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DECOMPOSITION OF COMPLETE BIPARTITE EVEN GRAPHS INTO CLOSED TRAILS

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Abstract. We prove that any complete bipartite graph $K_{a,b}$, where a, b are even integers, can be decomposed into closed trails with prescribed even lengths.

Keywords: complete bipartite graph, closed trail, arbitrarily decomposable graph

MSC 2000: 05C70

1. Introduction

In this paper we consider simple graphs only, and we use the standard notation of the graph theory.

A graph is said to be *even* if the degrees of all its vertices are even. By Euler's theorem, a connected even graph is Eulerian, i.e. contains a closed trail (a circuit) passing through all its edges (exactly once).

We denote by $\mathrm{Lct}(G)$ the set of all integers l such that there is a closed trail of length l in G and by $\mathrm{Sct}(G)$ the set of all sequences $(l_1, l_2, \ldots, l_p)$ such that $l_i \in \mathrm{Lct}(G)$, $i = 1, 2, \ldots, p$, and $\sum_{i=1}^{p} l_i = |E(G)|$. A connected even graph G is said to be *arbitrarily decomposable into closed trails* (ADCT for short) if, for any sequence $(l_1, l_2, \ldots, l_p) \in \mathrm{Sct}(G)$, G can be (edge-disjointly) decomposed into closed trails $T_1, T_2, \ldots, T_p$ of lengths $l_1, l_2, \ldots, l_p$, respectively.

A sequence of integers $(l_1, l_2, \ldots, l_p) \in \mathrm{Sct}(G)$ is said to be *realizable in G* if G can be (edge-disjointly) decomposed into closed trails $T_1, T_2, \ldots, T_p$ of lengths l_1, $l_2, \ldots, l_p$, respectively.

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So, a connected even graph G is ADCT if each sequence in $\mathrm{Sct}(G)$ is realizable in G.

The following theorem, in which M_{2k} is a matching of K_{2k} having k edges, is just a reformulation of a theorem by Balister [1].

Theorem 1. *If k is an integer, $k \geqslant 2$, then the graphs K_{2k-1} and $K_{2k} - M_{2k}$ are* ADCT.

Remark. The motivation and application of Theorem 1 can be found in problems concerning the vertex-distinguishing proper edge-colouring of a graph. This notion was introduced and studied by Burris and Schelp in [5] and, independently (the corresponding invariant is called there observability of a graph), by Černý, Horňák and Soták in [6]. See also [2], [3], [4] for recent results in this area.

The aim of the present paper is to prove that complete bipartite even graphs are arbitrarily decomposable into closed trails.

Theorem 2. *If a, b are positive even integers, then the graph $K_{a,b}$ is* ADCT.

2. Auxiliary and partial results

Let a, b be positive even integers. Clearly, $\mathrm{Lct}(K_{2,b}) = \{4i\colon\ i = 1, 2, \ldots, \frac{1}{2}b\}$ and, if $a, b \geqslant 4$, then $\mathrm{Lct}(K_{a,b}) = \{2i\colon\ i = 2, 3, \ldots, \frac{1}{2}(ab-4)\} \cup \{ab\}$.

Proposition 3. *If b is a positive even integer, then the graph $K_{2,b}$ is* ADCT.

P r o o f. The result follows from the fact that $K_{2,b}$ can be decomposed into $\frac{1}{2}b$ cycles C_4 which all share two common vertices. □

Lemma 4. *Let a, b^1, b^2 be positive even integers, let $b = b^1 + b^2$ and let a sequence $S^i = (l^i_1, l^i_2, \ldots, l^i_{p^i}) \in \mathrm{Sct}(K_{a,b^i})$ be realizable in K_{a,b^i}, $i = 1, 2$. Then the sequences $S^1 \cdot S^2 = (l^1_1, l^1_2, \ldots, l^1_{p^1}, l^2_1, l^2_2, \ldots, l^2_{p^2})$ and $S^1 + S^2 = (l^1_1 + l^2_1, l^1_2, l^1_3, \ldots, l^1_{p^1}, l^2_2, l^2_3, \ldots, l^2_{p^2})$ are realizable in $K_{a,b}$.*

