed with respect to the direct products.*

Let α be an infinite cardinal and let I be a linearly ordered set having the cardinality α. Further, let $\mathcal{A}$ be as in 2.5. Put

$$\mathcal{B}_i = \mathcal{A} \quad \text{for each } i \in I,$$
$$\mathcal{B} = \prod_{i\in I} \mathcal{B}_i.$$

Choose a fixed element $i(0) \in I$. For each $i \in I$ we define a binary operation

$$\mathrm{op}_{(i(0),i)}$$

on the set B_i as follows:

a) If $i \leqslant i(0)$ then $\mathrm{op}_{(i(0),i)}$ is the operation described in 2.5.

b) If $i > i(0)$ then $\mathrm{op}_{(i(0),i)}$ is defined as in 2.6.

Further, we define the binary operation $\mathrm{op}_{i(0)}$ on B by putting

$$x \,\mathrm{op}_{i(0)}\, y = z,$$

where

$$z_i = x_i \,\mathrm{op}_{(i(0),i)}\, y_i \quad \text{for each } i \in I.$$

3.3. Lemma. *For each $i(1) \in I$, $(\mathcal{B}, \mathrm{op}_{i(1)})$ is a product MV-algebra.*

P r o o f. This is a consequence of 3.1 (in view of 2.5 and 2.6). □

If $i(0)$ and $i(1)$ are distinct elements of I, then the operations $\mathrm{op}_{i(0)}$ and $\mathrm{op}_{i(1)}$ are distinct as well. Hence we have

$$P(\mathcal{B}) \geqslant \alpha.$$

Therefore we obtain

3.4. Theorem. *For each cardinal α there exists a cardinal β belonging to $\mathcal{P}$ such that $\beta \geqslant \alpha$.*

We conclude that $\mathcal{P}$ is a proper class.

Let G_1 be an abelian lattice ordered group with a strong unit u_1. Put $\mathcal{A}_1 = \Gamma(G_1, u_1)$. Further, let $0 < u_2 \in A_1$. The convex ℓ-subgroup of G_1 which is generated by u_2 will be denoted by G_2. Then u_2 is a strong unit in G_2. Consider the MV-algebra $\mathcal{A}_2 = \Gamma(G_2, u_2)$. The lattice $\ell(\mathcal{A}_2)$ is a convex sublattice of $\ell(\mathcal{A}_1)$. Under these assumptions $\mathcal{A}_2$ is said to be convexly embedded into the MV-algebra $\mathcal{A}_1$.

Let X be a lattice with the least element 0 and let X_1 be a sublattice of X such that

(i) $0 \in X_1$;

(ii) if $x \in X$ and $x \wedge x_1 = 0$ for each $x_1 \in X_1$, then $x = 0$.

Then X_1 will be called a dense sublattice of X (an analogous terminology is applied in the theory of lattice ordered groups).

3.5. Theorem. *Let $\mathcal{A}_1$ be an MV-algebra with* $\operatorname{card} A_1 > 1$. *Then there exists an MV-algebra $\mathcal{A}$ such that*

(i) *$\mathcal{A}_1$ is convexly embedded into $\mathcal{A}$;*

(ii) *$\ell(\mathcal{A}_1)$ is a dense sublattice of $\ell(\mathcal{A})$;*

(iii) *$\mathcal{A} \in \mathcal{M}_1$;*

(iv) $\operatorname{card} P(\mathcal{A}) \geqslant 2$.

Proof. There is an abelian lattice ordered group G with a strong unit u_1 such that $\mathcal{A}_1 = \Gamma(G, u_1)$. Then $\operatorname{card} G > 1$. Let $\mathcal{A}$ and H be as in 2.5. The convex ℓ-subgroup of H which is generated by the element u_1 is G. The lattice $\ell(\mathcal{A}_1)$ is a convex sublattice of the lattice $\ell(\mathcal{A})$. Hence (i) is valid.

