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ON GENERALIZATIONS OF OSTROWSKI INEQUALITY AND SOME RELATED RESULTS

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Abstract. Some generalizations of the Ostrowski inequality, the Milovanović-Pečarić-Fink inequality, the Dragomir-Agarwal inequality and the Hadamard inequality are given.

Keywords: Ostrowski inequality, Milovanović-Pečarić-Fink inequality, Dragomir-Agarwal inequality, Hadamard inequality

MSC 2000: 26D10, 26D15

1. Introduction

In 1938, Ostrowski [1] (see also [2, p. 468]) proved the following integral inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M \tag{1.1}$$

where $f\colon\ [a,b]\to\mathbb{R}$ is a differentiable function such that $|f'(x)|\leqslant M$ for all $x\in[a,b]$.

G. V. Milovanović and J. Pečarić [3] and A. M. Fink [4] (see also [2, p. 470]) have considered generalizations of (1.1) in the form

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}F_k(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant K(n,p,x)\|f^{(n)}\|_p \tag{1.2}$$

where $F_k(x)$ is defined by

$$F_k(x)=\frac{n-k}{k!\,(b-a)}[f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k] \tag{1.3}$$

so that they estimated a "two point expressions of f". For $n = 1$ the above sum is defined to be zero. As usual, let $1/p + 1/p' = 1$ with $p' = 1$ for $p = \infty$, $p' = \infty$ for $p = 1$, and

$$\|f\|_p = \left(\int_a^b |f(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}.$$

In fact, G. V. Milovanović and J. Pečarić have proved that ([2, p. 469])

$$K(n,\infty,x) = \frac{(x-a)^{n+1} + (b-x)^{n+1}}{n(n+1)!\,(b-a)} \tag{1.4}$$

while A. M. Fink gave the following generalization of this result ([2, p. 473]):

Theorem 1. *Let $f^{(n-1)}$ be absolutely continuous on (a, b) and let $f^{(n)} \in L_p(a, b)$. Then the inequality (1.2) holds with*

$$K(n,p,x) = \frac{[(x-a)^{np'+1} + (b-x)^{np'+1}]^{1/p'}}{n!\,(b-a)} B((n-1)p'+1, p'+1)^{1/p'}, \tag{1.5}$$

where $1 < p \leqslant \infty$, B is the beta function, and

$$K(n,1,x) = \frac{(n-1)^{n-1}}{n^n n!\,(b-a)} \max[(x-a)^n, (b-x)^n]. \tag{1.6}$$

Moreover, for $1 < p$ the inequality (1.2) is the best possible in the strong sense that for any $x \in (a, b)$ there is an f for which equality holds at x.

In fact, for $n = 1$ relation (1.6) becomes

$$K(1,1,x) = \frac{1}{b-a} \max[x-a, b-x]. \tag{1.7}$$

This result was recently obtained by S. S. Dragomir and S. Wang [5] in an equivalent form

$$K(1,1,x) = \frac{1}{2} + \frac{1}{b-a}\left|x - \frac{a+b}{2}\right|. \tag{1.8}$$

Of course, since $\max[(x-a)^n, (b-x)^n] = \max^n[(x-a), (b-x)]$, one can write (1.6) in an equivalent form

$$K(n,1,x) = \frac{(n-1)^{n-1}}{n!\,n^n(b-a)} \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^n. \tag{1.9}$$

Dragomir and Wang have also given various applications of their result. Moreover, Dragomir and Wang [6] also obtained (1.5) for $n = 1$, that is

$$K(1, p, x) = \frac{[(x-a)^{p'+1} + (b-x)^{p'+1}]^{1/p'}}{(b-a)(p'+1)^{1/p'}} \tag{1.10}$$

and gave various applications of this result.

In this paper we will give generalizations of the previous results as well as some related ones.

2. Some identities

Let (P_n) be a harmonic sequence of polynomials, that is $P_n' = P_{n-1}$, $n \geqslant 1$, $P_0 = 1$. Furthermore, let $I \subset \mathbb{R}$ be a segment and let $f\colon I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geqslant 1$. Then the following generalized Taylor formula is valid [7]:

$$\begin{aligned} f(y) = f(x) &+ \sum_{k=1}^{n-1} (-1)^k [P_k(x) f^{(k)}(x) - P_k(y) f^{(k)}(y)] \\ &+ (-1)^n \int_y^x P_{n-1}(t) f^{(n)}(t)\,\mathrm{d}t \end{aligned} \tag{2.1}$$

for $x, y \in I$. If we set $x = a$, $y = b$, $n = m + 1$ and replace $f(t)$ by $\int_a^t f(u)\,\mathrm{d}u$ in (2.1) we get

$$\begin{aligned} \int_a^b f(t)\,\mathrm{d}t = &\sum_{k=1}^{m} (-1)^k [P_k(a) f^{(k-1)}(a) - P_k(b) f^{(k-1)}(b)] \\ &+ (-1)^m \int_a^b P_m(t) f^{(m)}(t)\,\mathrm{d}t. \end{aligned} \tag{2.2}$$

