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ON ORDERED DIVISION RINGS

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Abstract. Prestel introduced a generalization of the notion of an ordering of a field, which is called a semiordering. Prestel's axioms for a semiordered field differ from the usual (Artin-Schreier) postulates in requiring only the closedness of the domain of positivity under $x \to xa^2$ for nonzero a, instead of requiring that positive elements have a positive product. In this work, this type of ordering is studied in the case of a division ring. It is shown that it actually behaves the same as in the commutative case. Further, it is shown that the bounded subring associated with that ordering is a valuation ring which is preserved under conjugation, so one can associate a natural valuation to a semiordering.

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1.

The investigation of ordered fields has a long tradition. They play an important part in many branches of mathematics. In [1], Prestel introduced a generalization of an ordering of a field, which is called a semiordering. Prestel's axioms for a semiordered field differ from the usual (Artin-Schreier) postulates in requiring only the closedness of the positive cone under $x \to xa^2$ for non-zero a, instead of requiring that positive elements have a positive product. This generalization of positive cones and orderings is based on the following observation. Very often one only uses the property $x > 0 \Rightarrow xa^2 > 0$ of an ordering together with $1 > 0$. This is especially the case if one deals with quadratic forms. In this work, this type of ordering is studied in the case of a division ring. It is shown that it actually behaves the same as in the commutative case. For example, a division ring admits a semiordering if and only if -1 is not a sum of products of squares. In fact, every semiordered division ring is ordered. Moreover, every archimedean semiordered division ring is an ordered field. Further, the existence of a natural valuation associated to a semiordering is investigated, this requires the study of the bounded subring associated to a given semiordering. This is the subring consisting of elements which are bounded by some rational number with respect to the semiordering. It is shown that the bounded subring is a valuation ring which is preserved under conjugation, so one can associate a natural valuation to a semiordering. To study extensions of semiordering to larger division rings, the notion of pre-semiordering is investigated. Finally, an example of a semiordering, which is not an ordering for a division ring is given.

2.

Throughout this work, D denotes a (not necessarily commutative) division ring, and $D^\bullet$ denotes its multiplicative group of non-zero elements.

Definition. A semiordering of a division ring D is an order relation $<$ such that D contains a subset P (the positive cone) satisfying

(1) $P + P \subset P$;
(2) $a \in P \Rightarrow ab^2 \in P$, for $0 \neq b \in D$;
(3) $0 \notin P$ and $1 \in P$;
(4) $P \cup \{0\} \cup -P = D$;

then, $a > b \Leftrightarrow a - b \in P$, and $P = \{a \in D / a > 0\}$.

Note that this definition is the same as in the commutative case (see [1]). Clearly any ordering of D is also a semiordering.

Let C be the subset of all finite sums of elements of the form $a_1^2 a_2^2 \dots a_k^2$ in D with every $a_i \neq 0$. Clearly, C is closed under sums and products. Also, C contains inverses (for $c \in C$, $c^{-2} \in C \Rightarrow c^{-1} = cc^{-2} \in C$). If D is a semiordered division ring, then for $a \in P$ and $c \in C$ we have $ac \in P$ (by applying conditions (1) and (2) of the above definition several times). Since $1 \in P$, then clearly $C \subset P$. Also, $-1 \notin P$ implies that $-1 \notin C$. So by [2], Theorem 1, D is an ordered division ring. Hence we have

Theorem 1. *Every semiordered division ring is ordered.*

Although a semiordered division ring is ordered, there is no guarantee that the given semiordering is an ordering. At the end of this work, an example of a semiordering which is not an ordering is given.

Corollary 2. *A division ring D admits a semiordering if and only if $-1 \notin C$.*

One can prove the following properties of semiorderings.