Let $x \in \mathcal{A}$ and suppose that $x \wedge x_1 = 0$ for each $x_1 \in A_1$. If $x(Z) \neq 0$, then $x \wedge x_1 = x_1$ for each $x_1 \in A_1$; hence $x(Z) = 0$. Suppose that $x(G) > 0$. Since u_1 is a strong unit in G we obtain $u_1 \wedge x(G) > 0$, whence

$$(0, u_1) \wedge x > 0$$

and $(0, u_1) \in A_1$. Thus we have arrived at a contradiction. Therefore $x = 0$. Hence (ii) is satisfied.

In view of 2.5, (iii) holds. Finally, according to 2.6, the condition (iv) is valid. □

4. Direct product decompositions

We denote by $\mathcal{M}_0$ the class of all algebraic systems $(\mathcal{A}, \cdot\,)$, where $\mathcal{A}$ is an MV-algebra and $\cdot$ is a DR-product defined on the set A. Then $x \cdot 0 = 0 \cdot x = 0$ for each $x \in A$.

For elements $(\mathcal{A}_i, \cdot\,)$ $(i \in I)$ of $\mathcal{M}_0$ the direct product

$$\prod_{i \in I} (\mathcal{A}_i, \cdot\,) \tag{1}$$

is defined in the standard way (i.e., all operations are performed componentwise).

Suppose that $(\mathcal{A}, \cdot)$ belongs to $\mathcal{M}_0$ and that

$$\varphi\colon (\mathcal{A}, \cdot) \to \prod_{i\in I}(\mathcal{A}_i, \cdot) \tag{2}$$

is an epimorphism. Then we say that φ (or the relation (2)) is a direct product decomposition of $(\mathcal{A}_i, \cdot)$.

An analogous terminology will be applied also for other types of algebraic structures.

If (2) is valid, $x \in A$, $\varphi(x) = (x_i)_{i\in I}$ and $i(0) \in I$, then we denote

$$\varphi_{i(0)}(x) = x_{i(0)}. \tag{3}$$

Suppose that (1) is valid. By dropping the operation $\cdot$ we infer that

$$\varphi\colon \mathcal{A} \to \prod_{i\in I} \mathcal{A}_i \tag{4}$$

is a direct product decomposition of the MV-algebra $\mathcal{A}$.

We say that the direct product decomposition (4) of $\mathcal{A}$ is determined by the direct product decomposition (2) of $(\mathcal{A}, \cdot)$.

The question arises whether each direct product decomposition of $\mathcal{A}$ is determined by some direct product decomposition of $(\mathcal{A}, \cdot)$.

Below we show by an example that the answer is negative in general. Further, we prove

4.1. Theorem. *Let $(\mathcal{A}, \cdot) \in \mathcal{M}_0$. The following conditions are equivalent:*

(i) *Each direct product decomposition of $\mathcal{A}$ is determined by some direct product decomposition of $(\mathcal{A}, \cdot)$.*

(ii) *Whenever b and c are complementary elements of the lattice $\ell(\mathcal{A})$, then*
  a) *the interval $[0, b]$ is closed with respect to the operation $\cdot$;*
  b) *if $b_1 \in [0, b]$ and $c_1 \in [0, c]$, then $b_1 \cdot c_1 = 0$.*

It is easy to verify that the condition (i) from 4.1 is equivalent to the condition

(∗) Whenever (4) is a direct product decomposition of $\mathcal{A}$, then
  a$_1$) for each $i \in I$ we can define an operation $\cdot$ on A_i such that $(\mathcal{A}_i, \cdot) \in \mathcal{M}_0$;
  b$_1$) if $i \in I$ and $x, y \in A$, then $\varphi_i(x \cdot y) = \varphi_i(x) \cdot \varphi_i(y)$.

For proving 4.1 we need some lemmas.