By integration, (2.1) becomes

$$\begin{aligned} \int_a^b f(y)\,\mathrm{d}y = (b-a)&\Big[f(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) f^{(k)}(x)\Big] \\ &- \sum_{k=1}^{n-1} (-1)^k \int_a^b P_k(y) f^{(k)}(y)\,\mathrm{d}y \\ &+ (-1)^n \int_a^b \int_y^x P_{n-1}(t) f^{(n)}(t)\,\mathrm{d}t\,\mathrm{d}y. \end{aligned} \tag{2.3}$$

Using (2.2), we have

$$\int_a^b f(y)\,\mathrm{d}y = (b-a)\Big[f(x) + \sum_{k=1}^{n-1}(-1)^k P_k(x) f^{(k)}(x)\Big] - \sum_{k=1}^{n-1}\Big[\sum_{j=1}^{k}(-1)^j[P_j(b)f^{(j-1)}(b) - P_j(a)f^{(j-1)}(a)] + \int_a^b f(t)\,\mathrm{d}t\Big] + (-1)^n \int_a^b \int_y^x P_{n-1}(t) f^{(n)}(t)\,\mathrm{d}t\,\mathrm{d}y,$$

that is,

$$n\int_a^b f(y)\,\mathrm{d}y = (b-a)\Big[f(x) + \sum_{k=1}^{n-1}(-1)^k P_k(x) f^{(k)}(x)\Big] - \sum_{k=1}^{n-1}(-1)^k (n-k)[P_k(b)f^{(k-1)}(b) - P_k(a)f^{(k-1)}(a)] + (-1)^n \int_a^b \int_y^x P_{n-1}(t) f^{(n)}(t)\,\mathrm{d}t\,\mathrm{d}y. \tag{2.4}$$

Using the notation

$$\widetilde{F_k} = \frac{(-1)^k (n-k)}{b-a}[P_k(a)f^{(k-1)}(a) - P_k(b)f^{(k-1)}(b)]$$

and

$$k(t,x) = \begin{cases} t-a & \text{if } t \in [a,x], \\ t-b & \text{if } t \in (x,b], \end{cases}$$

relation (2.4) becomes

$$\frac{1}{n}\Big[f(x) + \sum_{k=1}^{n-1}(-1)^k P_k(x) f^{(k)}(x) + \sum_{k=1}^{n-1}\widetilde{F_k}\Big] - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = \frac{(-1)^{n-1}}{n(b-a)}\int_a^b P_{n-1}(t) k(t,x) f^{(n)}(t)\,\mathrm{d}t. \tag{2.5}$$

The above sums are defined to be zero for $n = 1$.

For the harmonic sequence of polynomials

$$P_k(t) = \frac{(t-x)^k}{k!}, \quad k \geqslant 0,$$

relation (2.5) becomes a result from [4]:

$$\frac{1}{n}\Big[f(x)+\sum_{k=1}^{n-1}F_k(x)\Big]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = \frac{1}{n!\,(b-a)}\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)\,\mathrm{d}t \tag{2.6}$$

where $F_k(x)$ is defined by (1.3).

For the harmonic sequence of polynomials

$$P_k(t)=\frac{1}{k!}\Big(t-\frac{a+b}{2}\Big)^k,\quad k\geqslant 0,$$

relation (2.5) becomes

$$\begin{aligned}\frac{1}{n}\Big[f(x)&+\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\Big(x-\frac{a+b}{2}\Big)^k f^{(k)}(x)\\ &+\sum_{k=1}^{n-1}\frac{(b-a)^{k-1}(n-k)}{k!\,2^k}[f^{(k-1)}(a)-(-1)^k f^{(k-1)}(b)]\Big]\\ &-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\ =&\ \frac{1}{n!\,(b-a)}\int_a^b\Big(\frac{a+b}{2}-t\Big)^{n-1}k(t,x)f^{(n)}(t)\,\mathrm{d}t.\end{aligned} \tag{2.7}$$

