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INCIDENCE STRUCTURES OF TYPE (p, n)

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Abstract. Every incidence structure \mathcal{J} (understood as a triple of sets (G, M, I) , $I \subseteq G \times M$) admits for every positive integer p an incidence structure $\mathcal{J}^p = (G^p, M^p, I^p)$ where G^p (M^p) consists of all independent p -element subsets in G (M) and I^p is determined by some bijections. In the paper such incidence structures \mathcal{J} are investigated the \mathcal{J}^p 's of which have their incidence graphs of the simple join form. Some concrete illustrations are included with small sets G and M .

Keywords: incidence structures, independent sets

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Definition 1. Let G and M be sets and $I \subseteq G \times M$. Then the triple $\mathcal{J} = (G, M, I)$ is called an *incidence structure*.¹ If $A \subseteq G$, $B \subseteq M$, then we denote

$$A^\uparrow = \{m \in M \mid g I m \ \forall g \in A\}, \quad B^\downarrow = \{g \in G \mid g I m \ \forall m \in B\}.$$

Moreover, we denote $A^{\uparrow\downarrow} := (A^\uparrow)^\downarrow$, $B^{\downarrow\uparrow} := (B^\downarrow)^\uparrow$ for $A \subseteq G$, $B \subseteq M$.

Definition 2. An incidence structure $\mathcal{J}_1 = (G_1, M_1, I_1)$ is *embedded* into an incidence structure $\mathcal{J} = (G, M, I)$ if $G_1 \subseteq G$, $M_1 \subseteq M$ and $I_1 \subseteq I \cap (G_1 \times M_1)$. If $I_1 = I \cap (G_1 \times M_1)$, then \mathcal{J}_1 is a *substructure* of \mathcal{J} .

If we put $\mathcal{P}_G = \{A \subseteq G \mid A = A^{\uparrow\downarrow}\}$, then the pair $\mathcal{G} = (G, \mathcal{P}_G)$ is a (lower) closure space in which $X^{\uparrow\downarrow}$ is a closure of any subset $X \subseteq G$ in \mathcal{G} . A set $A \subseteq G$ is *independent* in \mathcal{G} if $a \notin (A - \{a\})^{\uparrow\downarrow}$ for all $a \in A$. In what follows we denote $A_a := A - \{a\}$.

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¹The triple (G, M, I) is called an incidence structure with regard to consecutive applications. The name “context” is used more frequently in literature—see [1] where the notation is taken from.

If $A \subseteq G$, then we put $X^A(a) := A_a^\uparrow - \{a\}^\uparrow$ for $a \in A$. Then $X^A(a) = \emptyset$ iff $A_a^\uparrow \subseteq \{a\}^\uparrow$ iff $a \in A_a^{\uparrow\downarrow}$. Hence the set A is independent in \mathcal{G} if and only if $X^A(a) \neq \emptyset$ for all $a \in A$. Moreover, $m \in X^A(a)$ iff $\{m\}^\downarrow \cap A = A_a$.

Let a non-empty subset $A \subseteq G$ be independent in \mathcal{G} . Then we put $\mathcal{X} = \{X^A(a) \mid a \in A\}$. For every choice $Q^A = \{m_a \in X^A(a) \mid X^A(a) \in \mathcal{X}\} \subseteq M$ from the set \mathcal{X} (which exists according to the axiom of choice) we define a map $\alpha: A \rightarrow Q^A$ by the formula $\alpha(a) = m_a$. This map is called an *A-norming map*.

If we put $\mathcal{P}_M = \{B \subseteq M \mid B = B^{\downarrow\uparrow}\}$, then $\mathcal{M} = (M, \mathcal{P}_M)$ is a (upper) closure space. A set $B \subseteq M$ is independent in \mathcal{M} if $m \notin (B - \{m\})^{\downarrow\uparrow} = B_m^{\downarrow\uparrow}$ for all $m \in M$. If $m \in B$, then we put $Y^B(m) = B_m^\downarrow - \{m\}^\downarrow$. B is independent in \mathcal{M} if and only if $Y^B(m) \neq \emptyset$ for all $m \in B$. Moreover, $a \in Y^B(m)$ iff $\{a\}^\uparrow \cap B = B_m$.

Let a non-empty set $B \subseteq M$ be independent in \mathcal{M} . Then we put $\mathcal{Y} = \{Y^B(m) \mid m \in B\}$. For every choice $Q^B = \{a_m \in Y^B(m) \mid Y^B(m) \in \mathcal{Y}\} \subseteq G$ we consider a map $\beta: B \rightarrow Q^B$ given by the formula $\beta(m) = a_m$. It will be called a *B-norming map*.

Theorem 1. *Let $A \subseteq G$, $B \subseteq M$ be independent sets in \mathcal{G} , \mathcal{M} , respectively. Then each A-norming map $A \rightarrow Q^A$ (each B-norming map $B \rightarrow Q^B$) is injective and the sets Q^A , Q^B are independent in \mathcal{M} , \mathcal{G} , respectively. (See [3].)*

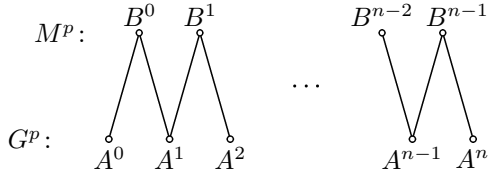
Remark 1. If $\alpha: A \rightarrow B$ is a map norming an independent set A of \mathcal{G} , then $\alpha^{-1}: B \rightarrow A$ is a map norming the independent set B of \mathcal{M} . Moreover, from $\alpha(a) = m_a$ for $a \in A$ we get $a \in Y^B(m_a)$.