4.2. Lemma. *Let $\mathcal{A}$ be an MV-algebra and let b, c be complementary elements of the lattice $\ell(\mathcal{A})$. Let the mapping*

$$\varphi\colon A \to [0, b] \times [0, c]$$

be defined by $\varphi(x) = (x \wedge b, x \wedge c)$ for each $x \in A$. Then φ is a direct product decomposition of the lattice $\ell(\mathcal{A})$.

Proof. This is an immediate consequence of the fact that the lattice $\ell(\mathcal{A})$ is distributive. □

As above, let $\mathcal{A} = \Gamma(G, u)$ and let b, c be as in 4.2. Let G_1 and G_2 be the convex ℓ-subgroups of G which are generated by the elements b and c, respectively. Then b is a strong unit in G_1 and, similarly, c is a strong unit in G_2. Denote

$$\mathcal{B} = \Gamma(G_1, b), \quad \mathcal{C} = \Gamma(G_2, c).$$

Hence $\ell(\mathcal{B}) = [0, b]$ and $\ell(\mathcal{C}) = [0, c]$ (where the intervals are taken with respect to the lattice $\ell(\mathcal{A})$).

4.3. Lemma. *Under the above assumptions and notation, we have a direct product decomposition*

$$\varphi\colon \mathcal{A} \to \mathcal{B} \times \mathcal{C}.$$

Proof. This is a consequence of 4.2 and of Theorem 3.5 in [8]. □

4.4. Lemma. *Let* $(\mathcal{A}, \cdot\,) \in \mathcal{M}_0$ *and let* (i), (ii) *be as in* 4.1. *Then* (i) $\Rightarrow$ (ii).

Proof. Let (i) be valid. Let b and c be complementary elements in $\ell(\mathcal{A})$. Consider the direct product decomposition φ from 4.3. Then (cf. $(*)$) we can define the binary operation $\cdot$ on B and on C such that $(\mathcal{B}, \cdot\,)$ and $(\mathcal{C}, \cdot\,)$ belong to $\mathcal{M}_0$; moreover, φ is a direct product decomposition of the groupoid $(A, \cdot\,)$.

a) Let $x, y \in [0, b]$. Hence $a \wedge b = x$, $x \wedge c = 0$, thus $\varphi(x) = (x, 0)$ and similarly, $\varphi(y) = (y, 0)$. Then

$$\varphi(x \cdot y) = (x \cdot y, 0)$$

(since, in view of (i), the operation $\cdot$ is performed componentwise; cf. also $(*)$). Therefore $x \cdot y$ must belong to $[0, b]$.

b) Let $b_1 \in [0, b]$, $c_1 \in [0, c]$. Then $\varphi(b_1) = (b_1, 0)$, $\varphi(c_1) = (0, c_1)$ and

$$\varphi(b_1 \cdot c_1) = (b_1, 0) \cdot (0, c_1) = 0.$$

Thus $b_1 \cdot c_1 = 0$. □

Now let us assume that $(\mathcal{A}, \cdot\,)$ is an element of $\mathcal{M}_0$ satisfying the condition (ii). Let us have a two factor direct product decomposition of $\mathcal{A}$

$$\varphi_1\colon \mathcal{A} \to \mathcal{B}_1 \times \mathcal{C}_1. \tag{5}$$

In view of the definition of the direct product decomposition, all the MV-operations are performed componentwise with respect to φ_1. It is well-known that the lattice operations $\vee$ and $\wedge$ can be expressed in terms of the operations $\oplus$, $*$ and $\neg$; hence $\vee$ and $\wedge$ are also performed componentwise.

Analogously as in (3) we denote

$$\varphi_1(x) = (x_{B_1}, x_{C_1})$$

for each $x \in A$. We obviously have $0_{B_1} = 0 = 0_{C_1}$.

We denote by b^1 and c^1 the greatest elements of $\ell(\mathcal{B}_1)$ and of $\ell(\mathcal{C}_1)$, respectively. Next, we put

$$b = \varphi_1^{-1}((b_1, 0)), \quad c = \varphi_1^{-1}((0, c_1)).$$

Then b and c are complementary elements of $\ell(\mathcal{A})$. Let $\mathcal{B}$ and $\mathcal{C}$ have the same meaning as above.