Let us transform relation (2.5) to a form suitable for harmonic sequences defined on the segment $[0,1]$. Set $f=h$, $x=u$, $a=0$ and $b=1$. We have

$$\frac{1}{n}\Big[h(u)+\sum_{k=1}^{n-1}(-1)^kP_k(u)h^{(k)}(u)+\sum_{k=1}^{n-1}H_k\Big]-\int_0^1 h(t)\,\mathrm{d}t = \frac{(-1)^{n-1}}{n}\int_0^1 P_{n-1}(t)\widetilde{k}(t,u)h^{(n)}(t)\,\mathrm{d}t \tag{2.8}$$

where $H_k=(-1)^k(n-k)[P_k(0)h^{(k-1)}(0)-P_k(1)h^{(k-1)}(1)]$ and

$$\widetilde{k}(t,u)=\begin{cases} t & \text{if } t\in[0,u],\\ t-1 & \text{if } t\in(u,1].\end{cases}$$

Now, for $h(t)=f(a+t(b-a))$ and $u=\frac{x-a}{b-a}$, we have $h^{(k)}(t)=(b-a)^kf^{(k)}(a+t(b-a))$ and $h^{(k)}(u)=(b-a)^kf^{(k)}(x)$. Further,

$$H_k=(-1)^k(n-k)(b-a)^{k-1}[P_k(0)f^{(k-1)}(a)-P_k(1)f^{(k-1)}(b)]$$

and

$$\int_0^1 P_{n-1}(t)\widetilde{k}(t,u)h^{(n)}(t)\,\mathrm{d}t$$
$$= (b-a)^n \int_0^1 P_{n-1}(t)\widetilde{k}\Big(t, \frac{x-a}{b-a}\Big) f^{(n)}(a+t(b-a))\,\mathrm{d}t$$
$$= (b-a)^{n-1} \int_a^b P_{n-1}\Big(\frac{y-a}{b-a}\Big)\widetilde{k}\Big(\frac{y-a}{b-a}, \frac{x-a}{b-a}\Big) f^{(n)}(y)\,\mathrm{d}y$$
$$= (b-a)^{n-2} \int_a^b P_{n-1}\Big(\frac{y-a}{b-a}\Big) k(y,x) f^{(n)}(y)\,\mathrm{d}y$$

since $\widetilde{k}\big(\frac{y-a}{b-a}, \frac{x-a}{b-a}\big) = \frac{1}{b-a}k(y,x)$. Therefore (2.8) becomes

$$\text{(2.9)}\qquad \frac{1}{n}\Big[f(x) + \sum_{k=1}^{n-1}(-1)^k(b-a)^k P_k\Big(\frac{x-a}{b-a}\Big) f^{(k)}(x) + \sum_{k=1}^{n-1} H_k\Big] - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = \frac{(-1)^{n-1}}{n}(b-a)^{n-2}\int_a^b P_{n-1}\Big(\frac{y-a}{b-a}\Big) k(y,x) f^{(n)}(y)\,\mathrm{d}y.$$

This identity is suitable for some harmonic sequences of polynomials. Let us give two examples: Bernoulli polynomials and Euler polynomials.

Bernoulli polynomials $B_n(t)$ can be defined by the formula

$$\frac{x\mathrm{e}^{tx}}{\mathrm{e}^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!}x^n, \quad |x| < 2\pi, \quad t \in \mathbb{R}.$$

They satisfy the relation [10, 23.1]: $B_n'(t) = nB_{n-1}(t)$, $n \in \mathbb{N}$.

The sequence $P_n(t) = \frac{1}{n!}B_n(t)$, $n \geqslant 0$, is a harmonic sequence of polynomials. The numbers $B_n = B_n(0)$, $n \geqslant 0$, are called the Bernoulli numbers. We also have $B_n(1) = B_n(0) = B_n$, $n \geqslant 2$, and $B_{2n+1} = 0$, $n \geqslant 1$.

Now, for $P_n(t) = \frac{1}{n!}B_n(t)$, $0 \leqslant t \leqslant 1$, formula (2.9) becomes

$$\text{(2.10)}\qquad \frac{1}{n}\Big[f(x) + \sum_{k=1}^{n-1}\frac{(-1)^k}{k!}(b-a)^k B_k\Big(\frac{x-a}{b-a}\Big) f^{(k)}(x) + \sum_{k=1}^{n-1} \widetilde{H_k}\Big] - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = \frac{(-1)^{n-1}}{n!}(b-a)^{n-2}\int_a^b B_{n-1}\Big(\frac{y-a}{b-a}\Big) k(y,x) f^{(n)}(y)\,\mathrm{d}y$$

where $\widetilde{H_k} = 0$ for k odd, and

$$\widetilde{H_k} = \frac{(n-k)(b-a)^{k-1}}{k!} B_k [f^{(k-1)}(a) - f^{(k-1)}(b)]$$

for k even, and B_k is the Bernoulli number.

