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Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 4, 851–860

Persistent URL: <http://dml.cz/dmlcz/127769>

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BERNSTEIN-TYPE OPERATORS ON THE HALF LINE

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(Received December 28, 1999)

Abstract. We define Bernstein-type operators on the half line $[0, +\infty[$ by means of two sequences of strictly positive real numbers. After studying their approximation properties, we also establish a Voronovskaja-type result with respect to a suitable weighted norm.

Keywords: Bernstein-Chlodovsky operators, approximation process, Voronovskaja-type formula

MSC 2000: 41A10, 41A36

1. INTRODUCTION AND NOTATION

In [4] Chlodovsky introduced and studied a sequence of positive linear operators $(C_n^*)_{n \geq 1}$ on the space $C([0, +\infty[)$ of all real valued continuous functions on the half line $[0, +\infty[$, defined by

$$(1.1) \quad C_n^* f(x) := \begin{cases} \sum_{k=0}^n f\left(\frac{b_n k}{n}\right) \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} & \text{if } 0 \leq x \leq b_n, \\ f(x) & \text{if } x > b_n, \end{cases}$$

where $(b_n)_{n \geq 1}$ is a divergent sequence of strictly positive real numbers.

Roughly speaking, the above operators, known as Bernstein-Chlodovsky operators, behave basically like the classical Bernstein ones on $[0, b_n]$, interpolating, in the meanwhile, the function f elsewhere.

A deeper analysis of their approximation properties was subsequently carried out in [8], [10] with respect to functions belonging to particular subspaces of $C([0, +\infty[)$. In this framework and without the assumption of completeness, it seems also useful to refer the reader to [6], [7], [9], [11] for general results concerning the approximation

of continuous functions on unbounded intervals and for some interesting extensions of the classical Korovkin's Theorem.

The purpose of this paper is to consider a generalization of Bernstein-Chlodovsky operators (1.1) by using two sequences $(b_n)_{n \geq 1}$ and $(c_n)_{n \geq 1}$ of strictly positive real numbers, satisfying particular assumptions.

As a consequence, our definition (2.1) actually turns out to be more flexible than (1.1), allowing to state, beyond classical approximation results, a Voronovskaja-type formula, which, as far as we know, cannot be stated for the classical C_n^* .

The corresponding differential operator is a rather general second-order one degenerating at the boundaries with coefficients depending on the sequences $(b_n)_{n \geq 1}$ and $(c_n)_{n \geq 1}$, and may be readily shown to be the generator of a strongly continuous positive contraction semigroup, due to some classical results stated in [5] and [12].

It would be perhaps interesting, falling, actually, within a wide program of investigation which has been inspiring the authors and other researchers in the last years, to prove that such a semigroup may be represented in terms of powers of the operators C_n as an application of the classical Trotter representation theorem [13] (see, also, [1], Proposition 1.6.7, p. 67): this may virtually justify further analysis in the concern.

As for the notation, throughout the paper, besides $C([0, +\infty[)$, we will sometimes deal with the subspace $UC_b([0, +\infty[)$ of all bounded uniformly continuous functions on $[0, +\infty[$ which is a Banach lattice, if endowed with the sup- norm $\|\cdot\|$.

For every $\alpha > 0$ we will be mainly concerned with the weighted space

$$(1.2) \quad E_\alpha^0 := \{f \in C([0, +\infty[) \mid \exists \lim_{x \rightarrow +\infty} \frac{f(x)}{1 + x^\alpha} = 0\},$$

which becomes a Banach lattice with respect to the norm

$$(1.3) \quad \|f\|_\alpha := \sup_{x \geq 0} \frac{|f(x)|}{1 + x^\alpha}.$$

Such spaces have been already considered in [2], [3] in which a worthy generalization of the classical Baskakov operators is studied.

As usual, for every integer $m \geq 1$, $C^m([0, +\infty[)$ is the vector space of all real valued m -times continuously differentiable functions on $[0, +\infty[$. For every $p \geq 0$, e_p is the test function defined by $e_p(x) := x^p$ ($x \geq 0$), whereas, for each $x \geq 0$, ψ_x is the function defined by $\psi_x(t) := t - x$ ($t \geq 0$).

The symbol $\omega(\cdot, \cdot)$ will denote the classical modulus of continuity, as usual.