Lemma 3. *Let D be any semiordered division ring, $a \in D$. Then*

(1) $a > 0$ *if and only if* $a^{-1} > 0$;
(2) *if* $a > 0$, *then* $d^2 a > 0$ *for* $0 \neq d \in D$ *and hence* $ca > 0$ *for* $c \in C$;
(3) *if* $a > 0$, *then* $ra > 0$ *for every* $r \in \mathbb{Q}^+$;
(4) *if* $a > 1$, *then* $a^{-1} < 1$;
(5) *if* $0 < a < b$, $a, b \in C$ *then* $a^{-1} > b^{-1}$;
(6) *if* $0 < a < b$, $a \in C$ *then* $a^2 < b^2$;
(7) *if* $0 < a < b$, $b \in C$ *then* $a^2 < b^2$;
(8) *if* $a > 0$, *then* $xax^{-1} > 0$ *for all* $x \in D^\bullet$.

Theorem 4. *A semiordering $>$ is an ordering of D if and only if for all $a, b \in D$ the inequality $0 < a < b$ implies $a^2 < b^2$.*

P r o o f. We first claim that for every $a, b \in D$, $a, b > 0$, we have $ba+ab > 0$. Since $a-b > 0$ or $b-a > 0$, we may assume that $0 < a < b$. Clearly, $a+b > b > b-a > 0$ and so $(a+b)^2 > b^2 > (b-a)^2$. Hence $2(ba+ab) > 0$. By Lemma 3, part (3), $ba+ab > 0$. Next, we claim that $bab > 0$ and $bab^{-1} > 0$ for every $a, b > 0$ in D. If $bab < 0$, then $-bab > 0$ and $a > 0$ implies $(-bab)a + a(-bab) > 0$. Thus $-(ba)^2 - (ab)^2 > 0$, which is a contradiction. Hence $bab > 0$ and $bab^{-1} = babb^{-2} > 0$. Now assume $ab < 0$ for $a, b > 0$. Then $-ab > 0$ and $-ba = b(-ab)b^{-1} > 0$. So $-ab - ba > 0$, which is a contradiction. Hence $ab > 0$ for every $a, b > 0$ and the semiordering is an ordering. □

A semiordering $<$ on a division ring D is called archimedean if for every $a \in D$ there is a natural number n such that $a < n$.

Proposition 5. *A semiordering $<$ on a division ring D is archimedean if and only if $\mathbb{Q}$ is dense in D with respect to $<$ (i.e., if $a < b$ then there is $r \in \mathbb{Q}$ such that $a < r < b$).*

P r o o f of Proposition 5 is similar to that for the case of a field (see [1]). □

Theorem 6. *Every archimedean semiordered division ring D is an ordered field.*

P r o o f. It is known that every archimedean ordered division ring D is an ordered field. Hence, to prove the theorem, it remains to show that D is ordered. Let $a, b \in D$, $a > b > 0$. Then $a + b > a - b > 0$, and there is $r \in \mathbb{Q}$ such that $a + b > r > a - b$. By Lemma 3, $(a + b)^2 > r^2 > (a - b)^2$. As in the proof of Theorem 4, one can show that $ab > 0$, i.e., D is ordered. □

3.

In this section, the notion of the order valuation of a semiordered division ring D will be studied. We will call $a \in D$ bounded if $a^2 \leqslant r$ for some $r \in \mathbb{Q}^+$. If $a^2 < r$ for every $r \in \mathbb{Q}^+$, we will call a an infinitesimal. Let V denote the set of all bounded elements of D, and let J denote the set of all infinitesimals in D. It will be established that V is a valuation ring in D and the multiplicative group U of invertible elements in V is formed by precisely those a which satisfy $r_1 \leqslant a^2 \leqslant r_2$ for some positive rationals r_1 and r_2 (call these elements units). Three more remarks will be needed.

Remark 7. For non-zero elements $a, b \in D$ and $r \in \mathbb{Q}^+$ we have

(1) $(a \pm b)^2 \leqslant 2(a^2 + b^2)$,

(2) $a^2 < r^2$ if and only if $-r < a < r$.

Remark 8.

(1) The set V of bounded elements is an additive $\mathbb{Q}$-subgroup of D.