Definition 3. Let us consider an incidence structure $\mathcal{J} = (G, M, I)$ and a positive integer $p \geq 2$. Let G^p and M^p be the sets of all independent sets of \mathcal{G} and \mathcal{M} , respectively, of cardinality p . Then $\mathcal{J}^p = (G^p, M^p, I^p)$ is an *incidence structure of independent sets* of \mathcal{J} , where $A I^p B$ if and only if there exists an A-norming map $\alpha: A \rightarrow B$ for $A \in G^p$, $B \in M^p$.

Remark 2. If $A \in G^p$, then $X^A(a) \neq \emptyset$ for all $a \in A$ and there exists a set $B \in M^p$ and a norming map $\alpha: A \rightarrow B$. Similarly for a set $B \in M^p$. Hence $A^\uparrow \neq \emptyset$, $B^\downarrow \neq \emptyset$ in \mathcal{J}^p for all $A \in G^p$, $B \in M^p$. If $G^p = \emptyset$, then $M^p = \emptyset$ and $\mathcal{J}^p = (\emptyset, \emptyset, \emptyset)$. For every incidence structure \mathcal{J} and for every $p \geq 2$ there exists a unique incidence structure \mathcal{J}^p .

Definition 4. $\mathcal{J} = (G, M, I)$ is said to be an *incidence structure of type (p, n)* , where $p > 1$, $n \geq 1$ are positive integers, if in $\mathcal{J}^p = (G^p, M^p, I^p)$ we have $G^p = \{A^0, \dots, A^n\}$, $M^p = \{B^0, \dots, B^{n-1}\}$ and $A^i I^p B^j$ iff $i = j$ or $i = j + 1$ for all $j \in \{0, \dots, n - 1\}$.

Remark 3. If \mathcal{J} is the structure of type (p, n) , then the incidence graph of the structure \mathcal{J}^p can be drawn in the form



and \mathcal{J}^p is called a *simple join*.

Theorem 2. If $\mathcal{J} = (G, M, I)$ is an incidence structure of type (p, n) , then

- (a) $|A^i \cap A^{i+1}| = p - 1$ for all $i \in \{0, \dots, n - 1\}$,
- (b) $|B^i \cap B^{i+1}| = p - 1$ for all $i \in \{0, \dots, n - 2\}$.

Proof. (a) Since $A^i, A^{i+1} \text{ I}^p B^i$ for all $i \in \{0, \dots, n - 1\}$, there exist norming mappings $\alpha_i: A^i \rightarrow B^i$, $\beta_i: B^i \rightarrow A^{i+1}$ and $\beta_i \alpha_i: A^i \rightarrow A^{i+1}$ is a bijective mapping of the sets A^i, A^{i+1} . We put $\alpha_i(a) = m_a$, $\beta_i(m_a) = a'$ for each $a \in A^i$. Since the inverse mapping $\alpha_i^{-1}: B^i \rightarrow A^i$, in which $\alpha_i^{-1}(m_a) = a$, is also norming, we get $a, a' \in Y^{B^i}(m_a)$ for each $a \in A^i$.

Let us suppose that there exist two distinct elements $b_1, b_2 \in A^{i+1} - A^i$. Then there exist distinct elements $a_1, a_2 \in A^i$ such that $\beta_i \alpha_i(a_1) = b_1$, $\beta_i \alpha_i(a_2) = b_2$. It is obvious that $a_1, b_1 \in Y^{B^i}(m_{a_1})$ and $a_2, b_2 \in Y^{B^i}(m_{a_2})$. If we put $A' = A_{a_i}^i \cup \{b_i\}$, then $|A'| = p$ and $A' \neq A^i, A^{i+1}$. We obtain $a \in Y^{B^i}(m_a)$ for all $a \in A_{a_i}^i$. The set A' is independent in \mathcal{G} and $\alpha: a \mapsto m_a$ for all $a \in A_{a_i}^i$, $b_1 \mapsto m_{a_1}$, is a norming mapping of the set A' to B^i . Hence $A' \text{ I}^p B^i$. However, this contradicts $A' \neq A^i, A^{i+1}$.

(b) It can be proved similarly to (a).

Notation. Since $|A^i \cap A^{i+1}| = p - 1$, we can put $R^i = A^i \cap A^{i+1}$, $A^i = \{a'_i\} \cup R^i$, $A^{i+1} = \{a_{i+1}\} \cup R^i$ for all $i \in \{0, \dots, n - 1\}$. In a similar way we put $Q^i = B^i \cap B^{i+1}$, $B^i = \{m'_i\} \cup Q^i$, $B^{i+1} = \{m_{i+1}\} \cup Q^i$.

Remark 4. In Theorems 3–7 we suppose that an incidence structure $\mathcal{J} = (G, M, I)$ of type (p, n) is given, where $G^p = \{A^0, \dots, A^n\}$, $M^p = \{B^0, \dots, B^{n-1}\}$, and all former notation is respected.

Theorem 3.