2. THE OPERATORS C_n

Let us consider two sequences $(b_n)_{n \geq 1}$ and $(c_n)_{n \geq 1}$ of strictly positive real numbers satisfying the following assumptions:

- 1) $b_n \rightarrow +\infty$ as $n \rightarrow +\infty$;
- 2) $b_n/n \rightarrow 0$, $b_n - c_n \rightarrow 0$ as $n \rightarrow +\infty$;
- 3) $b_n \leq c_n$ for every $n \geq 1$.

It immediately follows that, correspondingly, $c_n \rightarrow +\infty$ and $c_n/n \rightarrow 0$ as well, and, in addition, $b_n \approx c_n$ as $n \rightarrow +\infty$.

For every $n \geq 1$ and for every $f \in E_\alpha^0$ we set

$$(2.1) \quad C_n f(x) := \begin{cases} \sum_{k=0}^n f\left(\frac{c_n k}{n}\right) \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} & \text{if } 0 \leq x \leq b_n, \\ f(c_n) & \text{if } b_n < x \leq c_n, \\ f(x) & \text{if } x > c_n. \end{cases}$$

Since $C_n(f) = f$ in $[c_n, +\infty[$ by definition, we may refer to C_n as to a positive linear operator acting from E_α^0 into itself. Moreover, $C_n(e_0) = e_0$ and therefore $\|C_n\| = \|C_n(e_0)\| = 1$; in addition, a very simple computation shows that

$$(2.2) \quad C_n e_1(x) = \begin{cases} \frac{c_n}{b_n} x & \text{if } 0 \leq x \leq b_n, \\ c_n & \text{if } b_n < x \leq c_n, \\ x & \text{if } x > c_n, \end{cases}$$

$$(2.3) \quad C_n e_2(x) = \begin{cases} \frac{c_n^2}{b_n^2} x^2 + \frac{c_n^2}{nb_n^2} x(b_n - x) & \text{if } 0 \leq x \leq b_n, \\ c_n^2 & \text{if } b_n < x \leq c_n, \\ x^2 & \text{if } x > c_n, \end{cases}$$

$$(2.4) \quad C_n \psi_x(x) = \begin{cases} \left(\frac{c_n}{b_n} - 1\right) x & \text{if } 0 \leq x \leq b_n, \\ c_n - x & \text{if } b_n < x \leq c_n, \\ 0 & \text{if } x > c_n, \end{cases}$$

and

$$(2.5) \quad C_n \psi_x^2(x) = \begin{cases} \left(\frac{c_n}{b_n} - 1\right)^2 x^2 + \frac{c_n^2}{nb_n^2} x(b_n - x) & \text{if } 0 \leq x \leq b_n, \\ (c_n - x)^2 & \text{if } b_n < x \leq c_n, \\ 0 & \text{if } x > c_n. \end{cases}$$

An approximation result is indicated below.

Theorem 2.1. For every $f \in E_\alpha^0$ ($\alpha > 2$) we have

$$(2.6) \quad \lim_{n \rightarrow +\infty} \|C_n(f) - f\|_\alpha = 0,$$

i.e., the sequence $(C_n)_{n \geq 1}$ is a positive approximation process.

More precisely, for n large enough we have

$$(2.7) \quad \|C_n(f) - f\|_\alpha \leq 2\omega \left(f, \sqrt{(c_n - b_n)^2 + \frac{c_n^2}{nb_n}} \right).$$

Proof. Indeed, let us fix $n \geq 1$. On account of (2.2) and (2.3), we get

$$\begin{aligned} \frac{|C_n e_1(x) - x|}{1 + x^\alpha} &\leq \begin{cases} \left(\frac{c_n}{b_n} - 1 \right) & \text{if } 0 \leq x \leq b_n, \\ \frac{c_n - b_n}{1 + b_n^\alpha} & \text{if } b_n < x \leq c_n, \end{cases} \\ \frac{|C_n e_2(x) - x^2|}{1 + x^\alpha} &\leq \begin{cases} \left(\frac{c_n^2}{b_n^2} - 1 \right) + \frac{c_n^2}{nb_n} & \text{if } 0 \leq x \leq b_n, \\ \frac{2c_n(c_n - b_n)}{1 + b_n^\alpha} & \text{if } b_n < x \leq c_n. \end{cases} \end{aligned}$$

Now observe that each member on the right-hand side in the above estimates tends to 0 as $n \rightarrow +\infty$, as a consequence of the assumptions on the sequences $(b_n)_{n \geq 1}$ and $(c_n)_{n \geq 1}$. Moreover, by definition, $|C_n e_i(x) - e_i(x)| = 0$ whenever $x \in [c_n, +\infty[$ and therefore, since obviously $C_n(e_0) = e_0$, we have just shown that

$$\lim_{n \rightarrow +\infty} \|C_n(e_i) - e_i\|_\alpha = 0 \quad \text{for } i = 0, 1, 2,$$

which implies (2.6) on account of Korovkin's theorem (see, e.g., [1], Proposition 4.2.5, p. 215).

