A. D. Gadjiev; R. O. Efendiyev; E. Ibikli

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Czechoslovak Mathematical Journal, Vol. 53 (2003), No. 1, 45–53

Persistent URL: `http://dml.cz/dmlcz/127779`

ON KOROVKIN TYPE THEOREM IN THE SPACE OF LOCALLY INTEGRABLE FUNCTIONS

A. D. Gadjiev, R. O. Efendİyev, Baku, and E. İbİklİ, Ankara

(Received November 24, 1999)

Abstract. It is shown that a Korovkin type theorem for a sequence of linear positive operators acting in weighted space $L_{p,w}(\mathrm{loc})$ does not hold in all this space and is satisfied only on some subspace.

Keywords: linear positive operators, Korovkin type theorem, weighted $L_p(\mathrm{loc})$ spaces

MSC 2000: 41A36, 41A25

1.

A Korovkin type theorem for linear positive operators acting from $L_p(a,b)$ to $L_p(a,b)$ was studied in [4] and then some new results in this direction were established. We refer to the papers [1], [2], [3], [8], [10], [11], [12] [13]. Note that all the results just mentioned are devoted to the case of a finite interval $[a,b]$.

We consider the space of locally integrable functions on the entire real axis, and the sequences of linear positive operators defined in this space.

For $w(x) = 1 + x^2$, $-\infty < x < \infty$ and any fixed $h > 0$, we will denote by $L_{p,w}(\mathrm{loc})$ the space of measurable functions f satisfying the inequality

$$\left(\frac{1}{2h}\int_{x-h}^{x+h} |f(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \leqslant M_f w(x), \quad -\infty < x < \infty, \tag{1}$$

where $p \geqslant 1$ and M_f is a positive constant which depends on the function f. Obviously, $L_{p,w}(\mathrm{loc})$ is a linear normed space with norm

$$\|f\|_{p,w} = \sup_{-\infty<x<\infty} \frac{\left(\frac{1}{2h}\int_{x-h}^{x+h} |f(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}}{w(x)}$$

($\|f\|_{p,w}$ may depend also on h). To simplify notation, we need the following. For any finite real numbers a and b put

$$\|f; L_p(a,b)\| = \left(\frac{1}{b-a}\int_a^b |f(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}, \tag{2}$$

$$\|f; L_{p,w}(a,b)\| = \sup_{a\leqslant x\leqslant b} \frac{\|f; L_p(x-h,x+h)\|}{w(x)}, \tag{3}$$

$$\|f; L_{p,w}(|x|\geqslant a)\| = \sup_{|x|\geqslant a} \frac{\|f; L_p(x-h,x+h)\|}{w(x)}. \tag{4}$$

It follows that the norm in $L_{p,w}(\mathrm{loc})$ may be written in the form

$$\|f\|_{p,w} = \sup_{-\infty<x<\infty} \frac{\|f; L_p(x-h,x+h)\|}{w(x)}.$$

Let $L^k_{p,w}(\mathrm{loc})$ be the subspace of all functions $f \in L_{p,w}(\mathrm{loc})$ for which there exists a constant k_f such that

$$\lim_{|x|\to\infty} \frac{\|f - k_f w; L_p(x-h,x+h)\|}{w(x)} = 0. \tag{5}$$

In the case of $k_f = 0$ we will write $L^0_{p,w}(\mathrm{loc})$. Korovkin type theorems for sequence of linear positive operators acting in weighted space of continuous functions, defined on the entire real line, were studied in [5][1], [6]. We will study these theorems in the space $L_{p,w}(\mathrm{loc})$. The set of all linear positive operators acting from $L_{p,w}(\mathrm{loc})$ to $L_{p,w}(\mathrm{loc})$ will be denoted by $(L_{p,w}(\mathrm{loc}) \to L_{p,w}(\mathrm{loc}))^+$.

2.

We shall deal with the following problem.

Let the sequence of operators $L_n \in (L_{p,w}(\mathrm{loc}) \to L_{p,w}(\mathrm{loc}))^+$ satisfy the conditions:

i) The norms of these operators are uniformly bounded, that is,

$$\|L_n\| \leqslant C < \infty, \tag{6}$$

where C is a constant independent of n;

[1] A. D. Gadjiev = A. D. Gadžiev (also in other translation papers A. D. Gadzhiev, A. D. Gadziev).

ii) For $m = 0, 1, 2$

$$\lim_{n\to\infty} \|L_n(t^m; x) - x^m\|_{p,w} = 0, \tag{7}$$

where $L_n(t^m; x) := L_n(t^m)(x)$.

Is it possible to assert then that for each function $f \in L_{p,w}(\mathrm{loc})$

$$\lim_{n\to\infty} \|L_n(f; x) - f(x)\|_{p,w} = 0?$$

We show that the answer to this question is negative.