- 1. $\{a'_i\}^\uparrow \cap B^i = \{a_{i+1}\}^\uparrow \cap B^i$ for all $i \in \{0, \dots, n - 1\}$,
- 2. $\{m'_i\}^\downarrow \cap A^{i+1} = \{m_{i+1}\}^\downarrow \cap A^{i+1}$ for all $i \in \{0, \dots, n - 2\}$.

Proof. 1. There exist norming maps $\alpha_i: A^i \rightarrow B^i$, $\beta_i: B^i \rightarrow A^{i+1}$ for each $i \in \{0, \dots, n - 1\}$, where $\beta_i \alpha_i: A^i \rightarrow A^{i+1}$ is bijective. If $a \in A^i$, then we put

$\beta_i \alpha_i(a) = \beta_i(m_a) = \bar{a}$, where $a, \bar{a} \in Y^{B^i}(m_a)$. Assume that $a \in R^i$. If $a \neq \bar{a}$, then $a, \bar{a} \in A^{i+1}$ and $a, \bar{a} \in Y^{B^i}(m_a)$ implies a contradiction to the independence of A^{i+1} . Hence $a = \bar{a}$, $\beta_i \alpha_i(R^i) = R^i$ and $\beta_i \alpha_i(a'_i) = a_{i+1}$. This yields $a'_i, a_{i+1} \in Y^{B^i}(m_{a'_i})$ and thus $\{a'_i\}^\uparrow \cap B^i = \{a_{i+1}\}^\uparrow \cap B^i = B_{m_{a'_i}}^i$.

2. Since $A^{i+1} \text{ I P } B^i, B^{i+1}$ for each $i \in \{0, \dots, n-2\}$ by Definition 4, there exist norming mappings $\beta_i: B^i \rightarrow A^{i+1}$, $\alpha_{i+1}: A^{i+1} \rightarrow B^{i+1}$, where $\alpha_{i+1}\beta_i: B^i \rightarrow B^{i+1}$ is bijective. If we put $\alpha_{i+1}\beta_i(m) = \alpha_{i+1}(a_m) = \bar{m}$ for $m \in B^i$, then $m, \bar{m} \in X^{A^{i+1}}(a_m)$. Similarly to 1 we can show that $m'_i, m_{i+1} \in X^{A^{i+1}}(a_{m'_i})$. Thus $\{m'_i\}^\downarrow \cap A^{i+1} = \{m_{i+1}\}^\downarrow \cap A^{i+1} = A_{a_{m'_i}}^{i+1}$.

Theorem 4.

1. $a'_i \in \{m'_i\}^\downarrow \iff a'_i \notin \{m_{i+1}\}^\downarrow$,
2. $m'_i \in \{a'_{i+1}\}^\uparrow \iff m'_i \notin \{a_{i+2}\}^\uparrow$

for all $i \in \{0, \dots, n-2\}$.

Proof. 1. There exists a norming mapping $\alpha_i: A^i \rightarrow B^i$ because $A^i \text{ I P } B^i$. Since $m'_i \in B^i$, there exists an element $a' \in A^i$ such that $\alpha_i(a') = m'_i$. Then $m'_i \in X^{A^i}(a')$ and $\{m'_i\}^\downarrow \cap A^i = A_{a'}^i$. If we put $\alpha_i(a) = m_a$ for $a \in A_{a'}^i$, then $m_a \in X^{A^i}(a)$ and $Q^i = B^i \cap B^{i+1} = \{m_a \mid a \in A_{a'}^i\}$.

Let us assume that $a'_i \in \{m'_i\}^\downarrow, \{m_{i+1}\}^\downarrow$ or $a'_i \notin \{m'_i\}^\downarrow, \{m_{i+1}\}^\downarrow$. From Theorem 3 we obtain $\{m'_i\}^\downarrow \cap A^{i+1} = \{m_{i+1}\}^\downarrow \cap A^{i+1}$ and thus (by assumption) $\{m_{i+1}\}^\downarrow \cap A^i = \{m'_i\}^\downarrow \cap A^i = A_{a'}^i$. Hence $m_{i+1} \in X^{A^i}(a')$. From $B^{i+1} = \{m_{i+1}\} \cup Q^i$ it follows that $a' \mapsto m_{i+1}$, $a \mapsto m_a$ for $a \in A_{a'}^i$ is a norming mapping of A^i onto B^{i+1} . Thus $A^i \text{ I P } B^{i+1}$. It is a contradiction.

2. There exists a norming mapping $\beta_i: B^i \rightarrow A^{i+1}$ because $A^{i+1} \text{ I P } B^i$. Since $a'_{i+1} \in A^{i+1}$, there exists an element $m' \in B^i$ such that $\beta_i(m') = a'_{i+1}$. Then $a'_{i+1} \in Y^{B^i}(m')$ and $\{a'_{i+1}\}^\uparrow \cap B^i = B_{m'}^i$. If we put $\beta_i(m) = a_m$ for $m \in B_{m'}^i$, then $a_m \in Y^{B^i}(m)$ and $R^{i+1} = A^{i+1} \cap A^{i+2} = \{a_m \mid m \in B_{m'}^i\}$.