In order to establish (2.7), let us first note that by virtue of [1], Proposition 5.1.2, p. 268, a pointwise estimate

$$\begin{aligned} |C_n f(x) - f(x)| &\leq 2\omega(f, \sqrt{C_n \psi_x^2(x)}) \\ &= \begin{cases} 2\omega \left(f, \sqrt{\left(\frac{c_n}{b_n} - 1 \right)^2 x^2 + \frac{c_n^2}{nb_n^2} x(b_n - x)} \right) & \text{if } 0 \leq x \leq b_n, \\ 2\omega(f, c_n - x) & \text{if } b_n < x \leq c_n, \end{cases} \end{aligned}$$

holds true for any $f \in E_\alpha^0$. The uniform estimate (2.7) now immediately follows, since a straightforward computation yields for n large enough

$$\sup_{0 \leq x \leq c_n} \frac{\sqrt{C_n \psi_x^2(x)}}{1+x^\alpha} \leq \sqrt{(c_n - b_n)^2 + \frac{c_n^2}{nb_n}}.$$

□

The following two lemmas will be very useful in the sequel.

Lemma 2.2. *Let $(\varrho_n)_{n \geq 1}$ be a divergent sequence of strictly positive real numbers such that*

$$(2.8) \quad \varrho_n \frac{c_n}{n} \rightarrow 2a \quad \text{and} \quad \varrho_n \left(\frac{c_n}{b_n} - 1 \right) \rightarrow b \quad \text{as} \quad n \rightarrow +\infty,$$

where $a > 0$ $b \geq 0$. Then, if $\alpha \geq 4$, we have

$$(2.9) \quad \lim_{n \rightarrow +\infty} \frac{\varrho_n}{1+x^\alpha} \sum_{k=0}^n \binom{n}{k} \left(\frac{x}{b_n} \right)^k \left(1 - \frac{x}{b_n} \right)^{n-k} \left(c_n \frac{k}{n} - x \right)^4 = 0$$

uniformly on $[0, +\infty[$.

Proof. For any $n \geq 1$ and $x \geq 0$ a direct computation shows that

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} \left(\frac{x}{b_n} \right)^k \left(1 - \frac{x}{b_n} \right)^{n-k} \left(c_n \frac{k}{n} - x \right)^4 \\ &= x^4 \left[1 - 4 \frac{c_n}{b_n} + 6 \frac{c_n^2}{b_n^2} - 6 \frac{c_n^2}{nb_n^2} - 4 \frac{c_n^3(n-1)(n-2)}{n^2 b_n^3} + \frac{c_n^4(n-1)^4(n-2)}{n^5 b_n^4} \right] \\ &+ x^3 \left[6 \frac{c_n^2}{nb_n} - 12 \frac{c_n^3(n-1)}{n^2 b_n^2} + 3 \frac{c_n^4(n-1)^2}{n^3 b_n^3} - 3 \frac{c_n^4(n-1)}{n^3 b_n^3} + 3 \frac{c_n^4(n-1)^4}{n^5 b_n^3} \right] \\ &+ x^2 \left[-4 \frac{c_n^3}{n^2 b_n} + 6 \frac{c_n^4(n-1)}{n^3 b_n^2} + \frac{c_n^4(n-1)^3}{n^5 b_n^2} \right] + x \frac{c_n^4}{n^3 b_n}. \end{aligned}$$

