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ŁUKASIEWICZ TRIBES ARE ABSOLUTELY SEQUENTIALLY CLOSED BOLD ALGEBRAS

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Dedicated to the memory of my teacher Professor Josef Novák.

Abstract. We show that each sequentially continuous (with respect to the pointwise convergence) normed measure on a bold algebra of fuzzy sets (Archimedean MV-algebra) can be uniquely extended to a sequentially continuous measure on the generated Łukasiewicz tribe and, in a natural way, the extension is maximal. We prove that for normed measures on Łukasiewicz tribes monotone (sequential) continuity implies sequential continuity, hence the assumption of sequential continuity is not restrictive. This yields a characterization of the Łukasiewicz tribes as bold algebras absolutely sequentially closed with respect to the extension of probabilities. The result generalizes the relationship between fields of sets and the generated σ-fields discovered by J. Novák. We introduce the category of bold algebras and sequentially continuous homomorphisms and prove that Łukasiewicz tribes form an epireflective subcategory. The restriction to fields of sets yields the epireflective subcategory of σ-fields of sets.

Keywords: MV-algebra, bold algebra, field of sets, Łukasiewicz tribe, sequential convergence, sequential continuity, measure, extension of measures, sequential envelope, absolute sequentially closed bold algebra, epireflective subcategory

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In Section 1 we discuss continuity of measures on rings of sets. Since the usual sequential convergence in a ring of sets can be approximated by the monotone sequential convergence in the generated σ-ring, the measure extension theorem for rings of sets yields a simple proof of the fact (cf. [14]) that each bounded σ-additive measure on a ring of sets is sequentially continuous. In Section 2 we deal with sequentially continuous measures on bold algebras. Finally, in Section 3 we give a categorical

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characterization of the Łukasiewicz tribes as absolutely sequentially closed bold algebras.

1.

In measure theory and its applications, the monotone (sequential) continuity of a measure amounts to the σ-additivity. Indeed, let $\mathbb{A}$ be a ring of subsets of X and let m be an additive (finite) measure on $\mathbb{A}$. Then the following are equivalent:

(i) m is σ-additive, i.e., if $\langle A_n\rangle$ is a mutually disjoint sequence in $\mathbb{A}$ such that $\bigcup\limits_{n=1}^{\infty} A_n$ belongs to $\mathbb{A}$, then $m\Big(\bigcup\limits_{n=1}^{\infty} A_n\Big) = \sum\limits_{n=1}^{\infty} m(A_n)$;

(ii) m is sequentially continuous from above, i.e., if $A \in \mathbb{A}$ and $\langle A_n\rangle$ is a decreasing sequence in $\mathbb{A}$ such that $A = \bigcap\limits_{n=1}^{\infty} A_n$, then $m(A) = \lim\limits_{n\to\infty} m(A_n)$;

(iii) m is sequentially continuous from below, i.e., if $A \in \mathbb{A}$ and $\langle A_n\rangle$ is an increasing sequence in $\mathbb{A}$ such that $A = \bigcup\limits_{n=1}^{\infty} A_n$, then $m(A) = \lim\limits_{n\to\infty} m(A_n)$.

Monotone (sequential) continuity appears in various generalizations of measures: states on quantum logics, tribes, D-posets, MV-algebras, etc., see [20], [21], [9], [19], [18], [13], [8], [11], and the references therein.

Observe that, barring trivial cases, the monotone convergence on a ring of sets $\mathbb{A}$ is too fine to match the natural algebraic and topological structures of $\mathbb{A}$: if $\langle A_n\rangle$ and $\langle B_n\rangle$ are convergent monotone sequences, then the sequence $\langle A_n \div B_n\rangle$ of the symmetric differences $A_n \div B_n = (A_n \setminus B_n) \bigcup (B_n \setminus A_n)$, $n \in \mathbb{N}$, need not be monotone and the monotone convergence is strictly finer than the initial (or weak) convergence with respect to all σ-additive $\{0,1\}$-valued measures (there is a sequence $\langle A_n\rangle$ in $\mathbb{A}$ such that $0 = \lim\limits_{n\to\infty} m(A_n)$ for each σ-additive $\{0,1\}$-valued measure m and $\langle A_n\rangle$ fails to be decreasing to $\emptyset$). Hence the monotone convergence is not suitable in case we employ topological and categorical methods. We shall show that the usual convergence ($\langle A_n\rangle$ converges to A in $\mathbb{A}$ iff $A = \limsup A_n = \liminf A_n$, where $\limsup A_n = \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{n=k}^{\infty} A_n$ and $\liminf A_n = \bigcup\limits_{k=1}^{\infty} \bigcap\limits_{n=k}^{\infty} A_n$; this is equivalent to the pointwise convergence of the sequence $\langle \chi_{A_n}\rangle$ of characteristic functions to χ_A) is exactly what is needed. It turns rings of sets and their generalizations into continuous algebras (algebraic operations are sequentially continuous) and, which is more important, σ-additivity (and measurability, cf. [4]) can be characterized in terms of sequential continuity.

J. Novák proposed to study the extension of probabilities from a field $\mathbb{A}$ of sets over the generated σ-field $\sigma(\mathbb{A})$ from the topological viewpoint. He developed the theory of sequential envelopes—a theory resembling the Čech-Stone compactification and the Hewitt realcompactification (all are epireflections in the categorical language) and showed that $\sigma(\mathbb{A})$ is the sequential envelope of $\mathbb{A}$ with respect to the extension of probabilities (cf. [15], [16], [17]). The crucial point was to show that each probability (bounded σ-additive measure) on $\mathbb{A}$ is sequentially continuous with respect to the pointwise sequential convergence. A rather involved proof of this fact (attributed to M. Jiřina) appeared in [14]. We provide a simple proof based on the relationships between the monotone and the pointwise convergence.

Proposition 1.1. *Let $X \neq \emptyset$ be a set and let $\mathbb{A}$ be a σ-ring of subsets of X. A sequence $\langle B_n \rangle$ of elements of $\mathbb{A}$ converges to $B \in \mathbb{A}$ iff in $\mathbb{A}$ there are sequences $\langle A_n \rangle$ and $\langle C_n \rangle$ such that*
(c_1) *$\langle A_n \rangle$ is nondecreasing and $\langle C_n \rangle$ is nonincreasing,*
(c_2) *$A_n \subseteq B_n \subseteq C_n$, $n \in \mathbb{N}$,*
(c_3) *$B = \bigcup\limits_{n=1}^{\infty} A_n = \bigcap\limits_{n=1}^{\infty} C_n$.*

Proof. Necessity. Assume that $\langle B_n \rangle$ converges in $\mathbb{A}$ to B. Put $A_n = \bigcap\limits_{k=n}^{\infty} B_k$, $C_n = \bigcup\limits_{k=n}^{\infty} B_k$. Since $\mathbb{A}$ is a σ-ring, $\langle A_n \rangle$ and $\langle C_n \rangle$ are sequences in $\mathbb{A}$ and it is easy to verify that conditions (c_1), (c_2), (c_3) are satisfied.