Let us assume that $m'_i \in \{a'_{i+1}\}^\uparrow, \{a_{i+2}\}^\uparrow$ or $m'_i \notin \{a'_{i+1}\}^\uparrow, \{a_{i+2}\}^\uparrow$. From Theorem 3 we obtain $\{a'_{i+1}\}^\uparrow \cap B^{i+1} = \{a_{i+2}\}^\uparrow \cap B^{i+1}$ and thus (by assumption) $\{a_{i+2}\}^\uparrow \cap B^i = \{a_{i+1}\}^\uparrow \cap B^i = B_{m'}^i$. Hence $a_{i+2} \in Y^{B^i}(m')$. From $A^{i+2} = \{a_{i+2}\} \cup R^{i+1}$ it follows that $\beta: m'_i \mapsto a_{i+2}$, $m \mapsto a_m$ for $m \in B_{m'}^i$ is a norming mapping of B^i onto A^{i+2} . Thus $A^{i+2} \text{ I P } B^i$. It is a contradiction. \square

Remark 5. Since $a'_i \in \{m'_i\}^\downarrow$ iff $a'_i \notin \{m_{i+1}\}^\downarrow$, we obtain $m'_i \in \{a'_i\}^\uparrow$ iff $m_{i+1} \notin \{a'_i\}^\uparrow$. Similarly $a'_{i+1} \in \{m'_i\}^\downarrow$ iff $a_{i+2} \notin \{m'_i\}^\downarrow$.

Theorem 5. Let $m'_{i+1} = m_{i+1}$. If $a'_{i+1} = a_{i+1}$, then $a'_i \in \{m'_i\}^\downarrow$ iff $a'_i \notin \{m_{i+2}\}^\downarrow$. If $a'_{i+2} = a_{i+2}$, then $m'_i \in \{a'_{i+1}\}^\uparrow$ iff $m'_i \notin \{a_{i+3}\}^\uparrow$.

Proof. Accepting the former notation we have $B^i = \{m'_i\} \cup Q^i$, $B^{i+1} = \{m_{i+1}\} \cup Q^i = \{m'_{i+1}\} \cup Q^{i+1}$, $B^{i+2} = \{m_{i+2}\} \cup Q^{i+1}$. Moreover, $Q^i = Q^{i+1}$ and $B^{i+2} = \{m_{i+2}\} \cup Q^i$ because of $m'_{i+1} = m_{i+1}$.

a) Let us assume that $a'_{i+1} = a_{i+1}$. Then $R^i = R^{i+1}$ and $A^{i+2} = \{a_{i+2}\} \cup R^i$. By Theorem 3 we obtain $\{m'_i\}^\downarrow \cap A^{i+1} = \{m_{i+1}\}^\downarrow \cap A^{i+1}$, hence $\{m'_i\}^\downarrow \cap R^i = \{m_{i+1}\}^\downarrow \cap R^i$. Moreover, $\{m'_{i+1}\}^\downarrow \cap A^{i+2} = \{m_{i+2}\}^\downarrow \cap A^{i+2}$. Since $R^i \subseteq A^{i+2}$, we obtain $\{m'_{i+1}\}^\downarrow \cap R^i = \{m_{i+2}\}^\downarrow \cap R^i$ and the equality $m'_{i+1} = m_{i+1}$ implies that $\{m'_i\}^\downarrow \cap R^i = \{m_{i+2}\}^\downarrow \cap R^i$.

Let us assume that either $a'_i \in \{m'_i\}^\downarrow, \{m_{i+2}\}^\downarrow$ or $a'_i \notin \{m'_i\}^\downarrow, \{m_{i+2}\}^\downarrow$. Since $A^i = \{a'_i\} \cup R^i$, $A^{i+2} = \{a_{i+2}\} \cup R^i$, we get $\{m'_i\}^\downarrow \cap A^i = \{m_{i+2}\}^\downarrow \cap A^i$. Since $A^i \text{ I}^p B^i$, there exists a norming mapping $\alpha_i: A^i \rightarrow B^i$. Let $\alpha_i(a') = m'_i$, $\alpha_i(a) = m_a$ for $a \in A'_a$. Then $\alpha_i(A'_a) = Q^i$. From $\{m'_i\}^\downarrow \cap A^i = A'_a$, it follows that $m'_i, m_{i+2} \in X^{A^i}(a')$. Hence $a' \mapsto m_{i+2}$, $a \mapsto m_a$ for $a \in A'_a$ is a norming mapping of the set A^i onto $B^{i+2} = \{m_{i+2}\} \cup Q^i$, i.e. $A^i \text{ I}^p B^{i+2}$. It is a contradiction.

b) Let us assume that $a'_{i+2} = a_{i+2}$. Then $R^{i+1} = R^{i+2}$ and $A^{i+3} = \{a_{i+3}\} \cup R^{i+1}$. By Theorem 3 we obtain $\{a'_{i+1}\}^\uparrow \cap B^{i+1} = \{a_{i+2}\}^\uparrow \cap B^{i+1}$, hence $\{a'_{i+1}\}^\uparrow \cap Q^i = \{a_{i+2}\}^\uparrow \cap Q^i$. Moreover, $\{a'_{i+2}\}^\uparrow \cap B^{i+2} = \{a_{i+3}\}^\uparrow \cap B^{i+2}$. Since $Q^i \subseteq B^{i+2}$, we obtain $\{a'_{i+2}\}^\uparrow \cap Q^i = \{a_{i+3}\}^\uparrow \cap Q^i$ and the equality $a'_{i+2} = a_{i+2}$ implies that $\{a'_{i+1}\}^\uparrow \cap Q^i = \{a_{i+3}\}^\uparrow \cap Q^i$.