The other sequence important in this context is the sequence of Euler polynomials. These polynomials can be defined by the formula

$$\frac{2\mathrm{e}^{tx}}{\mathrm{e}^x + 1} = \sum_{n=0}^{\infty} \frac{E_n(t)}{n!} x^n, \quad |x| < \pi, \quad t \in \mathbb{R}.$$

They satisfy the relation [10, 23.1]: $E_n'(t) = nE_{n-1}(t)$, $n \in \mathbb{N}$.

The sequence $P_n(t) = \frac{1}{n!}E_n(t)$, $n \geqslant 0$, is a harmonic sequence of polynomials. Further, we have

$$E_n(0) = -E_n(1) = -\frac{2}{n+1}(2^{n+1} - 1)B_{n+1}, \quad n \in \mathbb{N}.$$

Now for $P_n(t) = \frac{1}{n!}E_n(t)$, $0 \leqslant t \leqslant 1$, formula (2.9) becomes

$$\begin{aligned} (2.11) \qquad & \frac{1}{n}\Bigg[f(x) + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!}(b-a)^k E_k\Big(\frac{x-a}{b-a}\Big) f^{(k)}(x) + \sum_{k=1}^{n-1} \widehat{H_k}\Bigg] \\ & - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \\ & = \frac{(-1)^{n-1}}{n!}(b-a)^{n-2} \int_a^b E_{n-1}\Big(\frac{y-a}{b-a}\Big) k(y,x) f^{(n)}(y)\,\mathrm{d}y, \end{aligned}$$

where $\widehat{H_k} = 0$ for k even, and

$$(2.12) \qquad \widehat{H_k} = \frac{2(2^{k+1} - 1)(n-k)}{(k+1)!}(b-a)^{k-1} B_{k+1}[f^{(k-1)}(a) + f^{(k-1)}(b)]$$

for k odd, and B_k is the Bernoulli number.

Relation (2.5) can be modified in another way, very useful in our context, by replacing $P_n(t)$ by $P_n(t-x)$. We get

$$\begin{aligned} (2.13) \qquad & \frac{1}{n}\Bigg[f(x) + \sum_{k=1}^{n-1}(-1)^k P_k(0) f^{(k)}(x) + \sum_{k=1}^{n-1} \widetilde{F_k}(x)\Bigg] \\ & - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \\ & = \frac{(-1)^{n-1}}{n(b-a)} \int_a^b P_{n-1}(t-x) k(t,x) f^{(n)}(t)\,\mathrm{d}t, \end{aligned}$$

where

$$\widetilde{F_k}(x) = \frac{(-1)^k(n-k)}{b-a}[P_k(a-x)f^{(k-1)}(a) - P_k(b-x)f^{(k-1)}(b)].$$

It is clear that (2.6) is a special case of this formula.

The notation of this section will be used throughout the rest of the paper.

3. Generalization of Milovanović-Pečarić-Fink inequality

Theorem 2. *Let $f\colon [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geqslant 1$ and $f^{(n)} \in L_p[a,b]$, $1 \leqslant p \leqslant \infty$. Then the inequality*

$$\left|\frac{1}{n}\left[f(x) + \sum_{k=1}^{n-1}(-1)^k P_k(x) f^{(k)}(x) + \sum_{k=1}^{n-1}\widetilde{F_k}\right] - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant C(n,p,x)\|f^{(n)}\|_p \tag{3.1}$$

holds for $x \in [a,b]$, and

$$C(n,p,x) = \frac{1}{n(b-a)}\|P_{n-1}k(\cdot,x)\|_{p'}, \tag{3.2}$$

where $1/p + 1/p' = 1$.

P r o o f. By (2.5) and Hölder's inequality we have

$$\begin{aligned}
&\left|\frac{1}{n}\left[f(x) + \sum_{k=1}^{n-1}(-1)^k P_k(x) f^{(k)}(x) + \sum_{k=1}^{n-1}\widetilde{F_k}\right] - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \\
&\quad= \left|\frac{(-1)^{n-1}}{n(b-a)}\int_a^b P_{n-1}(t)k(t,x)f^{(n)}(t)\,\mathrm{d}t\right| \\
&\quad\leqslant \frac{1}{n(b-a)}\int_a^b |P_{n-1}(t)k(t,x)f^{(n)}(t)|\,\mathrm{d}t \\
&\quad\leqslant \frac{1}{n(b-a)}\left[\int_a^b |P_{n-1}(t)k(t,x)|^{p'}\,\mathrm{d}t\right]^{1/p'}\left[\int_a^b |f^{(n)}(t)|^p\,\mathrm{d}t\right]^{1/p} \\
&\quad= C(n,p,x)\|f^{(n)}\|_p,
\end{aligned}$$

and (3.1) follows. □

Corollary 1. *Under the assumptions of the above theorem, we have*

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}F_k(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leqslant K(n,p,x)\|f^{(n)}\|_p$$

where $F_k(x)$ *is given by* (1.3) *and* $K(n,p,x)$ *by* (1.5).

