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ON ORDERED DIVISION RINGS

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Abstract. Prestel introduced a generalization of the notion of an ordering of a field, which is called a semiordering. Prestel's axioms for a semiordered field differ from the usual (Artin-Schreier) postulates in requiring only the closedness of the domain of positivity under \( x \to x a^2 \) for nonzero \( a \), instead of requiring that positive elements have a positive product. In this work, this type of ordering is studied in the case of a division ring. It is shown that it actually behaves the same as in the commutative case. Further, it is shown that the bounded subring associated with that ordering is a valuation ring which is preserved under conjugation, so one can associate a natural valuation to a semiordering.

Keywords: ordering, division ring

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1.

The investigation of ordered fields has a long tradition. They play an important part in many branches of mathematics. In [1], Prestel introduced a generalization of an ordering of a field, which is called a semiordering. Prestel's axioms for a semiordered field differ from the usual (Artin-Schreier) postulates in requiring only the closedness of the positive cone under \( x \to x a^2 \) for non-zero \( a \), instead of requiring that positive elements have a positive product. This generalization of positive cones and orderings is based on the following observation. Very often one only uses the property \( x > 0 \Rightarrow x a^2 > 0 \) of an ordering together with \( 1 > 0 \). This is especially the case if one deals with quadratic forms. In this work, this type of ordering is studied in the case of a division ring. It is shown that it actually behaves the same as in the commutative case. For example, a division ring admits a semiordering if and only if \(-1\) is not a sum of products of squares. In fact, every semiordered division
ring is ordered. Moreover, every archimedean semiordered division ring is an ordered field. Further, the existence of a natural valuation associated to a semiordering is investigated, this requires the study of the bounded subring associated to a given semiordering. This is the subring consisting of elements which are bounded by some rational number with respect to the semiordering. It is shown that the bounded subring is a valuation ring which is preserved under conjugation, so one can associate a natural valuation to a semiordering. To study extensions of semiordering to larger division rings, the notion of pre-semiordering is investigated. Finally, an example of a semiordering, which is not an ordering for a division ring is given.

2.

Throughout this work, $D$ denotes a (not necessarily commutative) division ring, and $D^*$ denotes its multiplicative group of non-zero elements.

**Definition.** A semiordering of a division ring $D$ is an order relation $<$ such that

1. $P + P \subseteq P$;
2. $a \in P \Rightarrow ab^2 \in P$, for $0 \neq b \in D$;
3. $0 \notin P$ and $1 \in P$;
4. $P \cup \{0\} \cup -P = D$;

then, $a > b \iff a - b \in P$, and $P = \{a \in D \mid a > 0\}$.

Note that this definition is the same as in the commutative case (see [1]). Clearly any ordering of $D$ is also a semiordering.

Let $C$ be the subset of all finite sums of elements of the form $a_1^2 a_2^2 \ldots a_k^2$ in $D$ with every $a_i \neq 0$. Clearly, $C$ is closed under sums and products. Also, $C$ contains inverses (for $c \in C$, $c^{-2} \in C \Rightarrow c^{-1} = cc^{-2} \in C$). If $D$ is a semiordered division ring, then for $a \in P$ and $c \in C$ we have $ac \in P$ (by applying conditions (1) and (2) of the above definition several times). Since $1 \in P$, then clearly $C \subseteq P$. Also, $-1 \notin P$ implies that $-1 \notin C$. So by [2], Theorem 1, $D$ is an ordered division ring. Hence we have

**Theorem 1.** Every semiordered division ring is ordered.

Although a semiordered division ring is ordered, there is no guarantee that the given semiordering is an ordering. At the end of this work, an example of a semiordering which is not an ordering is given.

**Corollary 2.** A division ring $D$ admits a semiordering if and only if $-1 \notin C$. 
One can prove the following properties of semiorderings.

**Lemma 3.** Let $D$ be any semiordered division ring, $a \in D$. Then

1. $a > 0$ if and only if $a^{-1} > 0$;
2. if $a > 0$, then $d^2a > 0$ for $0 \neq d \in D$ and hence $ca > 0$ for $c \in C$;
3. if $a > 0$, then $ra > 0$ for every $r \in \mathbb{Q}^+$;
4. if $a > 1$, then $a^{-1} < 1$;
5. if $0 < a < b$, $a, b \in C$ then $a^{-1} > b^{-1}$;
6. if $0 < a < b$, $a \in C$ then $a^2 < b^2$;
7. if $0 < a < b$, $b \in C$ then $a^2 < b^2$;
8. if $a > 0$, then $xax^{-1} > 0$ for all $x \in D^*$.