P r o o f. Consider vertex-disjoint graphs K_{a,b^1}, K_{a,b^2}, a decomposition of K_{a,b^i} into closed trails corresponding to S^i, $i = 1, 2$, and then identify (in an arbitrary way) pairs of vertices of parts of cardinality a. We obtain a decomposition of $K_{a,b}$ into closed trails corresponding to the sequence $S^1 \cdot S^2$. If the identification is chosen in such a way that trails T^1_1 in K_{a,b^1} of length l^1_1 and T^2_1 in K_{a,b^2} of length l^2_1 have a common vertex, what results can also be regarded as a decomposition corresponding to the sequence $S^1 + S^2$, because the union of T^1_1 and T^2_1 is a closed trail of length $l^1_1 + l^2_1$—see Fig. 1. □

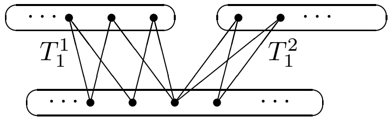

Figure 1. The partition set having a vertices of the graphs K_{a,b^1} and K_{a,b^2} has been chosen in such a way that the sets of vertices of T_1^1 and T_1^2 intersect.

Proposition 5. *If a, b are even integers with $a \geqslant 4$, $b \geqslant 4$ and $6 \mid ab$, then the graph $K_{a,b}$ can be decomposed into cycles C_6.*

Proof. Let the parts of the graph $K_{a,b}$ be $\{x_1, x_2, \ldots, x_a\}$ and $\{y_1, y_2, \ldots, y_b\}$. A decomposition of the graph $K_{6,b}$, $b \in \{4,6\}$, is presented in a $6 \times b$ matrix $M_{6,b}$, where the ith row and jth column entry indicates the number of the 6-cycle passing through the edge $x_i y_j$:

$$M_{6,4} = \begin{pmatrix} 1 & 1 & 3 & 3 \\ 3 & 1 & 1 & 3 \\ 1 & 4 & 1 & 4 \\ 3 & 2 & 3 & 2 \\ 4 & 2 & 2 & 4 \\ 4 & 4 & 2 & 2 \end{pmatrix}, \qquad M_{6,6} = \begin{pmatrix} 1 & 3 & 3 & 1 & 4 & 4 \\ 1 & 5 & 1 & 4 & 5 & 4 \\ 5 & 6 & 1 & 1 & 5 & 6 \\ 5 & 5 & 2 & 2 & 6 & 6 \\ 3 & 6 & 3 & 2 & 6 & 2 \\ 3 & 3 & 2 & 4 & 4 & 2 \end{pmatrix}.$$

Since $K_{b,a}$ is isomorphic to $K_{a,b}$, we may suppose that $6 \mid a$. Thus, $a = 6p$ and $b = 4q + 6r$ for appropriate integers p, q, r, $r \in \{0,1\}$. Using Lemma 4 we obtain a decomposition of $K_{6,4q+6r}$ or, equivalently, of $K_{4q+6r,6}$ (note that any closed trail of length 6 in a simple bipartite graph is in fact a cycle C_6). By Lemma 4 again this yields a decomposition of $K_{4q+6r,6p}$ and we are done. □

Theorem 6. *The graphs $K_{4,4}$, $K_{4,6}$ and $K_{6,6}$ are* ADCT.

Proof. (1) If a sequence from $\mathrm{Sct}(K_{4,4})$ contains only terms divisible by 4, it is realizable in $K_{4,4}$ because of Proposition 3 and Lemma 4. There are two other nondecreasing sequences in $\mathrm{Sct}(K_{4,4})$, namely (4,6,6) and (6,10). Consider a cycle C_6 in $K_{4,4}$. Evidently, the connected graph $K_{4,4}$–C_6 is even. It has 10 edges and is the union of cycles C_4 and C_6.

(2) Consider a sequence $S \in \mathrm{Sct}(K_{4,6})$. With respect to (1), Proposition 3 and Lemma 4, S is realizable in $K_{4,6}$ if all terms of S are divisible by 4, if there are terms in S whose sum is 8 or if $S \in \{(4,6,14),(4,10,10)\}$. For $S = (6,6,6,6)$ use Proposition 5. Finally, $S = (6,18)$ is realizable in $K_{4,6}$, because $K_{4,6}$–C_6 is a connected even graph.