Let us denote by $\alpha_n, \beta_n, \gamma_n, \delta_n$ the coefficients of the powers x^4, x^3, x^2 and x , respectively, in the above equality; since $\alpha \geq 4$ by assumption, for any $n \geq 1$ and $x \geq 0$ we have

$$\begin{aligned} & \frac{\varrho_n}{1+x^\alpha} \left| \sum_{k=0}^n \binom{n}{k} \left(\frac{x}{b_n} \right)^k \left(1 - \frac{x}{b_n} \right)^{n-k} \left(c_n \frac{k}{n} - x \right)^4 \right| \\ & \leq \frac{x^4}{1+x^\alpha} |\varrho_n \alpha_n| + \frac{x^3}{1+x^\alpha} |\varrho_n \beta_n| + \frac{x^2}{1+x^\alpha} |\varrho_n \gamma_n| + \frac{x}{1+x^\alpha} |\varrho_n \delta_n| \\ & \leq |\varrho_n \alpha_n| + |\varrho_n \beta_n| + |\varrho_n \gamma_n| + |\varrho_n \delta_n|. \end{aligned}$$

Now the assertion easily follows, because all sequences in the above last term tend to 0 as $n \rightarrow +\infty$ on account of (2.8) and the conditions on $(b_n)_{n \geq 1}$ and $(c_n)_{n \geq 1}$ stated just before the definition (2.1). \square

Lemma 2.3. *Under the assumptions (2.8), if $\alpha > 2$, we have*

$$(2.10) \quad \lim_{n \rightarrow +\infty} \|\varrho_n C_n(\psi_x) - be_1\|_{\alpha-1} = \lim_{n \rightarrow +\infty} \|\varrho_n C_n(\psi_x^2) - 2ae_1\|_\alpha = 0.$$

Proof. Let us choose $n \geq 1$; then, on account of (2.4) and (2.5), we get the estimates

$$\frac{|\varrho_n C_n \psi_x(x) - bx|}{1 + x^{\alpha-1}} \leq \begin{cases} \left| \varrho_n \left(\frac{c_n}{b_n} - 1 \right) - b \right| & \text{if } 0 \leq x \leq b_n, \\ \frac{\varrho_n(c_n - b_n) + bc_n}{1 + b_n^{\alpha-1}} & \text{if } b_n < x \leq c_n, \end{cases}$$

$$\frac{|\varrho_n C_n \psi_x^2(x) - 2ax|}{1 + x^\alpha} \leq \begin{cases} \varrho_n \left(\frac{c_n}{b_n} - 1 \right)^2 + \frac{\varrho_n c_n^2}{nb_n^2} + \left| \frac{\varrho_n c_n^2}{nb_n} - 2a \right| & \text{if } 0 \leq x \leq b_n, \\ \frac{\varrho_n(c_n - b_n)^2 + 2ac_n}{1 + b_n^\alpha} & \text{if } b_n < x \leq c_n, \end{cases}$$

and all terms on the right-hand sides tend to 0 as $n \rightarrow +\infty$. Now, in order to find out an estimate for $x > c_n$, let us first observe that the function $g(x) := x/(1+x^{\alpha-1})$ ($x \geq 0$) attains its maximum at a point, say x_0 , in $]0, +\infty[$. Of course there exists $k \in \mathbb{N}$ such that $c_n > x_0$ for any $n \geq k$ and g is strictly decreasing in $[c_n, +\infty[$. It immediately follows that for $n \geq k$ and $x \in [c_n, +\infty[$

$$\frac{|\varrho_n C_n \psi_x(x) - bx|}{1 + x^{\alpha-1}} = bg(x) \leq \frac{bc_n}{1 + c_n^{\alpha-1}},$$

where again the term on the right-hand side tends to 0 as $n \rightarrow +\infty$. Arguing similarly for $C_n \psi_x^2(x)$ gives (2.10). \square

Now we are ready to prove our main result, which states a Voronovskaja-type formula for the operators C_n .

Theorem 2.4. *For any $f \in C^2([0, +\infty[) \cap E_\alpha^0$ ($\alpha \geq 4$) such that $f'' \in UC_b([0, +\infty[)$ we have*

$$(2.11) \quad \lim_{n \rightarrow +\infty} \varrho_n(C_n f(x) - f(x)) = axf''(x) + bxf'(x) \quad \text{in } E_\alpha^0,$$

$(\varrho_n)_{n \geq 1}$, a and b being the same as those appearing in Lemma 2.2.

Proof. First of all, let us note that if $f \in C^2([0, +\infty[) \cap E_\alpha^0$ with $f'' \in UC_b([0, +\infty[)$, because of the identity

$$(1) \quad f'(x) = f'(0) + \int_0^x f''(s) \, ds \quad (x \geq 0),$$

for a suitable constant $K > 0$ one has

$$(2) \quad \frac{|f'(x)|}{1+x} \leq K \quad (x \geq 0).$$

Moreover, if $|f''(x)| \leq M$ for every $x \geq 0$, then obviously

$$(3) \quad |f'(x) - f'(y)| \leq M|x - y| \quad (x, y \geq 0).$$

We will show that (2.11) holds true on each of the intervals $[0, b_n]$, $]b_n, c_n]$, and $]c_n, +\infty[$, as suggested by the definition of our operators C_n .