Let us assume that either $m'_i \in \{a'_{i+1}\}^\uparrow, \{a_{i+3}\}^\uparrow$ or $m'_i \notin \{a'_{i+1}\}^\uparrow, \{a_{i+3}\}^\uparrow$. Then $\{a'_{i+1}\}^\uparrow \cap B^i = \{a_{i+3}\}^\uparrow \cap B^i$. By assumption $A^{i+1} \text{ I}^p B^i$, where $A^{i+1} = \{a'_{i+1}\} \cup R^{i+1}$. Hence there exists a norming mapping $\beta_i: B^i \rightarrow A^{i+1}$, where $\beta_i(m') = a'_{i+1}$ for a certain $m' \in B^i$ and $\beta_i(m) = a_m$ for $m \in B'_m$. Then $\beta_i(B'_m) = R^{i+1}$. From $\{a'_{i+1}\}^\uparrow \cap B^i = B'_m = \{a_{i+3}\}^\uparrow \cap B^i$ we get $a'_{i+1}, a_{i+3} \in Y^{B^i}(m')$. From $A^{i+3} = \{a_{i+3}\} \cup R^{i+1}$ we obtain that $m' \mapsto a_{i+3}$, $m \mapsto a_m$ for $m \in B'_m$ is a norming mapping of the set B^i onto A^{i+3} , i.e. $A^{i+3} \text{ I}^p B^i$. It is a contradiction. \square

Theorem 6. *If $0 \leq i \leq n-2$, then $a'_i \neq a_{i+1}, a'_{i+1}, a_{i+2}, a'_{i+2}$.*

Proof. Let us recall that $A^i = \{a'_i\} \cup R^i$, $A^{i+1} = \{a_{i+1}\} \cup R^i = \{a'_{i+1}\} \cup R^{i+1}$, $A^{i+2} = \{a_{i+2}\} \cup R^{i+1} = \{a'_{i+2}\} \cup R^{i+2}$.

1. If $a'_i = a_{i+1}$, then $A^{i+1} = A^i$. This is a contradiction.
2. Let $a'_i = a'_{i+1}$. If $a_{i+1} = a'_{i+1}$, then $a'_i = a_{i+1}$, a contradiction. If $a_{i+1} \neq a'_{i+1}$, then $a'_{i+1} \in R^i$ and $a'_i \in R^i$. This is a contradiction again.
3. We prove that $a'_i \neq a_{i+2}$. Since $A^i \text{ I}^p B^i$, there exists a norming mapping $\alpha_i: A^i \rightarrow B^i$, where $\alpha_i(a) = m_a$ for $a \in A^i$.

a) Let $a_{i+1} \neq a'_{i+1}$. From $a'_i \neq a'_{i+1}$ we obtain $m_{a'_i} \neq m_{a'_{i+1}}$. Hence $m_{a'_i} \neq m'_i$ or $m_{a'_{i+1}} \neq m'_i$. First assume that $m_{a'_{i+1}} \neq m'_i$. This yields $m_{a'_{i+1}} \in Q^i$ and $m_{a'_{i+1}} \in B^{i+1} = \{m_{i+1}\} \cup Q^i$. From $m_{a'_i} \in X^{A^i}(a'_i)$, $m_{a'_{i+1}} \in X^{A^i}(a'_{i+1})$ we obtain

$a'_i \text{ I } m_{a'_{i+1}}$ and $a'_{i+1} \not\text{I} m_{a'_{i+1}}$. By Theorem 3 $\{a'_{i+1}\}^\uparrow \cap B^{i+1} = \{a_{i+2}\}^\uparrow \cap B^{i+1}$, thus $a_{i+2} \not\text{I} m_{a'_{i+1}}$. Since $a'_i \text{ I } m_{a'_{i+1}}$, we get $a_{i+2} \neq a'_i$. If $m_{a'_i} \neq m'_i$, then we can proceed similarly.

b) Let $a_{i+1} = a'_{i+1}$. First we assume that $m'_i \in \{a'_i\}^\uparrow$. According to Theorem 3, $\{a'_i\}^\uparrow \cap B^i = \{a_{i+1}\}^\uparrow \cap B^i$, which implies $m'_i \in \{a_{i+1}\}^\uparrow$ and $m'_i \in \{a'_{i+1}\}^\uparrow$. From Theorem 4 we get $m'_i \notin \{a_{i+2}\}^\uparrow$. Hence $a'_i \text{ I } m'_i$, $a_{i+2} \not\text{I} m'_i$ and thus $a'_i \neq a_{i+2}$. If $m'_i \in \{a'_i\}^\uparrow$, then we can proceed similarly.

4. We show that $a'_i \neq a'_{i+2}$. If $a'_{i+2} = a_{i+2}$, then $a'_i \neq a'_{i+2}$ according to 3. Let $a'_{i+2} \neq a_{i+2}$. Then $a'_{i+2} \in R^{i+1}$. If $a_{i+1} = a'_{i+2}$, then $a'_i = a'_{i+2}$ implies $a'_i = a_{i+1}$. This is a contradiction to 1. Hence $a_{i+1} \neq a'_{i+2}$. From $a'_{i+2} \in R^{i+1}$ we obtain $a'_{i+2} \in R^i$ and thus $a'_i \neq a'_{i+2}$. \square

Remark 6. In an incidence structure of type (p, n) the case $a'_i = a_{i+3}$ is possible, as is shown in Fig. 5.