In view of (ii), the set B is closed with respect to the operation $\cdot$; let $x_1, x_2, x_3 \in [0, b]$ be such that $x_1 + x_2 \leqslant b$. Then $x_1 + x_2 \leqslant u$, whence $(x_1 + x_2) \cdot x_3 = x_1 \cdot x_3 + x_2 \cdot x_3$. Therefore the algebraic system $(\mathcal{B}, \cdot)$ belongs to $\mathcal{M}_0$. Analogously $(\mathcal{C}, \cdot) \in \mathcal{M}_0$.

The mapping $t \to t_{B_1}$ (where t runs over the set B) is an isomorphism of $\mathcal{B}$ onto $\mathcal{B}_1$. Similarly, the mapping $z \to z_{C_1}$ (with z running over C) is an isomorphism of $\mathcal{C}$ onto $\mathcal{C}_1$.

Let b_1' and b_2' belong to B_1. There are uniquely determined elements b_1 and b_2 in B such that

$$(b_i)_{B_1} = b_i' \quad (i = 1, 2).$$

Put $b_1 \cdot b_2 = b_3$. Then $b_3 \in B$. We define the operation $\cdot$ on B_1 by setting

$$b_1' \cdot b_2' = (b_3)_{B_1}. \tag{6}$$

Analogously we define the operation $\cdot$ on the set C_1.

Under this definition we have

$$(\mathcal{B}_1, \cdot) \in \mathcal{M}_0, \quad (\mathcal{C}_1, \cdot) \in \mathcal{M}_0.$$

We define the operation $\cdot$ on the set $B_1 \times C_1$ componentwise; then $(\mathcal{B}_1 \times \mathcal{C}_1, \cdot) \in \mathcal{M}_0$.

Further, whenever b_1 and b_2 are elements of B, then in view of (6) we have

$$\varphi_1(b_1 \cdot b_2) = \varphi(b_1) \cdot \varphi(b_2). \tag{6'}$$

For $x \in A$ we put

$$x_1 = x \wedge b, \quad x_2 = x \wedge c.$$

Then we have

$$(x_1)_{B_1} = x_{B_1} \wedge b^1 = x_{B_1},$$
$$(x_1)_{C_1} = x_{C_1} \wedge 0 = 0.$$

Similarly,

$$(x_2)_{B_1} = 0, \quad (x_2)_{C_1} = x_{C_1}.$$

Further, $x_1 \in B$, $x_2 \in C$ and $x_1 \wedge x_2 = 0$, $x_1 \vee x_2 = x$. Thus $x_1 \vee x_2 = x_1 + x_2 = x$. For $y \in A$ we apply analogous notation. Hence

$$x \cdot y = (x_1 + x_2) \cdot (y_1 + y_2) = x_1 \cdot y_1 + x_1 \cdot {}_2 + x_2 \cdot y_1 + x_2 \cdot y_2.$$

In virtue of the condition b) in (ii) we get

$$x_1 \cdot y_2 = 0 = x_2 \cdot y_1,$$

whence

$$x \cdot y = x_1 \cdot y_1 + x_2 \cdot y_2,$$
$$\varphi_1(x \cdot y) = \varphi_1(x_1 \cdot y_1) + \varphi_1(x_2 \cdot y_2).$$

In view of (6) and (6′) we obtain

$$\varphi_1(x_1 \cdot y_1) = \varphi_1(x_1) \cdot \varphi_1(y_1) = ((x_1)_{B_1}, 0) \cdot ((y_1)_{B_1}, 0)$$
$$= ((x_1)_{B_1} \cdot (y_1)_{B_1}, 0) = (x_{B_1} \cdot y_{B_1}, 0).$$