Sufficiency. Let $B \in \mathbb{A}$ and let $\langle B_n \rangle$ be a sequence in $\mathbb{A}$. Assume that $\langle A_n \rangle$ and $\langle C_n \rangle$ are sequences in $\mathbb{A}$ satisfying conditions (c_1), (c_2), (c_3). Clearly, $\liminf A_n \subseteq \liminf B_n \subseteq \limsup B_n \subseteq \limsup C_n$. Since $\liminf A_n = \bigcup\limits_{n=1}^{\infty} A_n$ and $\limsup C_n = \bigcap\limits_{n=1}^{\infty} C_n$, it follows that $B = \liminf B_n = \limsup B_n$. This completes the proof. □

Corollary 1.2. *Let $X \neq \emptyset$ be a set and let $\mathbb{A}$ be a σ-ring of subsets of X. Then the usual sequential convergence in $\mathbb{A}$ is the finest of all sequential convergences $\mathbb{K}$ in $\mathbb{A}$ such that each nondecreasing sequence $\langle A_n \rangle$ converges under $\mathbb{K}$ to $\bigcup\limits_{n=1}^{\infty} A_n$, each nonicreasing sequence $\langle C_n \rangle$ converges under $\mathbb{K}$ to $\bigcap\limits_{n=1}^{\infty} C_n$, and a sequence $\langle B_n \rangle$ converges under $\mathbb{K}$ to B whenever in $\mathbb{A}$ there are sequences $\langle A_n \rangle$ and $\langle C_n \rangle$ satisfying conditions* (c_1), (c_2), (c_3) *in Proposition* 1.1.

Corollary 1.3. *Let $X \neq \emptyset$ be a set, let $\mathbb{A}$ be a ring of subsets of X, and let $\sigma(\mathbb{A})$ be the generated σ-ring. A sequence $\langle B_n \rangle$ converges to B in $\mathbb{A}$ iff in $\sigma(\mathbb{A})$ there are sequences $\langle A_n \rangle$ and $\langle C_n \rangle$ satisfying conditions* (c_1), (c_2), (c_3) *in Proposition* 1.1.

Let $X \neq \emptyset$ be a set, let $\mathbb{A}$ be a ring of subsets of X, and let m be a finite unbounded finitely additive measure on $\mathbb{A}$. Then m fails to be sequentially continuous. Indeed (cf. [13]), in $\mathbb{A}$ there is a mutually disjoint sequence $\langle A_n \rangle$ such that $m(A_n) > 1$, $n \in \mathbb{N}$.

But $\langle A_n \rangle$ converges in $\mathbb{A}$ to $\emptyset$ and hence m fails to be sequentially continuous. Thus only a bounded finite measure on $\mathbb{A}$ can be sequentially continuous.

Proposition 1.4. *Let $X \neq \emptyset$ be a set, let $\mathbb{A}$ be a ring of subsets of X, and let m be a bounded σ-additive measure on $\mathbb{A}$. Then m is sequentially continuous.*

Proof. 1. Assume that $\mathbb{A}$ is a σ-ring. Let $\langle B_n \rangle$ be a sequence converging in $\mathbb{A}$ to B. According to Proposition 1.1, in $\mathbb{A}$ there are a nondecreasing sequence $\langle A_n \rangle$ and a nonincreasing sequence $\langle C_n \rangle$ such that $A_n \subseteq B_n \subseteq C_n$, $n \in \mathbb{N}$, $B = \bigcup_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} C_n$. Then $m(A_n) \leqslant m(B_n) \leqslant m(C_n)$, $n \in \mathbb{N}$, and since σ-additivity implies monotone continuity, necessarily $m(B) = \lim_{n\to\infty} m(B_n)$.

2. Let $\mathbb{A}$ be a ring and let $\sigma(\mathbb{A})$ be the generated σ-ring. Then there is a σ-additive measure m_σ on $\sigma(\mathbb{A})$ which extends m. It is known that m_σ is bounded and it is uniquely determined. It follows from the previous case that m_σ is sequentially continuous on $\sigma(\mathbb{A})$. Since the convergence in $\mathbb{A}$ is the restriction of the convergence in $\sigma(\mathbb{A})$, m is sequentially continuous on $\mathbb{A}$. This completes the proof. □

Corollary 1.5. *Let $X \neq \emptyset$ be a set, let $\mathbb{A}$ be a ring of subsets of X, and let m be a bounded finitely additive measure on $\mathbb{A}$. If m is sequentially continuous from above (from below), then m is sequentially continuous.*

2.

Information about MV-algebras, bold algebras of fuzzy sets, T-norms and T-tribes can be found in [1], [13], [21], [10], [2], [3]. Sequential convergence on MV-algebras has been studied in [7]. For the reader's convenience we recall here some basic facts.

Let $\mathcal{I}$ be the closed unit interval $[0, 1]$ carrying the usual MV-algebra operations and order and the usual convergence of sequences:

$$x \oplus y = \min\{1, x + y\},$$
$$x^* = 1 - x,$$
$$x \odot y = \max\{0, x + y - 1\} = (x^* \oplus y^*)^*,$$
$$x \vee y = \max\{x, y\} = (x^* \oplus y)^* \oplus y,$$
$$x \wedge y = \min\{x, y\} = (x^* \vee y^*)^*,$$
$$\lim_{n\to\infty} x_n = x.$$

Observe that both $\oplus$ and $\odot$ are commutative and associative.

Further, let X be a set and let $\mathcal{I}^X$ be the set $[0,1]I^X$ of all functions on X into $\mathcal{I}$ carrying the pointwise MV-algebra operations and order and the pointwise convergence of sequences, i.e., for each $x \in X$ put:

$(f \oplus g)(x) = f(x) \oplus g(x)$,
$f^*(x) = (f(x))^*$,
$(f \odot g)(x) = f(x) \odot g(x)$,
$(f \vee g)(x) = f(x) \vee g(x)$,
$(f \wedge g)(x) = f(x) \wedge g(x)$, and
$\mathrm{Lim}_{n\to\infty} f_n = f$ iff $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$.