**Theorem 4.** A semiordering $>$ is an ordering of $D$ if and only if for all $a, b \in D$ the inequality $0 < a < b$ implies $a^2 < b^2$.

**Proof.** We first claim that for every $a, b \in D$, $a, b > 0$, we have $ba + ab > 0$. Since $a-b > 0$ or $b-a > 0$, we may assume that $0 < a < b$. Clearly, $a+b > b-a > 0$ and so $(a+b)^2 > b^2 > (b-a)^2$. Hence $2(ba+ab) > 0$. By Lemma 3, part (3), $ba + ab > 0$. Next, we claim that $bab > 0$ and $bab^{-1} > 0$ for every $a, b > 0$ in $D$. If $bab < 0$, then $-bab > 0$ and $a > 0$ implies $(-bab)a + a(-bab) > 0$. Thus $-(ba)^2 - (ab)^2 > 0$, which is a contradiction. Hence $bab > 0$ and $bab^{-1} = bab^{-2} > 0$. Now assume $ab < 0$ for $a, b > 0$. Then $-ab > 0$ and $-ba = b(-ab)b^{-1} > 0$. So $-ab - ba > 0$, which is a contradiction. Hence $ab > 0$ for every $a, b > 0$ and the semiordering is an ordering.

A semiordering $<$ on a division ring $D$ is called archimedean if for every $a \in D$ there is a natural number $n$ such that $a < n$.

**Proposition 5.** A semiordering $<$ on a division ring $D$ is archimedean if and only if $\mathbb{Q}$ is dense in $D$ with respect to $<$ (i.e., if $a < b$ then there is $r \in \mathbb{Q}$ such that $a < r < b$).

**Proof** of Proposition 5 is similar to that for the case of a field (see [1]).

**Theorem 6.** Every archimedean semiordered division ring $D$ is an ordered field.

**Proof.** It is known that every archimedean ordered division ring $D$ is an ordered field. Hence, to prove the theorem, it remains to show that $D$ is ordered. Let $a, b \in D$, $a > b > 0$. Then $a + b > a - b > 0$, and there is $r \in \mathbb{Q}$ such that $a + b > r > a - b$. By Lemma 3, $(a+b)^2 > r^2 > (a-b)^2$. As in the proof of Theorem 4, one can show that $ab > 0$, i.e., $D$ is ordered.
In this section, the notion of the order valuation of a semiordered division ring $D$ will be studied. We will call $a \in D$ bounded if $a^2 \leq r$ for some $r \in \mathbb{Q}^+$. If $a^2 < r$ for every $r \in \mathbb{Q}^+$, we will call $a$ an infinitesimal. Let $V$ denote the set of all bounded elements of $D$, and let $J$ denote the set of all infinitesimals in $D$. It will be established that $V$ is a valuation ring in $D$ and the multiplicative group $U$ of invertible elements in $V$ is formed by precisely those $a$ which satisfy $r_1 \leq a^2 \leq r_2$ for some positive rationals $r_1$ and $r_2$ (call these elements units). Three more remarks will be needed.

**Remark 7.** For non-zero elements $a, b \in D$ and $r \in \mathbb{Q}^+$ we have

1. $(a \pm b)^2 \leq 2(a^2 + b^2)$,
2. $a^2 < r^2$ if and only if $-r < a < r$.

**Remark 8.**

1. The set $V$ of bounded elements is an additive $\mathbb{Q}$-subgroup of $D$.
2. The set $J$ of infinitesimals is an additive $\mathbb{Q}$-subgroup.

**Remark 9.** If $a$ is a positive element in a semiordered division ring $D$ and $0 \neq x \in D$, then

1. $a$ is bounded if and only if $a < r$ for some $r \in \mathbb{Q}^+$,
2. $a$ is unit if and only if $r_2 < a < r_1$ for some $r_1, r_2 \in \mathbb{Q}^+$,
3. $a$ is infinitesimal if and only if $a < r$ for every $r \in \mathbb{Q}^+$, and
4. $x$ is bounded (unit, infinitesimal) if and only if $x^2$ is bounded (respectively unit, infinitesimal).