Our main result is the following.

Theorem 1. *There exists a sequence of operators $L_n \in (L_{p,w}(\mathrm{loc}) \to L_{p,w}(\mathrm{loc}))^+$ satisfying conditions* (6), (7) *and there exists a function f^* in $L_{p,w}(\mathrm{loc})$ for which*

$$\overline{\lim_{n\to\infty}} \|L_n f^* - f^*\|_{p,w} \geqslant 2^{1-\frac{1}{p}}.$$

P r o o f. We define a sequence of operators L_n by the formulas

$$L_n(f; x) = \begin{cases} \dfrac{x^2}{(x+h)^2} f(x+h) & \text{for } (2n-2)h \leqslant x \leqslant (2n+1)h, \\ f(x) & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\|L_n f\|_{p,w} \leqslant 4\|f\|_{p,w},$$

that is, L_n are bounded operators belonging to $(L_{p,w}(\mathrm{loc}) \to L_{p,w}(\mathrm{loc}))^+$ and (6) holds.

Now, for $m = 0, 1$,

$$\|L_n(t^m; x) - x^m\|_{p,w} \leqslant \sup_{(2n-1)h \leqslant x \leqslant 2nh} \frac{(x+h)^m}{1+x^2} \leqslant \frac{(2n+1)^m h^m}{1+4h^2(2n-1)^2}$$

and therefore

$$\lim_{n\to\infty} \|L_n(t^m; x) - x^m\|_{p,w} = 0, \quad m = 0, 1.$$

Also, since $L_n(t^2; x) = x^2$, the conditions (7) hold.

Consider the function

$$f^*(x) = \begin{cases} x^2 & \text{if } x \in \bigcup\limits_{k=1}^{\infty} [(2k-1)h, 2kh), \\ -x^2 & \text{if } x \in \bigcup\limits_{k=0}^{\infty} (2kh, (2k+1)h], \\ 0 & \text{if } x < 0. \end{cases}$$

Then $f^* \in L_{p,w}(\mathrm{loc})$ and we get

$$\|L_n f^* - f^*\|_{p,w} = \sup_{(2n-1)h<x<(2n+1)h} \frac{\|L_n f^* - f^*; L_p((2n-1)h,(2n+1)h)\|}{w(x)}$$
$$\geqslant \frac{1}{w(2nh)} \left(\frac{1}{2h} \int_{(2n-1)h}^{2nh} \left| \frac{y^2}{(y+h)^2} f^*(y+h) - f^*(y) \right|^p \mathrm{d}y \right)^{\frac{1}{p}}$$
$$\geqslant 2^{1-\frac{1}{p}} \frac{(2n-1)^2h^2}{1+4n^2h^2}.$$

The theorem is proved. □

3.

Now we show that the above mentioned problem has a positive solution in the subset $L^k_{p,w}(\mathrm{loc})$. First we give the following simple proposition.

Lemma. *Let the sequence of operators $A_n \in (L_{p,w}(\mathrm{loc}) \to L_{p,w}(\mathrm{loc}))^+$ satisfy the three conditions:*

$$\lim_{n\to\infty} \|A_n(t^m; x) - x^m\|_{p,w} = 0 \quad (m = 0, 1, 2).$$

Then, for any continuous and bounded function f on the real axis,

$$\lim_{n\to\infty} \|A_n f - f; L_{p,w}(a,b)\| = 0$$

holds, where a and b are any real numbers.

The proof of this Lemma is conducted in the same way as in the case of the space $C(a,b)$.

Since f is uniformly continuous function on any closed interval, given $\varepsilon > 0$ there exists a positive $\delta = \delta(\varepsilon)$ such that

$$|f(t) - f(x)| < \varepsilon \quad \text{if } |t-x| < \delta, \quad x \in [a,b], \quad t \in \mathbb{R}.$$

Also, setting $M = \sup\limits_{x\in\mathbb{R}} |f(x)|$, we can write

$$|f(t) - f(x)| < 2M \quad \text{if } |t-x| \geqslant \delta \quad x \in [a,b], \quad t \in \mathbb{R}.$$

Therefore, from the basic inequality

$$|f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2}(t-x)^2,$$

where $-\infty < t < \infty$, $x \in [a, b]$, it follows that

$$\begin{aligned}\|A_n f - f; L_{p,w}(a,b)\| \leqslant \varepsilon + M\|A_n(1;x) - 1; L_{p,w}(a,b)\| \\ + \frac{2M}{\delta^2}\|A_n((t-x)^2;x); L_{p,w}(a,b)\|\end{aligned}$$

and the last two terms tend to zero as $n \to \infty$ by the conditions of the Lemma.