If $\mathcal{A}$ is a subalgebra of $\mathcal{I}^X$ ($\mathcal{A}$ contains the constant functions 1_X and 0_X and it is closed with respect to all operations and order and carries the pointwise convergence of sequences), then $\mathcal{A}$ is said to be a *bold algebra.* A bold algebra $\mathcal{A}$ such that for each sequence $\langle f_n\rangle$ in $\mathcal{A}$ also $\min\bigl\{1_X, \sum\limits_{n=1}^{\infty} f_n\bigr\}$ belongs to $\mathcal{A}$ is said to be a *Łukasiewicz tribe* or, simply, a *tribe.* We shall additionally utilize operations $\ominus$ and $\triangle$ defined as follows:

$$(f \ominus g)(x) = \max\{0, f(x) - g(x)\} = (f \odot g^*)(x),$$
$$(f \triangle g)(x) = \max\{(f \ominus g)(x), (g \ominus f)(x)\}.$$

It is known that each σ-complete MV-algebra is Archimedean (or semisimple) and each Archimedean MV-algebra can be represented by a bold algebra (the elements of which are fuzzy subsets of the underlying set of the bold algebra). If for each element f in a bold algebra $\mathcal{A} \subseteq \mathcal{I}^X$ we have $f(x) \in \{0,1\}$, $x \in X$, then $\mathcal{A}$ becomes a field of subsets of X (via characteristic functions).

Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a bold algebra. A map $m\colon \mathcal{A} \longrightarrow [0,1]$ is said to be a *measure* if it is normed and subtractive, i.e., $m(1_X) = 1$ and $m(g \ominus f) = m(g) - m(f)$ whenever $f, g \in \mathcal{A}$, $f \leqslant g$. This definition is not quite standard, but suites our purpose.

The main result of Section 2 is the following

Proposition 2.1. *Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a bold algebra, let $\sigma(\mathcal{A}) \subseteq \mathcal{I}^X$ be the generated Łukasiewicz tribe, and let $m\colon \mathcal{A} \longrightarrow [0,1]$ be a sequentially continuous measure. Then m can be extended to a sequentially continuous measure $m_\sigma\colon \sigma(\mathcal{A}) \longrightarrow [0,1]$ and the extension is uniquely determined.*

Remark 2.2. The proof of Proposition 2.1 is based on Theorem 1 in [8]. It is a general measure extension theorem for MV-algebras. Let $\mathcal{F}$ be an MV-σ-algebra (i.e. a σ-complete MV-algebra). Symbol $a_n \nearrow a$ means that $\langle a_n\rangle$ is an increasing sequence in $\mathcal{F}$ and $a = \bigvee\limits_{n=1}^{\infty} a_n$ and, similarly, $b_n \searrow b$ means that $\langle b_n\rangle$ is a decreasing sequence in $\mathcal{F}$ and $b = \bigwedge_{n=1}^{\infty} b_n$. Let $\mathcal{A}$ be a MV-subalgebra of $\mathcal{F}$ and let m be a map of $\mathcal{A}$ into $[0,1]$. We say that m is a *measure* if

(m$_1$) $m(1_{\mathcal{A}}) = 1$,

(m$_2$) if $a, b \in \mathcal{A}$, $a \leqslant b$, then $m(a) \leqslant m(b)$ and $m(b \ominus a) = m(b \odot a^*) = m(b) - m(a)$.

If, in addition, m satisfies

(m$_3$) if $\langle a_n \rangle$ is an increasing sequence in $\mathcal{A}$ such that $a_n \nearrow a$ and $a \in \mathcal{A}$, then $\lim_{n\to\infty} m(a_n) = m(a)$,

then m is said to be *sequentially continuous from below. Sequential continuity from above* is defined analogously. Recall that $\mathcal{F}$ is *σ-continuous* if $a_n \nearrow a$ and $b_n \searrow b$ implies $(a_n \vee b_n) \nearrow (a \vee b)$, and $a_n \nearrow a$ and $b_n \searrow b$ implies $(b_n \ominus a_n) \searrow (b \ominus a)$, $(a_n \ominus b_n) \nearrow (a \ominus b)$. Finally, consider the following condition

(*) if $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences in $\mathcal{A}$ such that $a_n \searrow a$, $b_n \nearrow b$, and $a \leqslant b$,

$$\text{then } \lim_{n\to\infty} m(b_n \ominus a_n) = \lim_{n\to\infty} m(b_n) - \lim_{n\to\infty} m(a_n).$$

Measure extension theorem (M. Jurečková). *Let $\mathcal{F}$ be a σ-continuous MV-σ-algebra, let $\mathcal{A}$ be an MV-subalgebra of $\mathcal{F}$, and let $m\colon \mathcal{A} \longrightarrow [0,1]$ be a measure sequentially continuous from below. Let $\mathcal{S}(\mathcal{A})$ be the generated MV-σ-algebra. If m satisfies (*), then m can be uniquely extended to a measure $m_\sigma\colon \mathcal{S}(\mathcal{A}) \longrightarrow [0,1]$ sequentially continuous from below.*

Remark 2.3. Since $\mathcal{I}^X$ is a MV-σ-algebra, to prove Proposition 2.1 it suffices to show that:

(a) $\mathcal{I}^X$ is σ-continuous,

(b) $\sigma(\mathcal{A}) = \mathcal{S}(\mathcal{A})$,

(c) m satisfies (*),

(d) $m_\sigma\colon \sigma(\mathcal{A}) \longrightarrow [0,1]$ is sequentially continuous.

Remark 2.4. We shall prove that on a Łukasiewicz tribe each measure sequentially continuous from below is sequentially continuous (with respect to the pointwise convergence). Hence the restriction to the generating bold algebra is sequentially continuous, too. Consequently, the assumption that $m\colon \mathcal{A} \longrightarrow [0,1]$ is sequentially continuous is not restrictive.

Before the proof of Proposition 2.1, we present a series of lemmas, some of them interesting on their own.

Lemma 2.5. *Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a bold algebra. Then the following are equivalent:*

(i) *$\mathcal{A}$ is a Łukasiewicz tribe;*

(ii) *If $\langle f_n \rangle$ is a nondecreasing (nonincreasing) sequence in $\mathcal{A}$ and $f = \operatorname{Lim}_{n\to\infty} f_n$, then $f \in \mathcal{A}$.*

P r o o f. (i) $\Rightarrow$ (ii). The assertion follows from Proposition 8.13 in [21].