Theorem 7. *If $0 \leq i \leq n - 3$, then $m'_i \neq m_{i+1}, m'_{i+1}, m_{i+2}, m'_{i+2}$.*

Proof. Analogous to Theorem 6. \square

Theorem 8. *Let $\mathcal{J} = (G, M, \text{I})$ be an incidence structure and $p > 1$ a positive integer. Let $A^i \subseteq G$, $|A^i| = p$ for $i \in \{0, \dots, n\}$ and $B^i \subseteq M$, $|B^i| = p$ for $i \in \{0, \dots, n - 1\}$, where $n \geq 1$. Let the following conditions be valid:*

1. *The sets A^0, B^0 are independent in \mathcal{G}, \mathcal{M} , respectively, and there exists a norming mapping $\alpha_0: A^0 \rightarrow B^0$.*
2. *$|A^i \cap A^{i+1}| = p - 1$, $|B^i \cap B^{i+1}| = p - 1$ for all possible i .*
3. (a) *$\{a'_i\}^\uparrow \cap B^i = \{a_{i+1}\}^\uparrow \cap B^i$, $i \in \{0, \dots, n - 1\}$.*
 (b) *$\{m'_i\}^\downarrow \cap A^{i+1} = \{m_{i+1}\}^\downarrow \cap A^{i+1}$, $i \in \{0, \dots, n - 2\}$*
with respect to the former notation.

Then all sets A^i, B^i are independent in \mathcal{G}, \mathcal{M} , respectively, and $A^i \text{ I}^p B^j$ for $i = j$, $i = j + 1$, $j \in \{0, \dots, n - 1\}$.

Proof. Let all the assumptions hold. If $A^i \in G^p$, $B^i \in M^p$ for a certain $i \in \{0, \dots, n - 2\}$ and a norming mapping $\alpha_i: A^i \rightarrow B^i$ exists, then $A^{i+1} \in G^p$, $B^{i+1} \in M^p$ and there exist norming mappings $\beta_i: B^i \rightarrow A^{i+1}$, $\alpha_{i+1}: A^{i+1} \rightarrow B^{i+1}$. We have $A^i = \{a'_i\} \cup R^i$, $A^{i+1} = \{a_{i+1}\} \cup R^i$, $R^i = A^i \cap A^{i+1}$ with respect to our notation. If we put $\alpha_i(a) = m_a$ for $a \in A^i$, then $a \in Y^{B^i}(m_a)$ and $\{a\}^\uparrow \cap B^i = B^i_{m_a}$. According to 3(a), $\{a'_i\}^\uparrow \cap B^i = \{a_{i+1}\}^\uparrow \cap B^i = B^i_{m_{a'_i}}$ and thus $a'_i, a_{i+1} \in Y^{B^i}(m_{a'_i})$. Since $a \in Y^{B^i}(m_a)$ for $a \in R^i$, the set A^{i+1} is independent in \mathcal{G} and $\beta_i: m_a \mapsto a$ for $a \in R^i$, $m_{a'_i} \mapsto a_{i+1}$ is a norming mapping of the set B^i onto A^{i+1} . Hence $A^{i+1} \text{ I}^p B^i$. Moreover, $m_a \in X^{A^{i+1}}(a)$ for $a \in R^i$ and $m_{a'_i} \in X^{A^{i+1}}(a_{i+1})$.

If we put $B^i = \{m'_i\} \cup Q^i$, $B^{i+1} = \{m_{i+1}\} \cup Q^i$, where $Q^i = B^i \cap B^{i+1}$, then $\alpha_i(a') = m'_i$ for a certain $a' \in A^i$. According to 3(b) we have $\{m'_i\}^\downarrow \cap A^{i+1} = \{m_{i+1}\}^\downarrow \cap A^{i+1} = A^{i+1}$, which implies $m'_i, m_{i+1} \in X^{A^{i+1}}(a')$. Let $a' \in R^i$. Then $\alpha_{i+1}: a \mapsto m_a$ for $a \in R^i_{a'}$, $a' \mapsto m_{i+1}$, $a_{i+1} \mapsto m_{a'_i}$ is a norming mapping of the set A^{i+1} onto B^{i+1} and B^{i+1} is independent in \mathcal{M} . If $a' = a_{i+1}$, then $\alpha_{i+1}: a \mapsto m_a$ for $a \in R^i$, $a_{i+1} \mapsto m_{i+1}$ is a norming mapping of A^{i+1} onto B^{i+1} again. Thus $A^{i+1} \text{ I}^p B^{i+1}$.

By assumption 1 we get $A^0 \in G^p$, $B^0 \in M^p$ and $A^0 \text{ I}^p B^0$. Hence $A^1 \in G^p$, $B^1 \in M^p$ and $A^1 \text{ I}^p B^0, B^1$. This yields $A^2 \in G^p$, $B^2 \in M^p$, $A^2 \text{ I}^p B^1, B^2$ and so on. \square

Remark 7. Let the assumptions from Theorem 8 be valid. If we put $G^p_1 = \{A^0, \dots, A^n\}$, $M^p_1 = \{B^0, \dots, B^{n-1}\}$ and $A^i \text{ I}^p_1 B^j$ iff $i = j$, $i = j + 1$, then the incidence structure $\mathcal{J}^p_1 = (G^p_1, M^p_1, I^p_1)$ is embedded into \mathcal{J}^p .