**Theorem 10.** Let $D$ be a semiordered division ring. Then

1. $V$ is a total subring of $D$, i.e., $V$ is a subring which contains $x$ or $x^{-1}$ for every $x \in D^\times$.
2. The set of non-units of the ring $V$ is precisely the ideal of infinitesimals and consequently, $J$ is the unique maximal ideal of $V$.

**Proof.** (1) Let $a, b \in V$, i.e., $a^2 \leq r_1, b^2 \leq r_2$ for some $r_1, r_2 \in \mathbb{Q}^+$. Then

$$(a - b)^2 \leq 2(a^2 + b^2) \leq 2(r_1 + r_2) = r \text{ for some } r \in \mathbb{Q}^+.$$ 

Hence, $a^2 + b^2 - (ab + ba) = (a - b)^2 \in V$ implies that $ab + ba \in V$. Thus,

$$baba = \frac{1}{2}[(ab + ba)b + b(ab + ba) - (b^2a + ab^2)] \in V.$$ 

Similarly, $ba^2b, ab^2a \in V$ and $(ab)^2 + (ba)^2 = a(bab) + (bab)a \in V$, so that

$$(ab - ba)^2 = (ab)^2 + (ba)^2 - [ab^2a + ba^2b] \in V.$$
By Remark 9, \(ab - ba \in V\). Finally, \(2ab = (ab + ba) + (ab - ba) \in V\) and so \(ab \in V\).

If \(x \in D, x \not\in V\), then \(x^2 > r\) for all \(r \in \mathbb{Q}^+\), and hence \(1 - rx^2 = (x^2 - r)x^2 > 0\). Then \(x^{-2} < 1/r\) and so \(x^{-1} \in V\). Thus \(V\) is a total subring.

(2) We show here that the units are precisely the invertible elements in \(V\). If \(x\) is a unit, then \(x \in V\) and \(x^2 \geq r\) for some \(r \in \mathbb{Q}^+\). Then \((x^2 - r)x^{-2} \geq 0\) and so \(x^{-2} \leq 1/r\). Hence \(x^{-1} \in V\) and \(x\) is invertible in \(V\). Conversely, if \(x\) is invertible in \(V\), then \(x^{-1} \in V\) for some \(r_1 \in \mathbb{Q}^+\). Also, \(x^{-1} \in V\) implies that \(x^{-2} \leq r_2\) for some \(r_2 \in \mathbb{Q}^+\). So, as above, \(x^2 \geq 1/r_2 = r'_2\). Hence \(r'_2 \leq x^2 \leq 1/r_1\) and \(x\) is a unit. \(\square\)

**Theorem 11.** If \(D\) is a semiordered division ring, then the bounded subring \(V\) is preserved under conjugation. Therefore, \(V\) is a valuation subring of \(D\).

**Proof.** Let \(a \in V, a > 0\). Then by Remark 9, \(a < r\) for some \(r \in \mathbb{Q}^+\). By Lemma 3 part (8), \(x(r - a)x^{-1} > 0\) for every \(x \in D^*\), so that \(xax^{-1} < r\). Since \(xax^{-1} > 0\), it follows that \(xax^{-1} \in V\) for every \(x \in D^*\). If \(a < 0\) in \(V\), then \(-a > 0\). Hence \(-xax^{-1} \in V\) and also \(xax^{-1} \in V\) for every \(x \in D^*\). \(\square\)

The bounded subring \(V\) of a semiordered division ring \(D\) is now a valuation ring. By standard construction, one can define a valuation whose valuation ring is precisely the bounded subring \(V\).

**Theorem 12.** In any semiordered division ring \(D\), the residue division ring \(\bar{D} = V/J\) has a semiordering which is archimedean, so \(\bar{D}\) is an archimedean ordered field.