(ii) $\Rightarrow$ (i). Assume (ii) and let $\langle g_n \rangle$ be a sequence in $\mathcal{A}$. Put $f_1 = g_1$ and, inductively, $f_{n+1} = f_n \oplus g_{n+1}$, $n \geqslant 1$. Then $\langle f_n \rangle$ is a nondecreasing sequence in $\mathcal{A}$. Since $\min\Big\{\sum_{n=1}^{\infty} g_n, 1\Big\} = \bigvee_{n=1}^{\infty} f_n$, it follows that $\mathcal{A}$ is a Łukasiewicz tribe. □

Lemma 2.6. *Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a Łukasiewicz tribe. Let $\langle g_n \rangle$ be a sequence in $\mathcal{A}$ and let $g \in \mathcal{A}$. Then $\operatorname{Lim}_{n\to\infty} g_n = g$ iff in $\mathcal{A}$ there are sequences $\langle f_n \rangle$ and $\langle h_n \rangle$ such that*

(C_1) *$\langle f_n \rangle$ is nondecreasing and $\langle h_n \rangle$ is nonincreasing,*

(C_2) $f_n \leqslant g_n \leqslant h_n$, $n \in \mathbb{N}$,

(C_3) $g = \bigvee_{n=1}^{\infty} f_n = \bigwedge_{n=1}^{\infty} h_n$.

P r o o f. Necessity. Assume that $\langle g_n \rangle$ converges to g in $\mathcal{A}$. Put $f_n = \bigwedge_{k=n}^{\infty} g_k$, $h_n = \bigvee_{k=n}^{\infty} g_k$. Since $\mathcal{A}$ is closed with respect to monotone limits, $\langle f_n \rangle$ and $\langle h_n \rangle$ are sequences in $\mathcal{A}$ and it is easy to see that conditions (C_1), (C_2), (C_3) are satisfied.

Sufficiency. Assume that (C_1), (C_2), (C_3) hold true. It is easy to verify that $\lim_{n\to\infty} g_n(x) = g(x)$ for each $x \in X$. Hence $\operatorname{Lim}_{n\to\infty} g_n = g$ and the proof is complete. □

Corollary 2.7. *Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a Łukasiewicz tribe. Then the pointwise convergence* Lim *in $\mathcal{A}$ is the finest of all sequential convergences $\mathbb{K}$ in $\mathcal{A}$ such that each nondecreasing sequence $\langle f_n \rangle$ converges under $\mathbb{K}$ to $\bigvee_{n=1}^{\infty} f_n$, each nonicreasing sequence $\langle h_n \rangle$ converges under $\mathbb{K}$ to $\bigwedge_{n=1}^{\infty}$ and a sequence $\langle g_n \rangle$ converges under $\mathbb{K}$ to g whenever in $\mathcal{A}$ there are sequences $\langle f_n \rangle$ and $\langle h_n \rangle$ satisfying conditions* (C_1), (C_2), (C_3) *in Lemma* 2.6.

Corollary 2.8. *Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a bold algebra. Then $\mathcal{A}$ is a Łukasiewicz tribe iff $\mathcal{A}$ is a sequentially closed subset of $\mathcal{I}^X$ (with respect to the pointwise convergence).*

Corollary 2.9. *Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a Łukasiewicz tribe and let $m\colon \mathcal{A} \longrightarrow [0,1]$ be a measure sequentially continuous from below. Then m is sequentially continuous (with respect to the pointwise convergence).*

P r o o f. Let $\operatorname{Lim}_{n\to\infty} g_n = g$ in $\mathcal{A}$. According to Lemma 2.6, in $\mathcal{A}$ there are sequences $\langle f_n \rangle$ and $\langle h_n \rangle$ satisfying conditions (C_1), (C_2), (C_3). Clearly, $f_n \leqslant g_n \leqslant h_n$ implies

$m(f_n) \leqslant m(g_n) \leqslant m(h_n)$, $n \in \mathbb{N}$. Since m is sequentially continuous both from below and from above, $\lim\limits_{n\to\infty} m(f_n) = \lim\limits_{n\to\infty} m(h_n) = m(g)$ implies $\lim\limits_{n\to\infty} m(g_n) = m(g)$. □

Lemma 2.10. *Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a bold algebra, let $\mathcal{S}(\mathcal{A})$ be the generated MV-σ-algebra and let $m\colon \mathcal{A} \longrightarrow [0,1]$ be a sequentially continuous measure.*

(i) *Let $\langle v_n\rangle$ and $\langle w_n\rangle$ be sequences in $\mathcal{A}$ such that $\operatorname*{Lim}\limits_{n\to\infty}(v_n \triangle w_n) = 0$. Then $\lim\limits_{n\to\infty} |m(v_n) - m(w_n)| = 0$.*

(ii) *Let $\langle v_n\rangle$ be a sequence in $\mathcal{A}$ such that for any two subsequences $\langle v'_n\rangle$, $\langle v''_n\rangle$ of $\langle v_n\rangle$ we have $\operatorname*{Lim}\limits_{n\to\infty}(v'_n \triangle v''_n) = 0_X$. Then $\langle m(v_n)\rangle$ is a Cauchy sequence.*

(iii) *Let $\langle a_n\rangle$, $\langle b_n\rangle$ be sequences in $\mathcal{A}$ such that $\langle a_n\rangle$ is decreasing, $\langle b_n\rangle$ is increasing, and $\bigwedge\limits_{n=0}^{\infty} a_n \leqslant \bigvee\limits_{n=0}^{\infty} nb_n$ in $\mathcal{S}(\mathcal{A})$. Then $\lim\limits_{n\to\infty} m(b_n \ominus a_n) = \lim\limits_{n\to\infty} m(b_n) - \lim\limits_{n\to\infty} m(a_n)$.*

Proof. (i) From $v_n \ominus w_n \leqslant (v_n \ominus w_n)^*$ we get $m(v_n \triangle w_n) = m((v_n \ominus w_n) \oplus (w_n \ominus v_n)) = m(v_n \ominus w_n) + m(w_n \ominus v_n)$. From the sequential continuity of m it follows that $\lim\limits_{n\to\infty} m(v_n \ominus w_n) = 0$, $\lim\limits_{n\to\infty} m(w_n \ominus v_n) = 0$. Observe that if $v, w \in \mathcal{A}$, then $v = (v \ominus w) \ominus \min\{v, w\}$ and $v \ominus w \leqslant (\min\{v, w\})^*$. Hence $\lim\limits_{n\to\infty} |m(v_n) - m(w_n)| = \lim\limits_{n\to\infty} |m(v_n \ominus w_n) + m(\min\{v_n, w_n\}) - m(w_n \ominus v_n) + m(\min\{v_n, w_n\})| = 0$.