Proof. Set $P_k(t)=\frac{1}{k!}(t-x)^k$, $k\geqslant 0$, in the theorem. The corollary is equivalent to Theorem 1 proved in [4], where we can find some additional interesting results concerning this inequality. □

Corollary 2. *Under the assumptions of Theorem* 2, *we have*

$$\begin{aligned}&\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\Big(x-\frac{a+b}{2}\Big)^k f^{(k)}(x)\right.\right.\\&\qquad\left.\left.+\sum_{k=1}^{n-1}\frac{(b-a)^{k-1}(n-k)}{k!\,2^k}[f^{(k-1)}(a)-(-1)^k f^{(k-1)}(b)]\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\\&\leqslant H(n,p,x)\|f^{(n)}\|_p,\end{aligned}$$

where $H(n,p,x)=\frac{1}{n(b-a)}\|P_{n-1}k(\cdot,x)\|_{p'}$.

Proof. Set $P_k(t)=\frac{1}{k!}\big(t-\frac{a+b}{2}\big)^k$, $k\geqslant 0$, in Theorem 2. □

Remark 1. The estimate $H(n,p,x)$ cannot be calculated easily. It can be roughly estimated by

$$H(n,p,x)\leqslant\frac{(b-a)^{n-1}}{2^{n-1}n!}.$$

One can easily see that $x\to H(n,p,x)$ has its maximum at $x=a$ or $x=b$ and minimum at $x=\frac{a+b}{2}$. This mimimum can be calculated as

$$H\Big(n,p,\frac{a+b}{2}\Big)=\frac{(b-a)^{n+1/p}}{2^n n!}B((n-1)p'+1,p'+1)^{1/p'},$$

where B is the beta function.

4. Inequalities of Dragomir-Agarwal type

S. S. Dragomir and R. P. Agarwal [8] have proved the following result:

Let $I \subset \mathbb{R}$ be an interval, $a, b \in I$, $a < b$, $f\colon I \to \mathbb{R}$ a differentiable function. If $|f'|^q$ is convex on $[a, b]$, where $1/p + 1/q = 1$, $1 < p$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{b-a}{2(p+1)^{1/p}}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \tag{4.1}$$

C. E. M. Pearce and J. Pečarić [9] have shown that the result can be improved, namely, the following inequality is valid for $q \geqslant 1$:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \tag{4.2}$$

Some similar results are also obtained in [9].

Here we will give some related results.

Theorem 3. *Let $I \subset \mathbb{R}$ be an interval, $a, b \in I$, $a < b$, $f\colon I \to \mathbb{R}$, $n \in \mathbb{N}$, $1/p + 1/p' = 1$, $p \geqslant 1$. Let $f^{(n-1)}$ be absolutely continuous on $[a, b]$ and such that $f^{(n)}(x)$ exists for all $x \in [a, b]$. Put*

$$\alpha(x) = \frac{\int_a^b \frac{t-a}{b-a}|P_{n-1}(t)k(t,x)|\,\mathrm{d}t}{\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t}, \qquad x \in (a, b).$$

(i) *If $|f^{(n)}|^{p'}$ is convex on $[a, b]$, then*

$$\begin{aligned} &\left|\frac{1}{n}\left[f(x) + \sum_{k=1}^{n-1}(-1)^k P_k(x) f^{(k)}(x) + \sum_{k=1}^{n-1}\widetilde{F_k}\right] - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \\ &\leqslant \frac{1}{n(b-a)}\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t \\ &\quad\times [\alpha(x)|f^{(n)}(b)|^{p'} + (1-\alpha(x))|f^{(n)}(a)|^{p'}]^{1/p'}. \end{aligned} \tag{4.3}$$

(ii) *If $|f^{(n)}|$ is concave on $[a, b]$, then*

$$\begin{aligned} &\left|\frac{1}{n}\left[f(x) + \sum_{k=1}^{n-1}(-1)^k P_k(x) f^{(k)}(x) + \sum_{k=1}^{n-1}\widetilde{F_k}\right] - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \\ &\leqslant \frac{1}{n(b-a)}\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t \cdot |f^{(n)}(\alpha(x)b + (1-\alpha(x))a)|. \end{aligned} \tag{4.4}$$