**Proof.** Let \(\bar{P} = \{a + J/a \text{ is a positive unit in } D\}\). Clearly, \(\bar{1} = 1 + J \in \bar{P}\) and \(0 \not\in \bar{P}\). It is straightforward to check that \(\bar{P}\) is a positive cone of some semiordering in \(\bar{D}\). Considering \(\bar{a} = a + J \in \bar{P}\), we have \(r_2 \leq a \leq r_1\) for some \(r_1, r_2 \in \mathbb{Q}^+\). Then \(\bar{r}_2 = r_2 + J \leq \bar{a} \leq \bar{r}_1 = r_1 + J\) and we can find a natural number \(n\) such that \(\bar{a} < n\). Hence, \(\bar{D}\) is an archimedean semiordered division ring. By Theorem 6, \(\bar{D}\) is an archimedean ordered field. \(\square\)

4.

As in the commutative case, the notion of a pre-positive cone is used to study extensions of a semiordering of a division ring \(D\) to larger division rings. A subset \(P \subset D\) is called a pre-positive cone if it satisfies conditions (1), (2) and (3) in the definition of a positive cone. A pre-positive cone \(P\) induces an order relation on \(D\), let us call it a pre-semiordering. Any positive cone of a semiordering is clearly a pre-positive cone. Also, any intersection of positive cones of \(D\) is a pre-positive cone of \(D\). In this section, assume that \(D\) is semiordered, that is \(-1 \not\in C\), or equivalently
0 \not\in C; then \( C \) is a pre-positive cone satisfying \( C \subseteq P \) and \( CP = PC = P \) for each positive cone \( P \).

**Theorem 13.** If \( P \) is a pre-positive cone, and if \( a \not\in P \), then there is a pre-positive cone \( P' \) containing \( P \), with \(-a \in P'\).

**Proof.** Let \( P' = P \cup -aC \cup (P + (-a)C) \). Clearly \(-a \in P'\). We will check axioms (1) to (3) for \( P' \). Since \( P \) and \( C \) are additive, it follows that \( P' + P' \subseteq P' \). Clearly \( 1 \in P' \), and to show that \( 0 \not\in P' \), it suffices to show that \( 0 \not\in P + (-a)C \). If \( 0 \in P + (-a)C \), then \( p - ac = 0 \) for some \( p \in P \) and \( c \in C \). Hence \( a = pc^{-1} \in PC = P \), which is a contradiction. As for axiom (2), this is evident. \( \square \)

**Theorem 14.** Any pre-positive cone \( P_0 \) of \( D \) can be extended to a positive cone \( P \).

**Proof.** By Zorn’s lemma, the set of all pre-positive cones extending \( P_0 \) contains a maximal pre-positive cone \( P \). If \( a \not\in P \) for some \( a \in D \), it follows by Theorem 13 that there is a pre-positive cone \( P' \) containing \( P \) and such that \(-a \in P' \). The maximality of \( P \) implies \( P' = P \), so that \(-a \in P \). Thus \( P \) is a positive cone. \( \square \)

**Corollary 15.** A pre-positive cone \( P \) is maximal (with respect to set theoretic inclusion) if and only if \( P \) is a positive cone.

**Theorem 16.** Let \( E \) be any division ring extension of \( D \). Let \( P \) be a positive cone of \( D \). Let \( P_1 \) be the set of elements in \( E \) which are expressible as sums of elements of the form \( \prod a_{ji}, c_{ji} \) (\( a_{ji} \in P \) and \( c_{ji} \in C_1 = \) the set of all finite sums of products of squares in \( E \) ). If \( 0 \not\in P_1 \), then \( P \) can be enlarged to a positive cone of \( E \).

**Proof.** Since \( 0 \not\in P_1 \), it follows that \( 0 \not\in C_1 \), and \( E \) is ordered. One can show that \( P_1 \) is a pre-positive cone of \( E \). Thus, by Theorem 14, \( P_1 \) can be extended to a positive cone of \( E \) which contains \( P \). \( \square \)

Exactly as for a semiordering, one can define bounded elements, infinitesimals and units at a given pre-semiordering of the division ring \( D \). For \( P_0 \), a pre-positive cone of some pre-semiordering, let \( V_0 \), \( J_0 \) denote the sets of all bounded elements and infinitesimals, respectively.