(ii) Clearly, (ii) is a straightforward consequence of (i).

(iii) It follows from (ii) that $\langle m(a_n)\rangle$ and $\langle m(b_n)\rangle$ are Cauchy sequences. Clearly, $\operatorname*{Lim}\limits_{n\to\infty}(a_n \ominus b_n) = 0_X$ and hence $\operatorname*{Lim}\limits_{n\to\infty}(\max\{b_n, a_n\} \ominus b_n) = 0_X$. Then

$$\lim_{n\to\infty} m(\max\{b_n, a_n\}) = \lim_{n\to\infty} (m(\max\{b_n, a_n\}) - m(b_n)) = 0$$

and hence

$$\lim_{n\to\infty} m(\max\{b_n, a_n\}) = \lim_{n\to\infty} m(b_n).$$

Further, $m(\max\{b_n, a_n\} \ominus a_n) = m(\max\{b_n, a_n\}) - m(a_n)$ and hence

$$\lim_{n\to\infty} m(\max\{b_n, a_n\} \ominus a_n) = \lim_{n\to\infty} m(b_n) - \lim_{n\to\infty} m(a_n).$$

Finally, $\operatorname*{Lim}\limits_{n\to\infty}((\max\{b_n, a_n\} \ominus a_n) \triangle (b_n \ominus a_n)) = 0_X$ and hence $\lim\limits_{n\to\infty} m(b_n \ominus a_n) = \lim\limits_{n\to\infty} m(b_n) - \lim\limits_{n\to\infty} m(a_n)$. This completes the proof. □

Proof (of Proposition 2.1). We shall proceed according to Remark 2.2 and Remark 2.3.

(a) Since all usual pointwise operations and the order on R^X are sequentially continuous with respect to the pointwise convergence, also the MV-algebra operations and the order on a bold algebra $\mathcal{A} \subseteq \mathcal{I}^X$ are sequentially continuous. In particular, $\mathcal{I}^X$ is σ-continuous.

(b) Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a bold algebra. Then $\sigma(\mathcal{A})$ is the intersection of all Łukasiewicz tribes containing $\mathcal{A}$ and $\mathcal{S}(\mathcal{A})$ is the intersection of all MV-σ-algebras containing $\mathcal{A}$. It follows from Lemma 2.5 that $\sigma(\mathcal{A}) = \mathcal{S}(\mathcal{A})$.

(c) According to Lemma 2.10 (iii), each sequentially continuous measure on a bold algebra satisfies condition (*).

(d) It follows from the previous points that $m\colon \mathcal{A} \longrightarrow [0,1]$ can be extended to a measure m_σ on $\sigma(\mathcal{A})$ and since m_σ is sequentially continuous from below, according to Corollary 2.9, m_σ is sequentially continuous. This completes the proof. □

Remark 2.11. Since it is natural to define a probability on bold algebras in such a way that each probability on a bold algebra $\mathcal{A} \subseteq \mathcal{I}^X$ can be uniquely extended to the generated Łukasiewicz tribe $\sigma(\mathcal{A})$ and each restriction of a probability on $\sigma(\mathcal{A})$ to $\mathcal{A}$ is a probability on $\mathcal{A}$, we propose the following

Definition 2.12. Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a bold algebra. A *probability* on $\mathcal{A}$ is a map $p\colon \mathcal{A} \longrightarrow [0,1]$ such that

(p_1) $p(1_X) = 1$,

(p_2) if $f, g \in \mathcal{A}$, $f \leqslant g$, then $p(g \ominus f) = p(g) - p(f)$,

(p_3) p is sequentially continuous (with respect to the pointwise convergence on $\mathcal{A}$).

Symbol $\operatorname{prob}(\mathcal{A})$ will denote the set of all probabilities on $\mathcal{A}$.

Remark 2.13. If $\mathcal{I}$ and $\mathcal{A} \subseteq \mathcal{I}^X$ are considered as D-posets, then probabilities on bold algebras are exactly sequentially continuous D-morphisms of $\mathcal{A}$ into $\mathcal{I}$.

We end this section with a useful characterization of the generated Łukasiewicz tribe $\sigma(\mathcal{A}) \subseteq \mathcal{I}^X$.

Proposition 2.14. *Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a bold algebra and let $\sigma(\mathcal{A})$ be the generated Łukasiewicz tribe. Then $\sigma(\mathcal{A})$ is the smallest sequentialy closed (with respect to the pointwise sequential convergence) subset of $\mathcal{I}^X$ which contains $\mathcal{A}$.*

Proof. For $\mathcal{B} \subseteq \mathcal{I}^X$, let $\operatorname{cl} \mathcal{B}$ be the set of all $f \in \mathcal{I}^X$ such that in $\mathcal{B}$ there exists a sequence $\langle f_n \rangle$ converging to f (in the pointwise sequential convergence). For each ordinal number α define $\operatorname{cl}^\alpha \mathcal{B}$ as follows: $\operatorname{cl}^0 \mathcal{B} = \mathcal{B}$, $\operatorname{cl}^\alpha \mathcal{B} = \operatorname{cl}(\operatorname{cl}^{\alpha-1} \mathcal{B})$ if α is an isolated ordinal number and $\operatorname{cl}^\alpha \mathcal{B} = \operatorname{cl}\Big(\bigcup\limits_{\beta<\alpha} \operatorname{cl}^\beta \mathcal{B}\Big)$ if α is a limit ordinal number. It is known (cf. [16] that each cl^α is a closure operator, $\operatorname{cl}(\operatorname{cl}^{\omega_1} \mathcal{B}) = \operatorname{cl}^{\omega_1} \mathcal{B}$, $\operatorname{cl}^{\omega_1}$ is a topological (idempotent) closure operator, and $\operatorname{cl}^{\omega_1} \mathcal{B}$ is the smallest sequentially closed subset of $\mathcal{I}^X$ which contains $\mathcal{B}$. We know (cf. (a) in the proof of Proposition 2.1) that the MV-algebra operations and the order in $\mathcal{I}^X$ are sequentially continuous. It follows that $\operatorname{cl} \mathcal{A} \subseteq \mathcal{I}^X$ is a bold algebra, too. Since $\bigcup\limits_{\beta<\beta} \operatorname{cl}^\alpha \mathcal{A}$ is always a bold algebra, it follows from Corollary 2.8 that $\mathrm{cl}^{\omega_1} \mathcal{A}$ is the smallest Łukasiewicz tribe in $\mathcal{I}^X$ which contains $\mathcal{A}$. This completes the proof. □

Remark 2.15. If $\mathbb{A}$ is a field of subsets of X, then the generated σ-field $\sigma(\mathbb{A})$ is the smallest sequentially closed subset in the field of all subsets of X which contains $\mathbb{A}$ or, in terms of characteristic functions, the smallest sequentially closed subset (with respect to the pointwise convergence) in $\{0,1\}^X$ which contains the characteristic functions of sets in $\mathbb{A}$ (cf. [17]).