Theorem 2. *If $A_n \in (L_{p,w}(\mathrm{loc}) \to L_{p,w}(\mathrm{loc}))^+$ is a sequence of operators satisfying the conditions* (6) *and* (7)*, then*

$$\lim_{n\to\infty} \|A_n f - f\|_{p,w} = 0$$

for each function $f \in L^k_{p,w}(\mathrm{loc})$.

P r o o f. Since $f \in L^k_{p,w}(\mathrm{loc})$ implies that $f - k_f$, $w \in L^0_{p,w}(\mathrm{loc})$, it is sufficient to prove the theorem for the function $f \in L^0_{p,w}(\mathrm{loc})$. For $\varepsilon > 0$, there exists a point x_0 such that the inequality

$$\left(\frac{1}{2h}\int_{x-h}^{x+h} |f(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} < \varepsilon w(x) \tag{8}$$

holds for all x, $|x| \geqslant x_0$.

By the well known Luzin Theorem (see, for example [7]), there exists a continuous function φ on the finite interval $[-x_0 - h, x_0 + h]$ such that the inequality

$$\|f - \varphi; L_p(-x_0, x_0)\| < \varepsilon \tag{9}$$

is fulfilled. Setting

$$\delta < \min\left\{\frac{2h\varepsilon^p}{M^p(x_0)}, h\right\}, \tag{10}$$

where $M(x_0) = \max\left\{\max\limits_{|x|\leqslant x_0+h} |\varphi(x)|, 1\right\}$, we can construct a continuous function g by the formulas

$$g(x) = \begin{cases} \varphi(x) & \text{if } |x| \leqslant x_0 + h, \\ 0 & \text{if } |x| \geqslant x_0 + h + \delta, \\ \text{linear} & \text{otherwise.} \end{cases}$$

Then, by (8), (9), (10) and the Minkowski inequality, we obtain

$$\begin{aligned}
\|f-g\|_{p,w} &\leqslant \|f-g; L_{p,w}(-x_0,x_0)\| + \|f-g; L_{p,w}(|x|\geqslant x_0+h+\delta)\| \\
&\quad + \|f-g; L_{p,w}(x_0,x_0+h+\delta)\| \\
&\quad + \|f-g; L_{p,w}(-x_0-h-\delta,-x_0)\| \\
&< 2\varepsilon + \|f-g; L_p(x_0-h,x_0+h)\| + \frac{1}{w(x_0)}\|f; L_p(x_0+h,x_0+2h+\delta)\| \\
&\quad + \|g; L_p(x_0+h,x_0+2h+\delta)\| + \frac{1}{w(x_0)}\|f; L_p(-x_0-2h-\delta,-x_0-h)\| \\
&\quad + \|g; L_p(-x_0-h-\delta,-x_0-h)\| \\
&\quad + \|f; L_p(-x_0-h,-x_0+h)\| \\
&< 4\varepsilon + 2M(x_0)\Big(\frac{\delta}{2h}\Big)^{\frac{1}{p}} + \frac{1}{w(x_0)}\|f; L_p(x_0+h,x_0+3h)\| \\
&\quad + \frac{1}{w(x_0)}\|f; L_p(-x_0-3h,-x_0-h)\|
\end{aligned}$$

and, on using the inequality (1),

$$\|f-g\|_{p,w} \leqslant 6\varepsilon + 2\varepsilon\frac{w(x_0+2h)}{w(x_0)} < C_1\varepsilon,$$

where $C_1 = 6 + 2(1+2h)^2$, since $w(x) = 1+x^2$.

Consequently, for each $f \in L^0_{p,w}(\mathrm{loc})$, there exists a continuous and bounded function g such that

$$\|f-g\|_{p,w} < C_1\varepsilon \tag{11}$$

for any $\varepsilon > 0$.

Now we can find a point $x_1 > x_0$ such that

$$w(x_1) > \frac{M(x_0)}{\varepsilon} \quad \text{and} \quad g(x) = 0 \ \text{ for } \ |x| > x_1, \tag{12}$$

where $M(x_0)$ is defined above. Then, by (6), (11) and the Lemma (cf., e.g., [9], pp. 28, 29),

$$\begin{aligned}
\|L_nf-f\|_{p,w} &\leqslant \|L_n(f-g)\|_{p,w} + \|L_ng-g\|_{p,w} + \|f-g\|_{p,w} \\
&\leqslant (C+1)\|f-g\|_{p,w} + \|L_ng-g\|_{p,w} \\
&\leqslant (C+1)\varepsilon + \|L_ng-g; L_{p,w}(-x_1,x_1)\| + \|L_ng; L_{p,w}(|x|\geqslant x_1)\| \\
&\leqslant (C+2)\varepsilon + \|L_ng; L_{p,w}(|x|\geqslant x_1)\|.
\end{aligned}$$