(3) Now let $S = (l_1, l_2, \ldots, l_p)$ be a nondecreasing sequence in $\mathrm{Sct}(K_{6,6})$. Using 2, Proposition 3 and Lemma 4 we see that S is realizable in $K_{6,6}$ if the sum of terms of S divisible by 4 is at least 8. Thus, we may suppose that if S has a term divisible by 4, it

is only $l_1 = 4$. If $S = (6,6,6,6,6,6)$, we are done by Proposition 5. So, suppose that i is the smallest index such that $l_i > 6$. Then it is easy to see that $s = \sum_{j=i}^{p} (l_i - 6) \geqslant 8$.

If $s \geqslant 12$, choose integers l'_j such that $4 \leqslant l'_j \leqslant l_j - 6$, $l'_j \equiv 0 \pmod 4$, $i \leqslant j \leqslant p$, and $\sum_{j=i}^{p} l'_j = 12$. Because of (2) the graph $K_{4,6}$ can be decomposed into closed trails $T_1, T_2, \ldots, T_p$ with lengths $l_1, l_2, \ldots, l_{i-1}, l_i - l'_i, l_{i+1} - l'_{i+1}, \ldots, l_p - l'_p$. Since $l_j - l'_j \equiv 2 \pmod 4$, $i \leqslant j \leqslant p$, and $p+1-i \leqslant 3$, in each trail T_j, $i \leqslant j \leqslant p$, we can pick a distinct vertex z_j from the part containing 6 vertices (so that $T_j \to z_j$ is an injection). Take a decomposition of $K_{2,6}$, sharing the part of 6 vertices with $K_{4,6}$, into closed trails $T'_i, T'_{i+1}, \ldots, T'_p$ of lengths $l'_i, l'_{i+1}, \ldots, l'_p$ in such a way that T'_j contains the vertex z_j, $i \leqslant j \leqslant p$. The union of T_j and T'_j is then a closed trail of length l_j, $i \leqslant j \leqslant p$, which shows that S is realizable in $K_{6,6}$.

If $s = 8$, then $l_1 = 4$. We proceed as above with $l'_j = l_j - 6$, $i \leqslant j \leqslant p$, with a decomposition of $K_{4,6}$ into closed trails of lengths $l_2, l_3, \ldots, l_{i-1}$ and a decomposition of $K_{2,6}$ into closed trails of lengths l_1 and l'_j, $i \leqslant j \leqslant p$. □

Proposition 7. *If* $a \in \{4,6,8\}$, *then the sequences* $(4a-2, 4a+2)$ *and* $(4, 4a-2, 4a-2)$ *are realizable in the graph* $K_{a,8}$.

Proof. Let the parts of the graph $K_{a,8}$ be $\{x_1, x_2, \ldots, x_a\}$ and $\{y_1, y_2, \ldots, y_8\}$. Consider a closed Eulerian trail in the subgraph of $K_{a,8}$ induced on the vertex set $\{x_1, x_2, \ldots, x_{a-2}, y_1, y_2, y_3, y_4\}$. Joining it with a closed trail of length 6 on the vertices x_1, y_5, x_2, y_6, x_3, y_7 results in a closed trail T of length $4a-2$. A closed trail T' on the vertices x_{a-1}, y_1, x_a, y_2 is edge-disjoint with T. Deleting the edges of T and T' (and possibly created isolated vertices) from $K_{a,8}$ we obtain a connected even graph G with $4a-2$ edges and $V(G) \cap V(T') \neq \emptyset$. Thus, the remaining trail(s) can be built up using T' and a closed Eulerian trail in G. □

Proposition 8. *The sequences* $S_4^6 = (4,6,6,6,6,6,6,6,6,6,6)$, $S_4^{10} = (4,10,10,10,10,10,10)$ *and* $S_{14}^{10} = (10,10,10,10,10,14)$ *are realizable in the graph* $K_{8,8}$.

Proof. Analogously as in the proof of Proposition 5 we present 8×8 matrices M_4^l, $l \in \{6,10\}$:

$$M_4^6 = \begin{pmatrix} 1 & 1 & 7 & 5 & 7 & 5 & 3 & 3 \\ 9 & 1 & 1 & 8 & 9 & 3 & 3 & 8 \\ 1 & 9 & 1 & 5 & 9 & 3 & 5 & 3 \\ 9 & 9 & 7 & \mathbf{1} & \mathbf{1} & 5 & 5 & 7 \\ 0 & 6 & 6 & \mathbf{1} & \mathbf{1} & 8 & 0 & 8 \\ 4 & 6 & 4 & 0 & 6 & 2 & 0 & 2 \\ 0 & 4 & 4 & 0 & 7 & 2 & 2 & 7 \\ 4 & 4 & 6 & 8 & 6 & 8 & 2 & 2 \end{pmatrix}, \quad M_4^{10} = \begin{pmatrix} 1 & 1 & 4 & 4 & 5 & 5 & 2 & 2 \\ \mathbf{1} & 1 & 1 & 4 & 4 & 2 & 2 & \mathbf{1} \\ 6 & 5 & 1 & 1 & 5 & 2 & 6 & 2 \\ \mathbf{1} & 2 & 3 & 1 & 1 & 2 & 3 & \mathbf{1} \\ 1 & 2 & 3 & 6 & 1 & 2 & 3 & 6 \\ 3 & 5 & 3 & 5 & 4 & 4 & 6 & 6 \\ 6 & 3 & 3 & 5 & 6 & 4 & 4 & 5 \\ 3 & 3 & 4 & 6 & 6 & 5 & 4 & 5 \end{pmatrix}.$$

The matrix M_4^l describes a decomposition of $K_{8,8}$ into closed trails with lengths corresponding to S_4^l in such a way that its ith row and jth column entry indicates the number of either the l-trail (of length l) or the 4-trail (if that entry is bold) passing through the edge $x_i y_j$; in M_4^6 0 stands instead of 10. The matrix M_4^{10} yields also the realizability of S_{14}^{10}: it is sufficient to join the (bold) 4-trail with one of 10-trails (note that no two trails described by M_4^{10} are vertex-disjoint). □

Proposition 9. *If a, b are even integers with $a \geqslant 4$, $b \geqslant 4$ and $10 \mid ab$, then the graph $K_{a,b}$ can be decomposed into closed trails of length* 10.

Proof. Without loss of generality we may suppose that $10 \mid a$. By Theorem 6, the sequence (6,10) is realizable in $K_{4,4}$, (4,10,10) in $K_{4,6}$ (and, equivalently, in $K_{6,4}$) and (6,10,10,10) in $K_{6,6}$. Thus, using Lemma 4, we see that the graphs $K_{4,10}$ and $K_{6,10}$ can be decomposed into closed trails of length 10. To conclude the proof we can proceed as in the proof of Proposition 5, since $a = 10p$ and $b = 4q + 6r$ for appropriate integers p, q, r, $r \in \{0, 1\}$. □

Lemma 10. *Let a, b be even integers with $b \geqslant a \geqslant 4$ and $b \geqslant 8$. If for any $b' \in \{b-8, b-6, b-4\}$ with $b' \geqslant 4$ the graph $K_{a,b'}$ is* ADCT, *so is the graph $K_{a,b}$.*

Proof. Consider a nondecreasing sequence $S = (l_1, l_2, \ldots, l_p) \in \mathrm{Sct}(K_{a,b})$. Put $s(j) := \sum_{i=1}^{j} l_i$ for $j = 0, 1, \ldots, p$, and let $q \in \{1, 2, \ldots, p\}$ be defined by inequalities $s(q-1) < 4a$ and $s(q) \geqslant 4a$.

(1) If $s(q) = 4a$, then the sequence $S^1 = (l_1, l_2, \ldots, l_q)$ is realizable in $K_{a,4}$ and the sequence $S^2 = (l_{q+1}, l_{q+2}, \ldots, l_p)$ in $K_{a,b-4}$. So, by Lemma 4, the sequence $S = S^1 \cdot S^2$ is realizable in $K_{a,b}$.

(2) If $s(q) = 4a + 2$, then clearly $l_q \geqslant 6$ and $s(q-1) \leqslant 4a - 4$.

(21) If $l_p \geqslant l_q + 2$, then the sequence $S^1 = (4a - s(q-1), l_1, l_2, \ldots, l_{q-1})$ is realizable in $K_{a,4}$ and $S^2 = (l_p - l_q + 2, l_q, l_{q+1}, \ldots, l_{p-1})$ in $K_{a,b-4}$. Since $4a - s(q-1) + l_p - l_q + 2 = l_p$, by Lemma 4 the sequence $S^1 + S^2 = (l_p, l_1, l_2, \ldots, l_{p-1}) \sim S$ is realizable in $K_{a,b}$. (We will write $S' \sim S''$ for sequences S' and S'' if one of them can be obtained from the other by permuting its terms.)