3.

In this section we describe the relationship between bold algebras and Łukasiewicz tribes. We will try to avoid the categorical formalism as much as possible but, at the same time, we will utilize the categorical machinery which makes the relationship transparent. Standard references on MV-algebras and category theory are [2] and [6], respectively.

Let BD be the category whose objects are bold algebras carrying the pointwise sequential convergence and whose morphisms are sequentially continuous MV-homomorphisms. If $\mathcal{A}$ and $\mathcal{B}$ are bold algebras, then $\hom(\mathcal{A}, \mathcal{B})$ will denote the set of all morphisms of $\mathcal{A}$ into $\mathcal{B}$. Clearly, if X is a singleton, then $\mathcal{I}^X$ is isomorphic to $\mathcal{I}$. In general, a bold algebra $\mathcal{A} \subseteq \mathcal{I}^X$ is a subobject of the product of the family $\{\mathcal{I}_x;\ \mathcal{I}_x = \mathcal{I},\ x \in X\}$, and each $x \in X$ defines a morphism $x\colon\ \mathcal{A} \longrightarrow \mathcal{I}$ by putting $x(f) = f(x)$, $f \in \mathcal{A}$; $\hom(\mathcal{A})$ will denote the set of all morphisms of $\mathcal{A}$ into $\mathcal{I}$ and $\mathrm{fix}(\mathcal{A})$ will denote its subset defined by the points of X ("fixing" $\mathcal{A}$).

Definition 3.1. Let $\mathcal{A} \subseteq \mathcal{I}^X$, $\mathcal{B} \subseteq \mathcal{I}^X$ be bold algebras. Let $\mathcal{C} \subseteq \mathcal{B}$ be a bold subalgebra and let φ be an isomorphism of $\mathcal{A}$ onto $\mathcal{C}$ in BD. We say that φ is a

(i) $\mathrm{fix}(\mathcal{A})$-*embedding* if for each $h \in \mathrm{fix}(\mathcal{A})$ there exists $\overline{h} \in \hom(\mathcal{B})$ such that $h = \overline{h} \circ \varphi$;

(ii) $\hom(\mathcal{A})$-*embedding* if for each $h \in \hom(\mathcal{A})$ there exists $\overline{h} \in \hom(\mathcal{B})$ such that $h = \overline{h} \circ \varphi$;

(iii) $\mathrm{prob}(\mathcal{A})$-*embedding* if for each $p \in \mathrm{prob}(\mathcal{A})$ there exists $\overline{p} \in \mathrm{prob}(\mathcal{B})$ such that $h = \overline{p} \circ \varphi$;

in such case we say that $\mathcal{A}$ is $\mathrm{fix}(\mathcal{A})$-*embedded*, $\hom(\mathcal{A})$-*embedded*, $\mathrm{prob}(\mathcal{A})$-*embedded*, respectively, in $\mathcal{B}$. If $\mathcal{C} = \varphi(\mathcal{A})$ is sequentially closed in $\mathcal{B}$ for each $\mathrm{fix}(\mathcal{A})$-embedding, $\hom(\mathcal{A})$-embedding, $\mathrm{prob}(\mathcal{A})$-embedding, respectively, $\varphi\colon\ \mathcal{A} \longrightarrow \mathcal{B}$, then $\mathcal{A}$ is said to be *absolutely sequentially closed* with respect to the extension of $\mathrm{fix}(\mathcal{A})$, $\hom(\mathcal{A})$, $\mathrm{prob}(\mathcal{A})$, respectively.

Lemma 3.2. *Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a bold algebra and let $h \in \mathrm{hom}(\mathcal{A})$. Then there exists a unique $h_\sigma \in \mathrm{hom}(\sigma(\mathcal{A}))$ such that $h_\sigma \restriction \mathcal{A} = h$.*

Proof. Since h is a sequentially continuous measure on $\mathcal{A}$, according to Proposition 2.1 there exists a unique sequentially continuous measure $h_\sigma\colon \sigma(\mathcal{A}) \longrightarrow [0,1]$ such that $h_\sigma \restriction \mathcal{A} = h$. It suffices to prove that h_σ is an MV-homomorphism into $\mathcal{I}$. By Proposition 2.14, $\sigma(\mathcal{A}) = \mathrm{cl}^{\omega_1} \mathcal{A}$. Assume that α is an ordinal number, $1 \leqslant \alpha \leqslant \omega_1$, and $h_\sigma \restriction \mathrm{cl}^\beta \mathcal{A}$ is an MV-homomorphism for each $\beta < \alpha$. Let α be isolated. If $f, g \in \mathrm{cl}^\alpha \mathcal{A}$, then in $\mathrm{cl}^{\alpha-1} \mathcal{A}$ there are sequences $\langle f_n\rangle$, $\langle g_n\rangle$ such that $\operatorname*{Lim}_{n\to\infty} f_n = f$ and $\operatorname*{Lim}_{n\to\infty} g_n = g$. Since h_σ is sequentially continuous, we have $\lim_{n\to\infty} h_\sigma(f_n) = h_\sigma(f)$, $\lim_{n\to\infty} h_\sigma(g_n) = h_\sigma(g)$ and $h_\sigma(f \oplus g) = \lim_{n\to\infty} h_\sigma(f_n \oplus g_n) = \lim_{n\to\infty} (h_\sigma(f_n) \oplus h_\sigma(g_n))$. Hence, in $\mathcal{I}$, $\lim_{n\to\infty} (h_\sigma(f_n) \oplus h_\sigma(g_n)) = \lim_{n\to\infty} h_\sigma(f_n) \oplus \lim_{n\to\infty} h_\sigma(g_n) = h_\sigma(f) \oplus h_\sigma(g)$. Clearly, $h_\sigma \restriction \mathrm{cl}^\alpha \mathcal{A}$ is an MV-homomorphism. If α is a limit ordinal number, we proceed analogously. The details are left out. Since $h_\sigma \restriction \mathrm{cl}^0 \mathcal{A} = h$ is an MV-homomorphism, it follows that h_σ is an MV-homomorphism of $\mathrm{cl}^{\omega_1} \mathcal{A} = \sigma(\mathcal{A})$ into $\mathcal{I}$. □

Proposition 3.3. *Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a bold algebra. Then the following are equivalent:*

(i) *$\mathcal{A}$ is a Łukasiewicz tribe;*
(ii) *$\mathcal{A}$ is absolutely sequentially closed with respect to the extension of* $\mathrm{fix}(\mathcal{A})$*;*
(iii) *$\mathcal{A}$ is absolutely sequentially closed with respect to the extension of* $\mathrm{hom}(\mathcal{A})$*;*
(iv) *$\mathcal{A}$ is absolutely sequentially closed with respect to the extension of* $\mathrm{prob}(\mathcal{A})$.