To start with, fix $n \geq 1$ and note that if $x \in [0, b_n]$, by virtue of Taylor's formula, for any $k = 0, 1, \dots, n$ there exists $d_{n,k,x}$ lying between x and $c_n k/n$ such that

$$\begin{aligned} f\left(\frac{c_n k}{n}\right) - f(x) &= f'(x)\left(c_n \frac{k}{n} - x\right) + \frac{f''(x)}{2}\left(c_n \frac{k}{n} - x\right)^2 \\ &\quad + \frac{f''(d_{n,k,x}) - f''(x)}{2}\left(c_n \frac{k}{n} - x\right)^2. \end{aligned}$$

After setting

$$(4) \quad \mu\left(x, \frac{c_n k}{n}\right) := \frac{f''(d_{n,k,x}) - f''(x)}{2},$$

we may therefore write

$$\varrho_n(C_n f(x) - f(x)) = \varrho_n f'(x) C_n \psi_x(x) + \frac{1}{2} \varrho_n f''(x) C_n \psi_x^2(x) + \varrho_n R_n(x),$$

where

$$R_n(x) = \sum_{k=0}^n \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} \mu\left(x, \frac{c_n k}{n}\right) \left(c_n \frac{k}{n} - x\right)^2.$$

It follows that

$$\begin{aligned} &\frac{1}{1+x^\alpha} |\varrho_n(C_n f(x) - f(x)) - ax f''(x) - bx f'(x)| \\ &\leq \frac{1}{1+x^\alpha} \left| \varrho_n \frac{1}{2} f''(x) C_n \psi_x^2(x) - ax f''(x) \right| \\ &\quad + \frac{1}{1+x^\alpha} |\varrho_n f'(x) C_n \psi_x(x) - bx f'(x)| + \frac{\varrho_n}{1+x^\alpha} |R_n(x)|, \end{aligned}$$

where the first two members on the right-hand side tend to 0 uniformly: simply apply Lemma 2.3, taking also into account that f'' is bounded by assumption and that

$$\begin{aligned} \frac{1}{1+x^\alpha} |\varrho_n f'(x) C_n \psi_x(x) - b x f'(x)| &\leq N \frac{|f'(x)|}{1+x} \|\varrho_n C_n(\psi_x) - b e_1\|_{\alpha-1} \\ &\leq N K \|\varrho_n C_n(\psi_x) - b e_1\|_{\alpha-1} \end{aligned}$$

by virtue of (2) (here N is a suitable positive constant).

Therefore, in order to establish (2.11) in $[0, b_n]$, it is sufficient to show that $\lim_{n \rightarrow +\infty} \varrho_n (1+x^\alpha)^{-1} |R_n(x)| = 0$ uniformly. To this aim, note that the assumptions on f together with the definition (4) ensure that $|\mu(x, t)| \leq M$ for every $(x, t) \in [0, b_n] \times [0, c_n]$ and that $\lim_{t \rightarrow x} \mu(x, t) = 0$ uniformly with respect to $x \in [0, b_n]$.

Now fix $\varepsilon > 0$ and choose $\delta > 0$ such that $|\mu(x, t)| < \varepsilon$ whenever $|x - t| < \delta$; then (2.9) and the second limit in (2.10) yield

$$\frac{\varrho_n}{1+x^\alpha} \left| \sum_{k=0}^n \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} \left(c_n \frac{k}{n} - x\right)^4 \right| < \frac{a\varepsilon\delta^2}{2M}$$

and

$$\frac{|\varrho_n C_n \psi_x^2(x) - 2ax|}{1+x^\alpha} < a/2$$

for every $x \in [0, b_n]$ if n is large enough, say $n \geq n_0$. It follows that for every $x \in [0, b_n]$ and $n \geq n_0$