(22) $l_p = l_q = l$.

(221) If there is $r \in \{1, 2, \ldots, q-1\}$ such that $6 \leqslant l_r \leqslant l - 2$, then the sequence $S^1 = (l_r - 2, l_1, l_2, \ldots, l_{r-1}, l_{r+1}, l_{r+2}, \ldots, l_q)$ is realizable in $K_{a,4}$, $S^2 = (l_{q+1} - l_r + 2, l_r, l_{q+2}, l_{q+3}, \ldots, l_p)$ in $K_{a,b-4}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(222) If the assumption of (221) is not true, then $l_i \in \{4, l\}$ for $i = 1, 2, \ldots, p$, and, clearly, $l \equiv 2 \pmod 4$.

(2221) $l_2 = 4$.

(22211) If $l = 6$, then the sequence $S^1 = (l_3, l_4, \ldots, l_{q+1})$ is realizable in $K_{a,4}$, $S^2 = (l_1, l_2, l_{q+2}, l_{q+3}, \ldots, l_p)$ in $K_{a,b-4}$ and $S \sim S^1 \cdot S^2$ in $K_{a,b}$.

(22212) If $l \geqslant 10$, then the sequence $S^1 = (6, l_3, l_4, \dots, l_q)$ is realizable in $K_{a,4}$, $S^2 = (l_{q+1} - 6, l_1, l_2, l_{q+2}, l_{q+3}, \dots, l_p)$ in $K_{a,b-4}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(2222) $l_2 = l$.

(22221) $l_1 = 4$.

(222211) If $b = 8$, then $4 + (p-1)l = 8a$ and $l \mid 8a - 4$. Since $a \in \{4, 6, 8\}$, this is possible only if either $a \in \{4, 6\}$, $l = 4a - 2$ and $S = (4, 4a-2, 4a-2)$ or $a = 8$, $l \in \{6, 10\}$ and $S = S_4^l$ so that we can use Propositions 7 and 8.

(222212) $b \geqslant 10$.

(2222121) If $l = 6$, then the sequence $S^1 = (l_2, l_3, \dots, l_{a+1})$ is realizable in $K_{a,6}$, $S^2 = (l_1, l_{a+2}, l_{a+3}, \dots, l_p)$ in $K_{a,b-6}$ and $S \sim S^1 \cdot S^2$ in $K_{a,b}$.

(2222122) If $l \geqslant 10$, then $s(q) = 4 + (q-1)l = 4a + 2$, q is even and $(q-2)l = 4a - 2 - l$, so that $tl = 6a - 3 - \frac{1}{2}l$ for $t = q - 1 + \frac{1}{2}(q-2)$. Thus, the sequence $S^1 = (\frac{1}{2}l - 1, l_1, l_2, \dots, l_{t+1})$ is realizable in $K_{a,6}$, $S^2 = (l_{t+2} - \frac{1}{2}l + 1, l_{t+3}, l_{t+4}, \dots, l_p)$ in $K_{a,b-6}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(22222) If $l_1 = l$, then $pl = ab$, $l \mid ab$, $ql = 4a + 2$, q is odd and $tl = 6a + 3 - \frac{1}{2}l$ for $t = q + \frac{1}{2}(q-1)$.

(222221) If $l \in \{6, 10\}$, we are done by Propositions 5 and 9.

(222222) If $l \geqslant 14$, then $b \geqslant 10$ ($8a$ for $a \in \{4, 6, 8\}$ does not have an appropriate divisor), the sequence $S^1 = (\frac{1}{2}l - 3, l_1, l_2, \dots, l_t)$ is realizable in $K_{a,6}$, $S^2 = (l_{t+1} - \frac{1}{2}l + 3, l_{t+2}, l_{t+3}, \dots, l_p)$ in $K_{a,b-6}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(3) $s(q) \geqslant 4a + 4$.

(31) If $s(q-1) \leqslant 4a - 4$, then the sequence $S^1 = (4a - s(q-1), l_1, l_2, \dots, l_{q-1})$ is realizable in $K_{a,4}$, $S^2 = (s(q) - 4a, l_{q+1}, l_{q+2}, \dots, l_p)$ in $K_{a,b-4}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(32) $s(q-1) = 4a - 2$.