Proof. (i) ⇒ (ii). Assume that $\mathcal{A} = \sigma(\mathcal{A})$. Let $\varphi\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a $\mathrm{fix}(\mathcal{A})$-embedding. Let $\langle f_n\rangle$ be a sequence in $\mathcal{A}$ such that the sequence $\langle \varphi(f_n)\rangle$ converges in $\mathcal{B}$. Then for each $h \in \mathrm{fix}(\mathcal{A})$ there exists $\overline{h} \in \mathrm{hom}(\mathcal{B})$ such that $h(f_n) = \overline{h}(\varphi(f_n))$, $n \in \mathbb{N}$, and the sequence $\langle h(f_n)\rangle$ converges in $\mathcal{I}$. Since $\mathrm{fix}(\mathcal{A})$ can be identified with X, the sequence $\langle f_n(x)\rangle$ converges in $\mathcal{I}$ for each $x \in X$. But $\mathcal{A} = \sigma(\mathcal{A}) \subseteq \mathcal{I}^X$ means that there exists $f \in \mathcal{A}$ such that $\operatorname*{Lim}_{n\to\infty} f_n = f$. Thus $\operatorname*{Lim}_{n\to\infty} \varphi(f_n) = \varphi(f)$ in $\varphi(\mathcal{A}) = \mathcal{C}$ and $\mathcal{C}$ is sequentially closed in $\mathcal{B}$. Hence (ii) holds true.

(ii) ⇒ (iii). Assume (ii). Let $\varphi\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a $\mathrm{hom}(\mathcal{A})$-embedding. From $\mathrm{fix}(\mathcal{A}) \subseteq \mathrm{hom}(\mathcal{A})$ it follows that φ is a $\mathrm{fix}(\mathcal{A})$-embedding, too. Thus $\varphi(\mathcal{A}) = \mathcal{C}$ is sequentially closed in $\mathcal{B}$ and (iii) holds true.

(iii) ⇒ (iv). Assume (iii). Let $\varphi\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a $\mathrm{prob}(\mathcal{A})$-embedding. Put $\mathcal{B}' = \sigma(\varphi(\mathcal{A})) \subseteq \mathcal{I}^X$. Let $h \in \mathrm{hom}(\mathcal{A})$. Then there exists $\overline{h} \in \mathrm{prob}(\mathcal{B})$ such that $h = \overline{h} \circ \varphi$ and clearly $\overline{h} \in \mathrm{hom}(\varphi(\mathcal{A}))$. According to Lemma 3.2, $\overline{h}$ can be uniquely extended to $\overline{h}_\sigma \in \mathrm{hom}(\mathcal{B}')$. Thus φ is a $\mathrm{hom}(\mathcal{A})$-embedding of $\mathcal{A}$ into $\mathcal{B}'$ and hence $\mathcal{C} = \varphi(\mathcal{A})$ is sequentially closed in $\mathcal{B}'$. Finally, since $\mathcal{C}$ is sequentially closed in $\mathcal{I}^X$, it is also sequentially closed in $\mathcal{B}$.

(iv) $\Rightarrow$ (i). Assume (iv). Since each probability $p \in \text{prob}(\mathcal{A})$ can be uniquely extended to a probability $p_\sigma \in \text{prob}(\sigma(\mathcal{A}))$, the identity id: $\mathcal{A} \longrightarrow \sigma(\mathcal{A})$ is a prob($\mathcal{A}$)-embedding. Thus $\mathcal{A}$ is sequentially closed in $\sigma(\mathcal{A})$ and hence $\mathcal{A} = \sigma(\mathcal{A})$. This completes the proof. □

Remark 3.4. Let $\mathcal{A} \subseteq \mathcal{I}^X$ be a bold algebra. Then $\sigma(\mathcal{A})$ is, in a natural way, a maximal bold algebra in which $\mathcal{A}$ is prob($\mathcal{A}$)-embedded in the following sense.

On the one hand, $\mathcal{A}$ is topologically dense in $\sigma(\mathcal{A}) = \text{cl}^{\omega_1} \mathcal{A} \subseteq \mathcal{I}^X$ and, due to the sequential continuity of probabilities, the values on $\mathcal{A}$ determine the values on $\sigma(\mathcal{A})$. Further, each $p \in \text{prob}(\mathcal{A})$ has a unique extension $p_\sigma \in \text{prob}(\sigma(\mathcal{A}))$ and if $\varphi\colon \sigma(\mathcal{A}) \longrightarrow \mathcal{B}$, $\mathcal{B} \subseteq \mathcal{I}^X$ is a prob($\sigma(\mathcal{A})$)-embedding, then the value of $\overline{p_\sigma}$, $p_\sigma = \overline{p_\sigma} \circ \varphi$, at $f \in \varphi(\sigma(\mathcal{A}))$ is determined by the values of p on $\mathcal{A}$. On the other hand, let $f \in \mathcal{I}^Y \setminus \varphi(\sigma(\mathcal{A}))$. Then f is sequentially remote from $\varphi(\sigma(\mathcal{A}))$, i.e., f is not a limit of any sequence in $\varphi(\sigma(\mathcal{A}))$ or any multisequence (cf. [12]) in $\varphi(\mathcal{A})$. Hence for each sequence in $\varphi(\sigma(\mathcal{A}))$, or a multisequence in $\varphi(\mathcal{A})$, there exists $y \in Y$ at which the sequence, or the multisequence, fails to converge to f. Consider the point probability $p_y \in \text{fix}(\mathcal{B})$, $p_y(g) = g(y)$. Its restriction to $\varphi(\mathcal{A})$ defines a probability $p \in \text{prob}(\mathcal{A})$ by putting $p(g) = p_y(\varphi(y))$, $g \in \mathcal{A}$. Then, for p and its extension p_y, the "topologically remote" element f is also "p_y-remote" (relative to the sequence, or the multisequence, in question). □