Theorems 2–7 can be used to construct incidence structures of type (p, n) , as is shown in the following example.

Example. Let us construct the incidence tables of some incidence structures of type $(3, 3)$. Let $\mathcal{J} = (G, M, I)$ be an incidence structure of type $(3, 3)$. Then $G^3 = \{A^0, A^1, A^2, A^3\}$, $M^3 = \{B^0, B^1, B^2\}$, where $A_i \subset G$ for $i \in \{0, 1, 2, 3\}$ and $B^i \subset M$ for $i \in \{0, 1, 2\}$. In what follows we suppose that $G = \bigcup_{i=0}^3 A^i$, $M = \bigcup_{i=0}^2 B^i$. From Theorem 2 we obtain $A^0 = \{a'_0\} \cup R^0$, $A^1 = \{a_1\} \cup R^0 = \{a'_1\} \cup R^1$, $A^2 = \{a_2\} \cup R^1 = \{a'_2\} \cup R^2$, $A^3 = \{a_3\} \cup R^2$ and $B^0 = \{m'_0\} \cup Q^0$, $B^1 = \{m_1\} \cup Q^0 = \{m'_1\} \cup Q^1$, $B^2 = \{m_2\} \cup Q^1$.

Moreover, we will assume that the following conditions are satisfied:

- (P1) $R^0 \neq R^1 \neq R^2 \neq R^0$,
- (P2) $Q^0 \neq Q^1$,
- (P3) $a_3 \neq a'_0$, $a_1 \neq a'_2$.

According to (P1), (P3) and Theorem 6, a'_i, a_j are distinct elements for all possible i, j . From $R^0 \neq R^1$ and $R^1 \neq R^2$ we obtain $a'_1 \in R^0$ and $a'_2 \in R^1$. The condition $a_1 \neq a'_2$ implies $a'_2 \in R^0$. Hence $R^0 = \{a'_1, a'_2\}$, $R^1 = \{a_1, a'_2\}$, $R^2 = \{a_1, a_2\}$. Similarly $m'_1 \in Q^0$. If we put $Q^0 = \{m'_1, m'_2\}$, then m'_i, m_j are distinct elements and $Q^1 = \{m_1, m'_2\}$. There exist a norming set $\alpha: A^0 \rightarrow B^0$ by assumptions.

1. Assume that $\alpha(a'_0) = m'_0$. We select such a notation that $\alpha(a'_1) = m'_1$, $\alpha(a'_2) = m'_2$ (see Tab. 1). By Theorem 3 we get $\{a'_0\}^\uparrow \cap B^0 = \{a_1\}^\uparrow \cap B^0$ and $\{m'_0\}^\downarrow \cap A^1 = \{m_1\}^\downarrow \cap A^1$. From Theorem 4, $a'_0 \notin \{m'_0\}^\downarrow$ implies $a'_0 \in \{m_1\}^\downarrow$ and thus $a'_0 \text{ I} m_1$. Moreover, $\{a'_1\} \cap B^1 = \{a_2\}^\uparrow \cap B^1$ by Theorem 3 and $m'_0 \in \{a'_1\}^\uparrow$ implies $m'_0 \notin \{a_2\}^\uparrow$ by Theorem 4. Thus $a_2 \not\text{I} m'_0$.

We know that $\{m'_1\}^\perp \cap A^2 = \{m_2\}^\perp \cap A^2$ and $a'_1 \notin \{m'_1\}^\perp$ implies $a'_1 \in \{m_2\}^\perp$. Thus $a'_1 \text{ I } m_2$. Finally, we obtain $\{a'_2\}^\uparrow \cap B^2 = \{a_3\}^\uparrow \cap B^2$ and $m'_1 \notin \{a_3\}^\uparrow$ because of $m'_1 \in \{a'_2\}^\uparrow$. Thus $a_3 \not\text{I } m'_1$.

It remains to decide about the incidence of elements a'_0, m_2 and a_3, m'_0 . If $a'_0 \not\text{I } m_2$, then for instance $A^0 \text{ I } B^1$. This is a contradiction and hence $a'_0 \text{ I } m_2$.

I	m'_0	m'_1	m'_2	m_1	m_2
a'_0		—	—	—	—
a'_1	—		—	—	—
a'_2	—	—		—	—
a_1		—	—		—
a_2			—	—	
a_3	?			—	—

Table 1.

My colleague Dr. V. Tichý has devised a computer program assigning to every incidence structure $\mathcal{J} = (G, M, I)$ for $|G|, |M| < 12$ all incidence structures \mathcal{J}^p of independent sets of \mathcal{J} . In the figures enclosed part a) shows the incidence table of the structure \mathcal{J} , parts b), c) show all independent sets in \mathcal{G}, \mathcal{M} , respectively, and part d) shows the incidence graph of the structure \mathcal{J}^p . Fig. 1 illustrates the described incidence structure \mathcal{J} for $a_3 \not\text{I } m'_0$ and Fig. 2 for $a_3 \text{ I } m'_0$. Both structures are of type (3,3).