Analogously,

$$\varphi_1(x_2 \cdot y_2) = (0, (x_2)_{C_1} \cdot (y_2)_{C_1}) = (0, x_{C_1} \cdot y_{C_1}).$$

Therefore

$$\varphi_1(x \cdot y) = (x_{B_1} \cdot y_{B_1}, x_{C_1} \cdot y_{C_1}).$$

Hence the operation $\cdot$ is performed componentwise with respect to the mapping φ_1. Thus we have

4.5. Lemma. *Let* $(\mathcal{A}, \cdot) \in \mathcal{M}_0$ *and suppose that the condition* (ii) *from* 4.1 *is satisfied. Let* (5) *be valid. Then we can define the binary operation* $\cdot$ *on* B_1 *and on* C_1 *such that*

(i) $(\mathcal{B}_1, \cdot)$ *and* $(\mathcal{C}_1, \cdot)$ *belong to* $\mathcal{M}_0$;

(ii) *the direct product decomposition*

$$\varphi_1\colon\ (\mathcal{A},\ \cdot\ ) \to (\mathcal{B}_1,\ \cdot\ ) \times (\mathcal{C}_1,\ \cdot\ ) \tag{7}$$

is valid.

Again, let $(\mathcal{A}, \cdot) \in \mathcal{M}_0$ and assume that the condition (ii) from 4.1 holds. Assume that

$$\varphi_2\colon\ \mathcal{A} \to \prod_{i\in I} \mathcal{A}_i \tag{5'}$$

is a direct product decomposition of $\mathcal{A}$. Let $i(0)$ be a fixed element of I. The case $I = \{i(0)\}$ being trivial we can suppose that the set $J = I \setminus \{i(0)\}$ is nonempty. Put

$$\mathcal{A}'_{i(1)} = \prod_{j\in J} \mathcal{A}_j.$$

In view of (5′) there exists a direct product decomposition

$$\varphi_{2i(0)}\colon\ \mathcal{A} \to \mathcal{A}_{i(0)} \times \mathcal{A}'_{i(0)} \tag{5''}$$

such that, for each $s \in A$, the component of x in $\mathcal{A}_{i(0)}$ with respect to (5′) is the same as the component of x in $\mathcal{A}_{i(0)}$ with respect to (5″).

Now we can apply Lemma 4.5 to the direct product decomposition (5″). In view of $(*)$ we obtain that the condition (i) from 4.1 is valid for $(\mathcal{A}, \cdot)$. Thus we have

4.6. Lemma. *Let* $(\mathcal{A}, \cdot) \in \mathcal{M}_0$ *and let* (i), (ii) *be as in* 4.1. *Then* (ii) $\Rightarrow$ (i).

From 4.4 and 4.6 we conclude that Theorem 4.1 is valid.

4.7. Example. Let $G_1 = G_2 = R$, $G = G_1 \times G_2$. For $g \in G$ we denote by g_i the component of g in G_i, $(i = 1, 2)$. Let $u = (1, 1)$, $\mathcal{A} = \Gamma(G, u)$. If $z_1, z_2 \in R$, then $z_1 z_2$ denotes the usual multiplication in R.

Let $x, y \in A$. Denote

$$t = \frac{1}{4}(x_1 y_1 + x_2 y_2).$$

Put $x \cdot y = (t, t)$. Then $(\mathcal{A}, \cdot)$ is an element of $\mathcal{M}_0$ which fails to satisfy the condition (ii) from 4.1. The MV-algebra $\mathcal{A}$ is directly decomposable, but the algebraic system $(\mathcal{A}, \cdot)$ is directly indecomposable.

4.8. Theorem. *Let* $(\mathcal{A}, \cdot) \in \mathcal{M}_0$ *and suppose that the condition* (ii) *from* 4.1 *is satisfied. Then any two direct product decompositions of* $(\mathcal{A}, \cdot)$ *have isomorphic refinements.*

P r o o f. This is a consequence of 4.1 and of [8], Corollary 3.6. □

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