P r o o f. (i) Let us use the identity (2.5), Hölder's inequality and Jensen's discrete inequality. We obtain

$$\begin{aligned}
&\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}(-1)^k P_k(x) f^{(k)}(x)+\sum_{k=1}^{n-1}\widetilde{F_k}\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \\
&\quad\leqslant \frac{1}{n(b-a)}\int_a^b |P_{n-1}(t)k(t,x)|\cdot|f^{(n)}(t)|\,\mathrm{d}t \\
&\quad\leqslant \frac{1}{n(b-a)}\left[\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t\right]^{1/p}\cdot\left[\int_a^b |P_{n-1}(t)k(t,x)|\cdot|f^{(n)}(t)|^{p'}\,\mathrm{d}t\right]^{1/p'} \\
&\quad= \frac{1}{n(b-a)}\left[\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t\right]^{1/p} \\
&\qquad\times\left[\int_a^b |P_{n-1}(t)k(t,x)|\cdot\left|f^{(n)}\Big(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\Big)\right|^{p'}\mathrm{d}t\right]^{1/p'} \\
&\quad\leqslant \frac{1}{n(b-a)}\left[\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t\right]^{1/p}\cdot\left[|f^{(n)}(a)|^{p'}\int_a^b |P_{n-1}(t)k(t,x)|\frac{b-t}{b-a}\,\mathrm{d}t\right. \\
&\qquad\left.+|f^{(n)}(b)|^{p'}\int_a^b |P_{n-1}(t)k(t,x)|\frac{t-a}{b-a}\,\mathrm{d}t\right]^{1/p'} \\
&\quad= \frac{1}{n(b-a)}\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t\cdot[\alpha(x)|f^{(n)}(b)|^{p'}+(1-\alpha(x))|f^{(n)}(a)|^{p'}]^{1/p'}.
\end{aligned}$$

(ii) Again by the identity (2.5) and Jensen's integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}(-1)^k P_k(x) f^{(k)}(x)+\sum_{k=1}^{n-1}\widetilde{F_k}\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \\
&\quad\leqslant \frac{1}{n(b-a)}\int_a^b |P_{n-1}(t)k(t,x)|\cdot|f^{(n)}(t)|\,\mathrm{d}t \\
&\quad\leqslant \frac{1}{n(b-a)}\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t\cdot\left|f^{(n)}\left(\frac{\int_a^b |P_{n-1}(t)k(t,x)|t\,\mathrm{d}t}{\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t}\right)\right| \\
&\quad= \frac{1}{n(b-a)}\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t \\
&\qquad\times\left|f^{(n)}\left(\frac{\int_a^b |P_{n-1}(t)k(t,x)|\big(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\big)\,\mathrm{d}t}{\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t}\right)\right| \\
&\quad= \frac{1}{n(b-a)}\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t\cdot|f^{(n)}(\alpha(x)b+(1-\alpha(x))a)|,
\end{aligned}$$

which proves our assertion. □

Corollary 3. *Let f be as in Theorem 3 (i). Then*

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}F_k(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{(x-a)^{n+1}+(b-x)^{n+1}}{n(n+1)!\,(b-a)}\cdot[\widetilde{\alpha}(x)|f^{(n)}(b)|^{p'}+(1-\widetilde{\alpha}(x))|f^{(n)}(a)|^{p'}]^{1/p'},$$

where $F_k(x)$ is given by (1.3) *and $\widetilde{\alpha}(x)$ by*

$$\widetilde{\alpha}(x)=\frac{2(x-a)[(x-a)^{n+1}+(b-x)^{n+1}]+n(b-a)(b-x)^{n+1}}{(n+2)(b-a)[(x-a)^{n+1}+(b-x)^{n+1}]}.$$

Let f be as in Theorem 3 (ii). Then

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}F_k(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{(x-a)^{n+1}+(b-x)^{n+1}}{n(n+1)!\,(b-a)}\cdot|f^{(n)}(\widetilde{\alpha}(x)b+(1-\widetilde{\alpha}(x))a)|.$$