Remark 3.5. Let $\{\mathcal{A}_t;\ t \in T\}$ be a family of bold algebras. Then the product bold algebra $\mathcal{A}$ in BD is the usual product, i.e., the set of all families $\{a_t \in \mathcal{A}_t;\ t \in T\}$, considered as mappings $a = \{a_t \in \mathcal{A}_t;\ t \in T\}$ of the disjoint union $X = \biguplus_{t\in T} X_t$ into $\mathcal{I}$ defined by $a(x) = a_t(x)$, $x \in X_t$, $t \in T$, carrying the pointwise MV-operations and the pointwise sequential convergence, together with the family $\{\text{pr}_t\colon \mathcal{A} \longrightarrow \mathcal{A}_t;\ t \in T\}$ of projections, where $\text{pr}_s(\{a_t \in \mathcal{A}_t;\ t \in T\}) = a_s$, $s \in T$. In particular, if $\mathcal{A}_x = \mathcal{I}$, for each $x \in X$, then $\mathcal{I}^X$ is the product bold algebra. Recall that each projection is a sequentially continuous MV-homomorphism and $\mathcal{A}$ has the following characteristic property. If $\mathcal{A}'$ is a bold algebra and for each $t \in T$ there exists (in BD) a morphism $\varphi_t\colon \mathcal{A}' \longrightarrow \mathcal{A}_t$, then there exists a unique morphism $\varphi\colon \mathcal{A}' \longrightarrow \mathcal{A}$ such that $\text{pr}_t \circ \varphi = \varphi_t$ for each $t \in T$.

Proposition 3.6. *Let $\mathcal{A}$ be a bold algebra and let φ be a sequentially continuous MV-homomorphism of $\mathcal{A}$ into a Łukasiewicz tribe $\mathcal{B}$. Then φ can be uniquely extended to a sequentially continuous MV-homomorphism $\varphi_\sigma\colon \sigma(\mathcal{A}) \longrightarrow \mathcal{B}$.*

Proof. Clearly, the natural evaluation ev: $\mathcal{B} \longrightarrow \mathcal{I}^{\text{hom}(\mathcal{B})}$ defined by $\text{ev}(b) = \{h(b);\ h \in \text{hom}(\mathcal{B})\}$ is a hom($\mathcal{B}$)-embedding. To simplify the notation, we identify $\mathcal{B}$ with its isomorphic image $\text{ev}(\mathcal{B}) \subseteq \mathcal{I}^{\text{hom}(\mathcal{B})}$. Then $\mathcal{B}$ is a sequentially closed subset of $\mathcal{I}^{\hom(\mathcal{B})}$, φ can be considered as a sequentially continuous MV-homomorphism into $\mathcal{I}^{\hom(\mathcal{B})}$, and each projection $\mathrm{pr}_h\colon \mathcal{I}^{\hom(\mathcal{B})} \longrightarrow \mathcal{I}$, $h \in \hom(\mathcal{B})$, yields a sequentially continuous MV-homomorphism $\mathrm{pr}_h \circ\varphi\colon \mathcal{A} \longrightarrow \mathcal{I}$. According to Lemma 3.2, $\mathrm{pr}_h \circ\varphi$ can be uniquely extended to a sequentially continuous MV-homomorphism $(\mathrm{pr}_h \circ\varphi)_\sigma\colon \sigma(\mathcal{A}) \longrightarrow \mathcal{I}$. Since $\mathcal{I}^{\hom(\mathcal{B})}$ is a product, there exists a unique sequentially continuous MV-homomorphism $\varphi_\sigma\colon \sigma(\mathcal{A}) \longrightarrow \mathcal{I}^{\hom(\mathcal{B})}$ such that $\mathrm{pr}_h \circ\varphi_\sigma = (\mathrm{pr}_h \circ\varphi)_\sigma$ for each $h \in \hom(\mathcal{B})$. Since $\mathcal{A}$ is sequentially dense in $\sigma(\mathcal{A})$, in fact $\sigma(\mathcal{A}) = \mathrm{cl}^{\omega_1} \mathcal{A}$, it follows that the restriction of φ_σ to $\mathcal{A}$ is equal to φ. Finally, $\mathcal{B} = \mathrm{cl}\,\mathcal{B}$ in $\mathcal{I}^{\hom(\mathcal{B})}$ implies that $\varphi_\sigma(\sigma(\mathcal{A})) \subseteq \mathcal{B}$. This completes the proof. □

Let ABD be the subcategory of BD consisting of absolutely sequentially closed bold algebras, i.e., Łukasiewicz tribes. Lemma 2.5, Corollary 2.8 and Proposition 3.2 provide characterizations of Łukasiewicz tribes. It follows from Proposition 3.6 that the embedding $\mathcal{A} \hookrightarrow \sigma(\mathcal{A})$ yields a functor σ from BD to ABD. Since the pointwise convergence has unique limits and $\mathcal{A}$ is sequentially dense in $\sigma(\mathcal{A}) = \mathrm{cl}^{\omega_1} \mathcal{A}$, if $\varphi, \varphi'\colon \sigma(\mathcal{A}) \longrightarrow \sigma(\mathcal{B})$ are two morphisms such that $\varphi(f) = \varphi'(f)$ for each $f \in \mathcal{A}$, then $\varphi = \varphi'$ (cf. [16]).

Corollary 3.7. *$\sigma\colon BD \longrightarrow ABD$ is an epireflector.*

Remark 3.8. Fields of sets are special bold algebras. Indeed, if $\mathbb{A}$ is a field of subsets of X, then $\mathbb{A} \subseteq \{0,1\}^X$ can be identified with the corresponding bold algebra $\mathcal{A} \subseteq \mathcal{I}^X$ and the generated σ-field $\sigma(\mathbb{A})$, as the smallest sequentially closed subset of $\{0,1\}^X$ containing $\mathbb{A}$, can be identified with the generated Łukasiewicz tribe $\sigma(\mathbb{A})$, as the smallest sequentially closed subset of $\mathcal{I}^X$ containing $\mathcal{A}$. It is known (cf. Frič [5]) that a field of sets $\mathbb{A}$ is a σ-field iff $\mathbb{A}$ is absolutely sequentially closed and the embedding $\mathbb{A} \hookrightarrow \sigma(\mathbb{A})$ yields an epireflection. The embedding of the category FS of fields of sets and sequentially continuous (Boolean) homomorphisms into BD preserves the embedding of σ-fields into Łukasiewicz tribes, i.e., Corollary 3.7 generalizes the epireflection of FS into AFS, the subcategory of absolutely sequentially closed fields of sets.

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