Since $|g(x)| \leqslant M(x_0)$ for all $x \in \mathbb{R}$, we can write

$$\begin{aligned}\|L_n g; L_{p,w}(|x| \geqslant x_1)\| &\leqslant M(x_0)\|L_n 1; L_{p,w}(|x| \geqslant x_1)\| \\ &\leqslant M(x_0)\|L_n 1 - 1\|_{p,w} + M(x_0)\|1; L_{p,w}(|x| \geqslant x_1).\end{aligned}$$

Therefore

$$\|L_n f - f\|_{p,w} \leqslant (C+2)\varepsilon + M(x_0)\|L_n 1 - 1\|_{p,w} + \frac{M(x_0)}{w(x_1)}.$$

In view of (7) and (12), the proof is completed. □

4.

Theorem 2 gives a way of approximating all functions in $L_p(\mathrm{loc})$. Namely, we have the following result.

Theorem 3. *Let a sequence of linear operators $A_n \in (L_{p,w}(\mathrm{loc}) \to L_{p,w}(\mathrm{loc}))^+$ satisfy the conditions* i) *and* ii). *Then, for all functions $f \in L_{p,w}(\mathrm{loc})$,*

$$\lim_{n\to\infty} \sup_{-\infty<x<\infty} \frac{\|A_n f - f; L_p(x-h, x+h)\|}{1+|x|^{2-\varepsilon}} = 0,$$

where $\varepsilon > 0$ is any positive number (ε cannot be zero).

P r o o f. By the condition (6) for any fixed x_0

$$a_n = \sup_{|x|>x_0} \frac{\|A_n f - f; L_p(x-h, x+h)\|}{1+x^2}$$

is bounded, if $f \in L_{p,w}(\mathrm{loc})$.

Also by the Lemma for any fixed x_0,

$$b_n = \sup_{|x|\leqslant x_0} \frac{\|A_n f - f; L_p(x-h, x+h)\|}{1+x^2}$$

tends to zero as $n \to \infty$. Indeed, by the Luzin Theorem there exists a continuous function φ_1 on the interval $[-x_0 - h, x_0 + h]$ such that the inequality

$$\|f - \varphi_1; L_p(-x_0 - h, x_0 + h)\| < \varepsilon \tag{13}$$

is fulfilled. Moreover, setting

$$\begin{aligned}\varphi_1(x) &= \varphi_1(-x_0 - h) \quad \text{if} \quad x \leqslant x_0 - h,\\ \varphi_1(x) &= \varphi_1(x_0 + h) \qquad \text{if} \quad x \geqslant x_0 + h,\end{aligned}$$

we see that φ_1 is continuous and bounded function on the whole real axis, for which the Lemma holds. Now,

$$\begin{aligned} b_n &\leqslant \|A_n f - f; L_p(-x_0 - h, x_0 + h)\| \\ &\leqslant (1 + \|A_n\|)\|f - \varphi_1; L_p(-x_0 - h, x_0 + h)\| + \|A_n\varphi_1 - \varphi_1; L_p(-x_0 - h, x_0 + h)\| \end{aligned}$$

and $\lim\limits_{n\to\infty} b_n = 0$ by (6), (13) and the Lemma.

All that remains to see is that

$$\sup_{-\infty<x<\infty} \frac{\|A_n f - f; L_p(x - h, x + h)\|}{1 + |x|^{2+\varepsilon}} \leqslant (1 + x_0^2)b_n + a_n \sup_{|x|\geqslant x_0} \frac{1 + x^2}{1 + |x|^{2+\varepsilon}}$$

and the proof is completed since the last statement concerning ε follows from Theorem 1.

Acknowledgment. We are very thankful to the referee for the attentive reading of the manuscript and for valuable comments and suggestions.

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Authors' addresses: A. D. Gadjiev, Institut of Mathematics and Mechanics, Academy of Sciences of The Azerbaijan Republic, Azerbaijan Academy of National Aviation, Baku, Azerbaijan; currently: Ankara University Faculty of Sciences, Department of Mathematics, 06100 Tandogan, Ankara, Turkey, e-mail: haciyev@science.ankara.edu.tr; R. O. Efendiyev, Institut of Mathematics and Mechanics, Academy of Sciences of the Azerbaijan Republic, Baku, Azerbaijan, e-mail: frteb@aas.ab.az; E. İbikli, Ankara University, Faculty of Sciences, Department of Mathematics, 06100 Tandogan, Ankara, Turkey, e-mail: ibikli@science.ankara.edu.tr.