Proof. Set $P_k(t)=\frac{1}{k!}(t-x)^k$, $k\geqslant 0$. Then

$$\int_a^b |P_{n-1}(t)k(t,x)|\,\mathrm{d}t=\frac{(x-a)^{n+1}+(b-x)^{n+1}}{(n+1)!}$$

and

$$\int_a^b (t-a)|P_{n-1}(t)k(t,x)|\,\mathrm{d}t = \frac{2(x-a)[(x-a)^{n+1}+(b-x)^{n+1}]+n(b-a)(b-x)^{n+1}}{(n+2)!},$$

which proves our assertion. □

Corollary 4. *Let f be as in Theorem 3. Put*

$$A=\frac{1}{n}\left[f\Big(\frac{a+b}{2}\Big)+\sum_{k=1}^{n-1}\frac{(n-k)(b-a)^{k-1}}{2^k k!}[f^{(k-1)}(a)-(-1)^k f^{(k-1)}(b)]\right] -\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t.$$

(i) *If* $|f^{(n)}|^{p'}$ *is convex on* $[a,b]$, *then*

$$|A| \leqslant \frac{(b-a)^n}{2^n n(n+1)!}\left[\frac{|f^{(n)}(a)|^{p'}+|f^{(n)}(b)|^{p'}}{2}\right]^{1/p'}.$$

(ii) *If* $|f^{(n)}|$ *is concave on* $[a,b]$, *then*

$$|A| \leqslant \frac{(b-a)^n}{2^n n(n+1)!}\left|f^{(n)}\Big(\frac{a+b}{2}\Big)\right|.$$

Proof. The result follows by putting $x=\frac{1}{2}(a+b)$ in Corollary 3. □

Remark 2. For $n=1$ the inequalities of the above theorem become

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{b-a}{4}\left[\frac{|f'(b)|^{p'}+|f'(a)|^{p'}}{2}\right]^{1/p'}$$

and

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{b-a}{4}\left|f'\Big(\frac{a+b}{2}\Big)\right|.$$

These inequalities have been proved in [9].

5. Inequalities of Hadamard type

The Hadamard inequalities for convex functions are one of the cornerstones of mathematical analysis: if $f\colon [a,b]\to\mathbb{R}$ is a convex function, then

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \leqslant \frac{f(a)+f(b)}{2}.$$

Here we will give some generalizations of these inequalities. We use the same notation as above. Further, to simplify notation, we denote the expression

$$(-1)^{n-1}\left[\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}(-1)^k P_k(0) f^{(k)}(x)+\sum_{k=1}^{n-1}\widetilde{F_k}(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right]$$

by $J_n(x)$ and let

$$S_n(x)=\int_a^b P_{n-1}(t-x)k(t,x)\,\mathrm{d}t.$$

Theorem 4. *Suppose that*

$$P_{n-1}(t-x)k(t,x) \geqslant 0, \quad \textit{for all} \quad t \in [a,b]. \tag{5.1}$$

If $f^{(n)}(t) \geqslant 0$ for every $t \in [a,b]$, then $J_n(x) \geqslant 0$. If $f^{(n)}(t) \leqslant 0$ for every $t \in [a,b]$, then $J_n(x) \leqslant 0$. Moreover, if the reverse inequality holds in (5.1)*, then we obtain the reverse inequalities for $J_n(x)$.*

Proof. The identity (2.13) can be written as

$$J_n(x) = \frac{1}{n(b-a)} \int_a^b P_{n-1}(t-x)k(t,x)f^{(n)}(t)\,\mathrm{d}t.$$

Our assertion follows immediately from this relation. □

Theorem 5. *Let $f^{(n)}$ be convex on $[a,b]$ and let*

$$P_{n-1}(t-x)k(t,x) \geqslant 0 \quad \textit{or} \quad P_{n-1}(t-x)k(t,x) \leqslant 0$$

for every $t \in [a,b]$. Then

$$f^{(n)}(\beta(x)b + (1-\beta(x))a) \leqslant n(b-a)\frac{J_n(x)}{S_n(x)} \leqslant \beta(x)f^{(n)}(b) + (1-\beta(x))f^{(n)}(a),$$

where

$$\beta(x) = \frac{1}{(b-a)S_n(x)} \int_a^b (t-a)P_{n-1}(t-x)k(t,x)\,\mathrm{d}t.$$

If $f^{(n)}$ is concave on $[a,b]$ the reverse inequality holds.