(2) The set J of infinitesimals is an additive $\mathbb{Q}$-subgroup.

Remark 9. If a is a positive element in a semiordered division ring D and $0 \neq x \in D$, then

(1) a is bounded if and only if $a < r$ for some $r \in \mathbb{Q}^+$,

(2) a is unit if and only if $r_2 < a < r_1$ for some $r_1, r_2 \in \mathbb{Q}^+$,

(3) a is infinitesimal if and only if $a < r$ for every $r \in \mathbb{Q}^+$, and

(4) x is bounded (unit, infinitesimal) if and only if x^2 is bounded (respectively unit, infinitesimal).

Theorem 10. *Let D be a semiordered division ring. Then*

(1) *V is a total subring of D, i.e., V is a subring which contains x or x^{-1} for every $x \in D^\bullet$.*

(2) *The set of non-units of the ring V is precisely the ideal of infinitesimals and consequently, J is the unique maximal ideal of V.*

Proof. (1) Let $a, b \in V$, i.e., $a^2 \leqslant r_1$, $b^2 \leqslant r_2$ for some $r_1, r_2 \in \mathbb{Q}^+$. Then

$$(a-b)^2 \leqslant 2(a^2+b^2) \leqslant 2(r_1+r_2) = r \quad \text{for some} \quad r \in \mathbb{Q}^+.$$

Hence, $a^2 + b^2 - (ab + ba) = (a-b)^2 \in V$ implies that $ab + ba \in V$. Thus,

$$bab = \frac{1}{2}[(ab+ba)b + b(ab+ba) - (b^2a + ab^2)] \in V.$$

Similarly, $ba^2b, ab^2a \in V$ and $(ab)^2 + (ba)^2 = a(bab) + (bab)a \in V$, so that

$$(ab-ba)^2 = (ab)^2 + (ba)^2 - [ab^2a + ba^2b] \in V.$$

By Remark 9, $ab - ba \in V$. Finally, $2ab = (ab + ba) + (ab - ba) \in V$ and so $ab \in V$.

If $x \in D$, $x \notin V$, then $x^2 > r$ for all $r \in \mathbb{Q}^+$, and hence $1 - rx^{-2} = (x^2 - r)x^{-2} > 0$. Then $x^{-2} < 1/r$ and so $x^{-1} \in V$. Thus V is a total subring.

(2) We show here that the units are precisely the invertible elements in V. If x is a unit, then $x \in V$ and $x^2 \geqslant r$ for some $r \in \mathbb{Q}^+$. Then $(x^2 - r)x^{-2} \geqslant 0$ and so $x^{-2} \leqslant 1/r$. Hence $x^{-1} \in V$ and x is invertible in V. Conversely, if x is invertible in V, then $x^2 \leqslant r_1$ for some $r_1 \in \mathbb{Q}^+$. Also, $x^{-1} \in V$ implies that $x^{-2} \leqslant r_2$ for some $r_2 \in \mathbb{Q}^+$. So, as above, $x^2 \geqslant 1/r_2 = r_2'$. Hence $r_2' \leqslant x^2 \leqslant r_1$ and x is a unit. □

Theorem 11. *If D is a semiordered division ring, then the bounded subring V is preserved under conjugation. Therefore, V is a valuation subring of D.*

Proof. Let $a \in V$, $a > 0$. Then by Remark 9, $a < r$ for some $r \in \mathbb{Q}^+$. By Lemma 3 part (8), $x(r - a)x^{-1} > 0$ for every $x \in D^\bullet$, so that $xax^{-1} < r$. Since $xax^{-1} > 0$, it follows that $xax^{-1} \in V$ for every $x \in D^\bullet$. If $a < 0$ in V, then $-a > 0$. Hence $-xax^{-1} \in V$ and also $xax^{-1} \in V$ for every $x \in D^\bullet$. □

The bounded subring V of a semiordered division ring D is now a valuation ring. By standard construction, one can define a valuation whose valuation ring is precisely the bounded subring V.