(321) $l = l_p \geqslant l_{q-1} + 2$.

(3211) If there is $r \in \{q, q+1, \dots, p\}$ such that $l_r = l_{q-1} + 2$, then the sequence $S^1 = (l_1, l_2, \dots, l_{q-2}, l_r)$ is realizable in $K_{a,4}$, $S^2 = (l_{q-1}, l_q, \dots, l_{r-1}, l_{r+1}, l_{r+2}, \dots, l_p)$ in $K_{a,b-4}$ and $S \sim S^1 \cdot S^2$ in $K_{a,b}$.

(3212) If $l_p \geqslant l_{q-1} + 6$, then the sequence $S^1 = (l_{q-1} + 2, l_1, l_2, \dots, l_{q-2})$ is realizable in $K_{a,4}$, $S^2 = (l_p - l_{q-1} - 2, l_{q-1}, l_q, \dots, l_{p-1})$ in $K_{a,b-4}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(3213) If the assumptions of (3211) and (3212) are not fulfilled, then $l_{q-1} = l - 4$, $l_i \in \{l - 4, l\}$ for $i = q, q+1, \dots, p$, and $l \geqslant 10$.

(32131) If $l_1 \leqslant l - 6$, then the sequence $S^1 = (l_1 + 2, l_2, l_3, \dots, l_{q-1})$ is realizable in $K_{a,4}$, $S^2 = (l_p - l_1 - 2, l_q, l_{q+1}, \dots, l_{p-1})$ in $K_{a,b-4}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(32132) If $l_1 = l - 4$, then $(q-1)(l-4) = 4a - 2$, hence q is even and $l \equiv 2 \pmod 4$.

(321321) If $l_2 = l - 4$, then $p \geqslant 3$.

(3213211) If $l_{p-1} = l$, then the sequence $S^1 = (l_{p-1} - 6, l_p, l_3, l_4, \dots, l_{q-1})$ is realizable in $K_{a,4}$, $S^2 = (6, l_q, l_{q+1}, \dots, l_{p-2})$ in $K_{a,b-4}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(3213212) If $l_{p-1} = l - 4$, then $(p-1)(l-4) + l = ab$ and $l - 4 \mid ab - 4$.

(32132121) If $b = 8$, then $l - 4 \mid 8a - 4$. Since $a \in \{4, 6, 8\}$ and $p \geqslant 3$, the only possibility is $a = 8$, $l = 14$, $S = S^{10}_{14}$ and we are done by Proposition 8.

(32132122) If $b \geqslant 10$, then $t(l-4) = 6a - 1 - \frac{1}{2}l$ for $t = q - 1 + \frac{1}{2}(q-2)$, the sequence $S^1 = (\frac{1}{2}l + 1, l_1, l_2, \ldots, l_t)$ is realizable in $K_{a,6}$, $S^2 = (l_{t+1} - \frac{1}{2}l - 1, l_{t+2}, l_{t+3}, \ldots, l_p)$ in $K_{a,b-6}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(321322) If $l_2 = l$, then $q = 2$, $l - 4 = 4a - 2$, $l - 4 + (p-1)l = ab$ and $l \mid ab + 4$.

(3213221) If $b = 8$, then $l \mid 8a + 4$, which yields as possible just the pairs $(a, l) = (4, 18), (6, 26), (8, 34)$ and the sequence $S = (4a - 2, 4a + 2)$. Thus, we are done by Proposition 7.

(3213222) If $b \geqslant 10$, then the sequence $S^1 = (2a - 2, l_2)$ is realizable in $K_{a,6}$, $S^2 = (l_3 - 2a + 2, l_1, l_4, l_5, \ldots, l_p)$ in $K_{a,b-6}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(322) If $l = l_p = l_{q-1}$, then also $l_{p-1} = l$.

(3221) If there is $r \in \{1, 2, \ldots, q-2\}$ such that $l_r = l - 2$, then the sequence $S^1 = (l_p, l_1, l_2, \ldots, l_{r-1}, l_{r+1}, l_{r+2}, \ldots, l_{q-1})$ is realizable in $K_{a,4}$, $S^2 = (l_q, l_{q+1}, \ldots, l_{p-1})$ in $K_{a,b-4}$ and $S \sim S^1 \cdot S^2$ in $K_{a,b}$.