Proof. Let (5.1) hold. Then $S_n(x) \geqslant 0$ and by applying Jensen's integral inequality to the relation (2.13) we have

$$\begin{aligned}
J_n(x) &= \frac{1}{n(b-a)} \int_a^b P_{n-1}(t-x)k(t,x)f^{(n)}(t)\,\mathrm{d}t \\
&\geqslant \frac{1}{n(b-a)} S_n(x) \cdot f^{(n)}\Big(\frac{1}{S_n(x)} \int_a^b P_{n-1}(t-x)k(t,x)t\,\mathrm{d}t\Big) \\
&= \frac{1}{n(b-a)} S_n(x) \cdot f^{(n)}\Big(\frac{1}{S_n(x)} \int_a^b P_{n-1}(t-x)k(t,x)\Big(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\Big)\,\mathrm{d}t\Big) \\
&= \frac{1}{n(b-a)} S_n(x) \cdot f^{(n)}(\beta(x)b + (1-\beta(x))a).
\end{aligned}$$

On the other hand, by applying discrete Jensen's inequality to relation (2.13), we have

$$\begin{aligned}
J_n(x) &= \frac{1}{n(b-a)} \int_a^b P_{n-1}(t-x)k(t,x)f^{(n)}(t)\,\mathrm{d}t \\
&= \frac{1}{n(b-a)} \int_a^b P_{n-1}(t-x)k(t,x)f^{(n)}\Big(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\Big)\,\mathrm{d}t \\
&\leqslant \frac{1}{n(b-a)} S_n(x) \cdot (\beta(x)f^{(n)}(b) + (1-\beta(x))f^{(n)}(a)),
\end{aligned}$$

which proves our assertion in this case. If the reverse inequality holds in (5.1), apply the same calculations to $-J_n(x)$ and $-S_n(x)$. If $f^{(n)}$ is concave on $[a,b]$, apply the above arguments to $-f^{(n)}$. □

The important case of the harmonic sequence of polynomials $P_k(t) = \frac{1}{k!}t^k$, $k \geqslant 0$, admits explicit calculations. In this case we have

$$\begin{aligned}
J_n(x) &= (-1)^{n-1}\left[\frac{1}{n}\left[f(x) + \sum_{k=1}^{n-1}\widetilde{F_k}(x)\right] - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right], \\
\widetilde{F_k}(x) &= \frac{n-k}{k!\,(b-a)}[f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k]
\end{aligned}$$

and

$$S_n(x) = \frac{1}{(n+1)!}[(a-x)^{n+1} - (b-x)^{n+1}].$$

If n is odd, then $P_{n-1}(t-x)k(t,x)$ changes its sign on $[a,b]$ (except for $x=a$ or $x=b$). If n is even, then

$$\begin{aligned}
P_{n-1}(t-x)k(t,x) &\leqslant 0 \quad \text{for all } t \in [a,b], \\
S_n(x) &= \frac{-1}{(n+1)!}[(x-a)^{n+1} + (b-x)^{n+1}]
\end{aligned}$$

and

$$S_n\Big(\frac{a+b}{2}\Big) = -\frac{(b-a)^{n+1}}{2^n(n+1)!}$$

and the above theorem applies. If $x=a$ or $x=b$ the theorem applies for every n.

For every n we have

$$S_n(b) = \frac{(-1)^{n-1}}{(n+1)!}(b-a)^{n+1} \quad \text{and} \quad S_n(a) = -\frac{(b-a)^{n+1}}{(n+1)!}.$$

Corollary 5. *Let $f^{(n)}$ be convex on $[a,b]$ and let n be even. Then*

$$\begin{aligned}
&f^{(n)}(\widetilde{\alpha}(x)b + (1-\widetilde{\alpha}(x))a)\\
&\quad \leqslant \frac{n(b-a)(n+1)!}{(x-a)^{n+1}+(b-x)^{n+1}}\left[\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}\widetilde{F_k}(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right]\\
&\quad \leqslant \widetilde{\alpha}(x)f^{(n)}(b)+(1-\widetilde{\alpha}(x))f^{(n)}(a)
\end{aligned}$$

where $\widetilde{\alpha}(x)$ is defined in Corollary 3.

Proof. The result follows by putting $P_k(t)=\frac{1}{k!}t^k$, $k\geqslant 0$, in Theorem 5. □

Corollary 6. *Let $f^{(n)}$ be convex on $[a,b]$ and let n be even. Then*

$$\begin{aligned}
f^{(n)}\Big(\frac{a+b}{2}\Big) &\leqslant \frac{2^n n(n+1)!}{(b-a)^n}\cdot\left[\frac{1}{n}\left[f\Big(\frac{a+b}{2}\Big)\right.\right.\\
&\quad \left.+\sum_{k=1}^{n-1}\frac{(b-a)^{k-1}(n-k)}{2^k k!}[f^{(k-1)}(a)-(-1)^k f^{(k-1)}(b)]\right]\\
&\quad \left.-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right]\\
&\leqslant \frac{f^{(n)}(a)+f^{(n)}(b)}{2}.
\end{aligned}$$

Proof. The result follows by putting $x=\frac{1}{2}(a+b)$ in Corollary 5. □

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