$$\begin{aligned} &\frac{\varrho_n}{1+x^\alpha} |R_n(x)| \\ &\leq \frac{\varepsilon \varrho_n}{1+x^\alpha} \left| \sum_{\substack{k=0 \\ |c_n k/n - x| < \delta}}^n \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} \left(c_n \frac{k}{n} - x\right)^2 \right| \\ &\quad + \frac{\varrho_n}{1+x^\alpha} \left| \sum_{\substack{k=0 \\ |c_n k/n - x| \geq \delta}}^n \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} \mu\left(x, \frac{c_n k}{n}\right) \left(c_n \frac{k}{n} - x\right)^2 \right| \\ &\leq \frac{\varepsilon \varrho_n}{1+x^\alpha} C_n \psi_x^2(x) + \frac{\varrho_n M}{\delta^2 (1+x^\alpha)} \left| \sum_{k=0}^n \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} \left(c_n \frac{k}{n} - x\right)^4 \right| \\ &\leq \varepsilon \frac{|\varrho_n C_n \psi_x^2(x) - 2ax|}{1+x^\alpha} + \frac{2a\varepsilon x}{1+x^\alpha} + \frac{a\varepsilon}{2} \leq 3a\varepsilon. \end{aligned}$$

Therefore $\limsup_{n \rightarrow +\infty} \varrho_n (1+x^\alpha)^{-1} |R_n(x)| \leq 3a\varepsilon$ and, consequently, since ε is arbitrary, the proof is complete in this first case.

Now, if $x \in]b_n, c_n]$, since $C_n f(x) = f(c_n)$ by the definition (2.1), applying Taylor's formula together with (2.4) and (2.5) gives

$$\begin{aligned} & \frac{1}{1+x^\alpha} |\varrho_n(f(c_n) - f(x)) - axf''(x) - bxf'(x)| \\ &= \frac{1}{1+x^\alpha} \left| \varrho_n f'(c_n)(c_n - x) \right. \\ & \quad \left. - \varrho_n \frac{1}{2} f''(d_{n,x})(c_n - x)^2 - axf''(x) - bxf'(x) \right| \\ &\leq \frac{1}{1+x^\alpha} |\varrho_n C_n \psi_x(x) f'(c_n) - bxf'(x)| \\ & \quad + \frac{1}{1+x^\alpha} \left| \varrho_n C_n \psi_x^2(x) \frac{f''(d_{n,x})}{2} - axf''(x) \right| := I_1 + I_2, \end{aligned}$$

$d_{n,x}$ being a suitable point between x and c_n . Next we show that each I_i tends to 0 uniformly; indeed, on account of (3), for a suitable $N > 0$ we have

$$\begin{aligned} I_1 &\leq \frac{1}{1+x^\alpha} |\varrho_n C_n \psi_x(x) f'(c_n) - bxf'(c_n)| + \frac{1}{1+x^\alpha} |bxf'(c_n) - bxf'(x)| \\ &\leq \frac{N|f'(c_n)|}{1+b_n} \|\varrho_n C_n(\psi_x) - be_1\|_{\alpha-1} + \frac{Mbc_n(c_n - b_n)}{1+b_n^\alpha}, \end{aligned}$$

and the term on the right-hand side tends to 0 due to the first limit in (2.10) and to (2), because $|f'(c_n)|/(1+b_n) \approx |f'(c_n)|/(1+c_n)$ as $n \rightarrow +\infty$.

Similarly, since $|f''(x)| \leq M$ for every $x \geq 0$ by assumption, we get

$$\begin{aligned} I_2 &\leq \frac{1}{1+x^\alpha} \left| \varrho_n C_n \psi_x^2(x) \frac{f''(d_{n,x})}{2} - axf''(d_{n,x}) \right| + \frac{1}{1+x^\alpha} |axf''(d_{n,x}) - axf''(x)| \\ &\leq \frac{M}{2} \|\varrho_n C_n(\psi_x^2) - 2ae_1\|_\alpha + \frac{2Mac_n}{1+b_n^\alpha}, \end{aligned}$$

which easily yields $I_2 \rightarrow 0$, too, because of the second limit in (2.10).

At last, when $x > c_n$ and therefore $C_n f(x) = f(x)$ by definition, we have, for n large enough and a suitable $N > 0$ (see the last part of the proof of Lemma 2.3)

$$\frac{1}{1+x^\alpha} |axf''(x) + bxf'(x)| \leq \frac{Mac_n}{1+c_n^\alpha} + \frac{N|f'(x)|}{1+x} \cdot \frac{bc_n}{1+c_n^{\alpha-1}},$$

where again the term on the right-hand side tends to 0 because of (2).

The proof of the theorem is now complete. □

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