Theorem 12. *In any semiordered division ring D, the residue division ring $\overline{D} = V/J$ has a semiordering which is archimedean, so $\overline{D}$ is an archimedean ordered field.*

Proof. Let $\overline{P} = \{a + J/a \text{ is a positive unit in } D\}$. Clearly, $\bar{1} = 1 + J \in \overline{P}$ and $\bar{0} \notin \overline{P}$. It is straightforward to check that $\overline{P}$ is a positive cone of some semiordering in $\overline{D}$. Considering $\bar{a} = a + J \in \overline{P}$, we have $r_2 \leqslant a \leqslant r_1$ for some $r_1, r_2 \in \mathbb{Q}^+$. Then $\bar{r}_2 = r_2 + J \leqslant \bar{a} \leqslant \bar{r}_1 = r_1 + J$ and we can find a natural number n such that $\bar{a} < n$. Hence, $\overline{D}$ is an archimedean semiordered division ring. By Theorem 6, $\overline{D}$ is an archimedean ordered field. □

4.

As in the commutative case, the notion of a pre-positive cone is used to study extensions of a semiordering of a division ring D to larger division rings. A subset $P \subset D$ is called a pre-positive cone if it satisfies conditions (1), (2) and (3) in the definition of a positive cone. A pre-positive cone P induces an order relation on D, let us call it a pre-semiordering. Any positive cone of a semiordering is clearly a pre-positive cone. Also, any intersection of positive cones of D is a pre-positive cone of D. In this section, assume that D is semiordered, that is $-1 \notin C$, or equivalently

$0 \notin C$; then C is a pre-positive cone satisfying $C \subset P$ and $CP = PC = P$ for each positive cone P.

Theorem 13. *If P is a pre-positive cone, and if $a \notin P$, then there is a pre-positive cone P' containing P, with $-a \in P'$.*

Proof. Let $P' = P \cup -aC \cup (P + (-a)C)$. Clearly $-a \in P^{'}$. We will check axioms (1) to (3) for P'. Since P and C are additive, it follows that $P' + P' \subset P'$. Clearly $1 \in P'$, and to show that $0 \notin P'$, it suffices to show that $0 \notin P + (-a)C$. If $0 \in P+(-a)C$, then $p-ac = 0$ for some $p \in P$ and $c \in C$. Hence $a = pc^{-1} \in PC = P$, which is a contradiction. As for axiom (2), this is evident. □

Theorem 14. *Any pre-positive cone P_0 of D can be extended to a positive cone P.*

Proof. By Zorn's lemma, the set of all pre-positive cones extending P_0 contains a maximal pre-positive cone P. If $a \notin P$ for some $a \in D$, it follows by Theorem 13 that there is a pre-positive cone P' containing P and such that $-a \in P'$. The maximality of P implies $P' = P$, so that $-a \in P$. Thus P is a positive cone. □

Corollary 15. *A pre-positive cone P is maximal (with respect to set theoretic inclusion) if and only if P is a positive cone.*

Theorem 16. *Let E be any division ring extension of D. Let P be a positive cone of D. Let P_1 be the set of elements in E which are expressible as sums of elements of the form $\prod_i a_{j_i} c_{j_i}$ ($a_{j_i} \in P$ and $c_{j_i} \in C_1$ = the set of all finite sums of products of squares in E). If $0 \notin P_1$, then P can be enlarged to a positive cone of E.*

Proof. Since $0 \notin P_1$, it follows that $0 \notin C_1$, and E is ordered. One can show that P_1 is a pre-positive cone of E. Thus, by Theorem 14, P_1 can be extended to a positive cone of E which contains P. □

Exactly as for a semiordering, one can define bounded elements, infinitesimals and units at a given pre-semiordering of the division ring D. For P_0, a pre-positive cone of some pre-semiordering, let V_0, J_0 denote the sets of all bounded elements and infinitesimals, respectively.