2. Assume that $\alpha(a'_0) \neq m'_0$. Let for instance $\alpha(a'_0) = m'_2, \alpha(a'_1) = m'_1, \alpha(a'_2) = m'_0$. Fig. 3 shows such an incidence structure \mathcal{J} of type (3,3) which is assigned similarly to 1.

Incidence structures in Figs. 1, 2, 3 are not isomorphic.

Figs. 4, 5 illustrate incidence structure of type (3,3), in which conditions $(P_1), (P_2)$ are satisfied but $a_3 \neq a'_0, a_1 = a'_2$, and $a_3 = a'_0, a_1 \neq a'_2$, respectively.

An incidence structure of type (5,4), where $R^0 = R^1$ and $Q^1 = Q^2$, is in Fig. 6.

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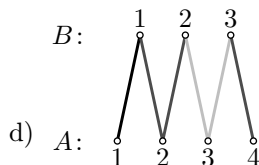
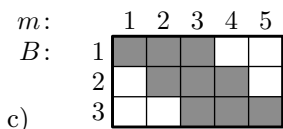
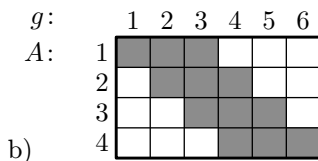
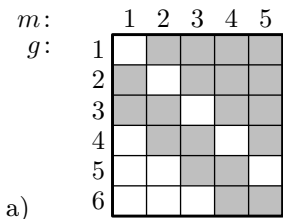


Fig. 1

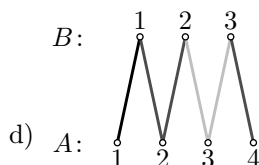
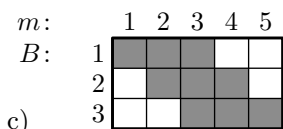
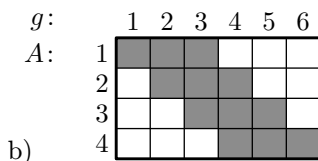
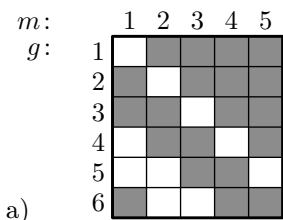


Fig. 2

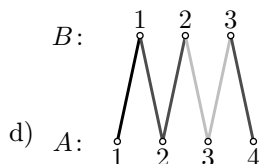
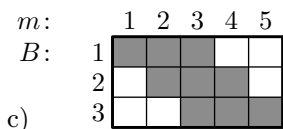
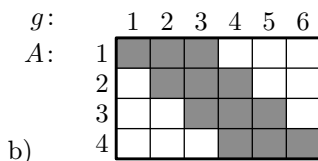
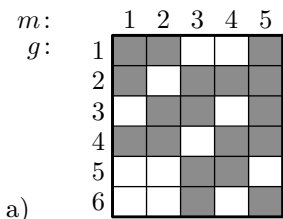
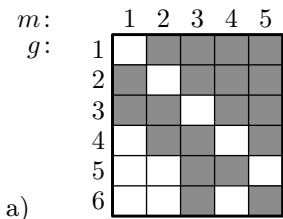
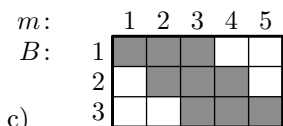


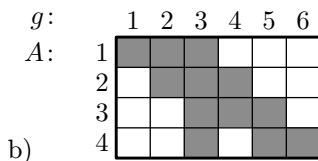
Fig. 3



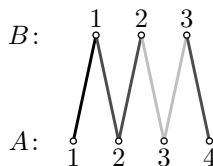
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c)

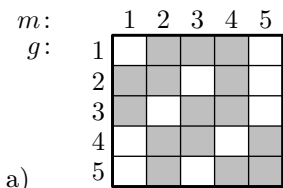


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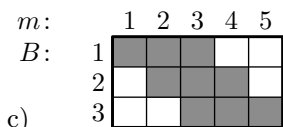


d)

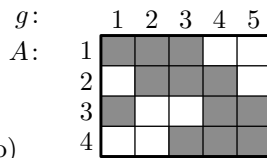
Fig. 4



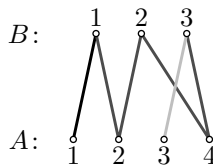
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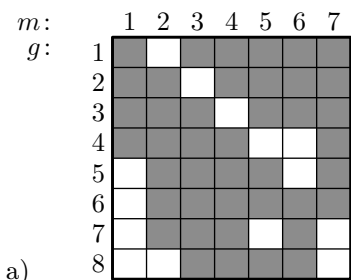


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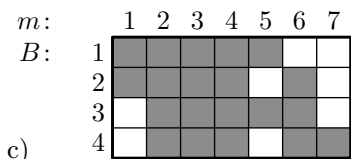


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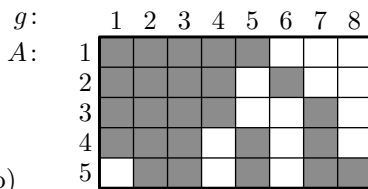
Fig. 5



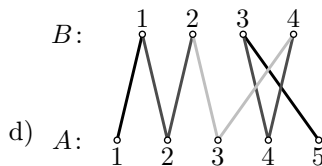
a)



c)



b)



d)

Fig. 6