**Theorem 17.** Let \( (P_i)_{i \in I} \) be the family of positive cones containing a given pre-positive cone \( P_0 \) of the division ring \( D \). Let \( V_i \), \( J_i \) be the subring of bounded elements and the ideal of infinitesimals, respectively, attached to the semiordering induced by \( P_i \). Let \( U_i \) be the group of units of the ring \( V_i \). Then

(i) \( \bigcap_i V_i = V_0 \),
(ii) $\bigcap_i J_i = J_0$,
(iii) $\bigcap_i U_i = U_0$.

Proof. We prove here (i), for (ii) and (iii) we would use similar arguments. Clearly $V_0 \subset \bigcap V_i$. Conversely, if $a \notin V_0$, we show that $a \notin V_i$ for some $i$. From $a \notin V_0$ it follows that $a^2 > r$ for every positive rational $r$, that is $a^2 - r \in P_0$ for every rational $r$. Let $P_r = P_0 \cup (a^2 - r)C \cup (P_0 + (a^2 - r)C)$ and $P'_0 = \bigcup P_r$. One can show that $P'_0$ is a pre-positive cone of $D$ containing $P_0$ and $a^2 - r$ for every rational $r$. By Theorem 14, $P'_0$ can be extended to a positive cone $P$. Clearly $P_0 \subset P$ and $a^2 - r \in P$ for every rational $r$. Thus $a$ is not bounded in $P$, i.e., $a \notin V_i$ for some $i$. \hfill $\Box$

To a certain extent, Theorem 17 reduces the treatment of a pre-semiordering to that of a semiordering. For instance, one has

**Corollary 18.** For any pre-positive cone $P_0$ of $D$, the bounded subring $V_0$ is preserved under conjugation.

5.

Finally, in this section an example of a semiordering, which is not an ordering for a division ring will be given. Start with a semiordered commutative field $F$ (e.g., $\mathbb{R}$). Construct the field $F((x))$ of $F$-coefficient formal Laurent series in the single indeterminate $x$. Next, form the division ring $D$ of formal Laurent series in an indeterminate $y$, coefficients in $F((x))$ written on the left, according to the relation $yx = 2xy$. Note that the characteristic of $F$ is not 2 (actually the characteristic of any ordered field is zero). Clearly, the centre of $D$ is $F$. It will be shown that $D$ has a semiordering, which extends that of $F$, and this semiordering is not an ordering.

Consider $G = \mathbb{Z} \times \mathbb{Z}$ as an abelian group under componentwise addition, ordered lexicographically by

$$(m, n) > 0 \text{ or } < 0 \text{ according as } m > 0 \text{ or } m < 0,$$

and

$$(0, n) > 0 \text{ or } < 0 \text{ according as } n > 0 \text{ or } n < 0.$$  

Define $\omega(\alpha) = (p, q) \in G$ for $\alpha \in D$, where $x^p y^q$ is the monomial of the smallest power in the element $\alpha$ in $D$. This is a valuation on $D$, whose residue field $\overline{D}$ can be identified with the centre $F$. Now, the semiordering of the residue field $F$ will be lifted to $D$.  

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By the proof of Theorem 12, the positive cone $P$ of $D$ is expected to contain all $\alpha$ such that $\alpha + J$ is positive in $\overline{D} = F$ and $\alpha = u$ a unit in $D$. In fact, every $\alpha \in D$ where $\omega(\alpha) = (p, q)$, can be written in the form $ux^qy^p$ for a unit $u$ in $D$. Since $yx = 2xy$, it follows that $x^2y^2 = (\frac{1}{\sqrt{2}}xy)^2$. So every element $\alpha \in D$ can be written as a product of an element of the form $u, ux, uy$ or $uxy$ (where $u$ is a unit in $D$) and a non-zero square in $D$.

Let $M = \{u, ux, uy: u$ is a unit in $D$ and $u + J > 0$ in $\overline{D}\} \cup \{uxy: u$ is a unit in $D$ and $u + J < 0$ in $\overline{D}\}$. Take $P = M \cup MD^\bullet$. It is a routine work to check that $P$ is a positive cone of a semiordering of $D$ which extends the semiordering of $F$. Clearly, $x > 0$ and $y > 0$ but $xy < 0$ and hence $P$ is not closed under product, i.e., $P$ is not an ordering.

References


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