Theorem 17. *Let $(P_i)_{i \in I}$ be the family of positive cones containing a given pre-positive cone P_0 of the division ring D. Let V_i , J_i be the subring of bounded elements and the ideal of infinitesimals, respectively, attached to the semiordering induced by P_i. Let U_i be the group of units of the ring V_i. Then*

(i) $\bigcap_i V_i = V_0$,

(ii) $\bigcap_i J_i = J_0$,

(iii) $\bigcap_i U_i = U_0$.

P r o o f. We prove here (i), for (ii) and (iii) we would use similar arguments. Clearly $V_0 \subset \bigcap V_i$. Conversely, if $a \notin V_0$, we show that $a \notin V_i$ for some i. From $a \notin V_0$ it follows that $a^2 > r$ for every positive rational r, that is $a^2 - r \in P_0$ for every rational r. Let $P_r = P_0 \cup (a^2 - r)C \cup (P_0 + (a^2 - r)C)$ and $P_0' = \bigcup P_r$. One can show that P_0' is a pre-positive cone of D containing P_0 and $a^2 - r$ for every rational r. By Theorem 14, P_0' can be extended to a positive cone P. Clearly $P_0 \subset P$ and $a^2 - r \in P$ for every rational r. Thus a is not bounded in P, i.e., $a \notin V_i$ for some i. □

To a certain extent, Theorem 17 reduces the treatment of a pre-semiordering to that of a semiordering. For instance, one has

Corollary 18. *For any pre-positive cone P_0 of D, the bounded subring V_0 is preserved under conjugation.*

5.

Finally, in this section an example of a semiordering, which is not an ordering for a division ring will be given. Start with a semiordered commutative field F (e.g., $\mathbb{R}$). Construct the field $F((x))$ of F-coefficient formal Laurent series in the single indeterminate x. Next, form the division ring D of formal Laurent series in an indeterminate y, coefficients in $F((x))$ written on the left, according to the relation $yx = 2xy$. Note that the characteristic of F is not 2 (actually the characteristic of any ordered field is zero). Clearly, the centre of D is F. It will be shown that D has a semiordering, which extends that of F, and this semiordering is not an ordering.

Consider $G = \mathbb{Z} \times \mathbb{Z}$ as an abelian group under componentwise addition, ordered lexicographically by

$$(m, n) > 0 \text{ or } < 0 \text{ according as } m > 0 \text{ or } m < 0,$$

and

$$(0, n) > 0 \text{ or } < 0 \text{ according as } n > 0 \text{ or } n < 0.$$

Define $\omega(\alpha) = (p, q) \in G$ for $\alpha \in D$, where $x^q y^p$ is the monomial of the smallest power in the element α in D. This is a valuation on D, whose residue field $\overline{D}$ can be identified with the centre F. Now, the semiordering of the residue field F will be lifted to D.

By the proof of Theorem 12, the positive cone P of D is expected to contain all α such that $\alpha + J$ is positive in $\overline{D} = F$ and $\alpha = u$ a unit in D. In fact, every $\alpha \in D$ where $\omega(\alpha) = (p, q)$, can be written in the form $ux^q y^p$ for a unit u in D. Since $yx = 2xy$, it follows that $x^2y^2 = (\frac{1}{\sqrt{2}}xy)^2$. So every element $\alpha \in D$ can be written as a product of an element of the form u, ux, uy or uxy (where u is a unit in D) and a non-zero square in D.

Let $M = \{u, ux, uy\colon\ u$ is a unit in D and $u + J > 0$ in $\overline{D}\} \cup \{uxy\colon\ u$ is a unit in D and $u + J < 0$ in $\overline{D}\}$. Take $P = M \cup MD^{\bullet 2}$. It is a routine work to check that P is a positive cone of a semiordering of D which extends the semiordering of F. Clearly, $x > 0$ and $y > 0$ but $xy < 0$ and hence P is not closed under product, i.e., P is not an ordering.

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