(3222) If $l_1 \leqslant l - 6$, then the sequence $S^1 = (l_1 + 2, l_2, l_3, \ldots, l_{q-1})$ is realizable in $K_{a,4}$, $S^2 = (l_p - l_1 - 2, l_q, l_{q+1}, \ldots, l_{p-1})$ in $K_{a,b-4}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(3223) If the assumptions of (3221) and (3222) are not fulfilled, then $l_i \in \{l-4, l\}$ for $i = 1, 2, \ldots, p$, and consequently $l \equiv 2 \pmod 4$.

(32231) If $l_1 = l - 4$, then $l \geqslant 10$.

(322311) If $l_2 = l - 4$, then the sequence $S^1 = (l_{p-1} - 6, l_p, l_3, l_4, \ldots, l_{q-1})$ is realizable in $K_{a,4}$, $S^2 = (6, l_q, l_{q+1}, \ldots, l_{p-2})$ in $K_{a,b-4}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(322312) If $l_2 = l$, then $l - 4 + (p-1)l = ab$. Since $1 < q - 1 < p$, we have $p \geqslant 3$ and so $b \geqslant 10$ (as in (3213221), the assumption $b = 8$ would lead to $p = 2$). Moreover, $l - 4 + (q-2)l = 4a - 2$, q is even and $tl = 6a + 3 - \frac{1}{2}l$ for $t = q - 1 + \frac{1}{2}(q-2)$.

(3223121) If $l = 10$, then $b \geqslant 12$ and $6 + 10(2q - 3) = 8a$. So, the sequence $S^1 = (l_1, l_2, \ldots, l_{2q-2})$ is realizable in $K_{a,8}$, $S^2 = (l_{2q-1}, l_{2q}, \ldots, l_p)$ in $K_{a,b-8}$ and $S = S^1 \cdot S^2$ in $K_{a,b}$.

(3223122) If $l \geqslant 14$, then the sequence $S^1 = (\frac{1}{2}l - 3, l_1, l_2, \ldots, l_t)$ is realizable in $K_{a,6}$, $S^2 = (l_{t+1} - \frac{1}{2}l + 3, l_{t+2}, l_{t+3}, \ldots, l_p)$ in $K_{a,b-6}$ and $S \sim S^1 + S^2$ in $K_{a,b}$.

(32232) If $l_1 = l$, then $pl = ab$ and $l \mid ab$.

(322321) If $l \in \{6, 10\}$, then we are done by Propositions 5 and 9.

(322322) If $l \geqslant 14$, then necessarily $b \geqslant 10$ (the assumption $b = 8$ would mean $8 \mid p$ and $l \leqslant a \leqslant b$). Moreover, $(q-1)l = 4a - 2$, q is even and $tl = 6a - 3 - \frac{1}{2}l$ for $t = q - 1 + \frac{1}{2}(q-2)$. Thus, the sequence $S^1 = (\frac{1}{2}l + 3, l_1, l_2, \ldots, l_t)$ is realizable in $K_{a,6}$, $S^2 = (l_{t+1} - \frac{1}{2}l - 3, l_{t+2}, l_{t+3}, \ldots, l_p)$ in $K_{a,b-6}$ and $S \sim S^1 + S^2$ in $K_{a,b}$. $\square$

3. Proof of the main theorem

With respect to Proposition 3 it is sufficient to show that for any even integer $a \geqslant 4$ the following statement $S(a)$ is true: For any even integer $b \geqslant 4$ the graph $K_{a,b}$ is ADCT.

We proceed by induction on a. Because of Theorem 6 the graphs $K_{4,4}$, $K_{4,6}$, $K_{6,4}$ and $K_{6,6}$ are ADCT. Thus, by induction on b using Lemma 10, the statements $S(4)$ and $S(6)$ are true.

So, suppose that $a \geqslant 8$ and $S(a')$ is true for every even integer a' with $4 \leqslant a' \leqslant a - 2$. If b is an even integer with $4 \leqslant b \leqslant a - 2$, then the graph $K_{a,b}$ isomorphic to $K_{b,a}$ is ADCT by $S(b)$. Now, assume that b is an even integer with $b \geqslant a$ and that for every even integer b' with $4 \leqslant b' \leqslant b - 2$ the graph $K_{a,b'}$ is ADCT. Then, by Lemma 10, the graph $K_{a,b}$ is ADCT, which shows that $S(a)$ is true.

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