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Precovers

Czechoslovak Mathematical Journal, Vol. 53 (2003), No. 1, 191–203

Persistent URL: http://dml.cz/dmlcz/127790

Czechoslovak Mathematical Journal, 53 (128) (2003), 191–203

PRECOVERS

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(Received February 28, 2000)

Abstract. Let $\mathscr{G}$ be an abstract class (closed under isomorpic copies) of left R-modules. In the first part of the paper some sufficient conditions under which $\mathscr{G}$ is a precover class are given. The next section studies the $\mathscr{G}$-precovers which are $\mathscr{G}$-covers. In the final part the results obtained are applied to the hereditary torsion theories on the category on left R-modules. Especially, several sufficient conditions for the existence of σ-torsionfree and σ-torsionfree σ-injective covers are presented.

Keywords: precover, cover, (pre)cover class of modules, hereditary torsion theory, relatively injective modules

MSC 2000: 16D90, 16S90, 16D50

Throughout this paper R denotes a ring with identity and $\sigma = (\mathscr{T}, \mathscr{F})$ a hereditary torsion theory in the category of left R modules, R-mod. An R-module M is said to be σ-injective if $\operatorname{Ext}^1_R(T, M) = 0$ for any σ-torsion module T.

In order to study the structure of a module, it is useful to approximate the module using the so-called $\mathscr{G}$-cover, where $\mathscr{G}$ is a class of left R-modules. The crucial question is the existence of such covers (cf. [9]). Associated to a torsion theory σ there exist two important classes of modules, the class of σ-torsionfree modules and the class of σ-torsionfree σ-injective modules (cf. [6], [7], [8]). In this note we consider the problem of existence of covers for a general class of modules and we apply our results to the case of the above mentioned two classes.

This work has been initiated while the first author was visiting the University of Almeria. The first author has been partially supported by the Grant Agency of the Charles University, grant #GAUK 10/97/B–MAT/MFF and also by the institutional grant CEZ # J13/98: 113 200 007. The second author has been partially supported by PB98-1005 from DGES.

1. Auxiliary results

Recall that a class of modules is said to be *abstract*, if it is closed under isomorphic copies, *co-abstract*, if its members are pairwise non-isomorphic, *hereditary*, if it is abstract and closed under submodules and *inductive*, if it is closed under unions of chains. We further say that the homomorphisms $f\colon F\to M$ and $g\colon G\to M$ are M*-equivalent*, if there is an isomorphism $\pi\colon F\to G$ such that $g\pi=f$.

Lemma 1.1. *Let $F=\bigoplus\limits_{\delta\in D}F_\delta$ be a direct sum of modules and $f\colon F\to M$ an arbitrary homomorphism. Then there is a subset $D'\subseteq D$ such that $F=U\oplus V$, where $U=\bigoplus\limits_{\delta\in D'}F_\delta$, $V\subseteq\operatorname{Ker}f$ and for $\delta,\varepsilon\in D'$, $\delta\neq\varepsilon$, the homomorphisms $f\big|F_\delta$ and $f\big|F_\varepsilon$ are not M-equivalent.*

P r o o f. For the sake of simplicity denote $f_\delta=f\big|F_\delta$ for every $\delta\in D$ and we define an equivalence relation $\sim$ on D in such a way that $\delta\sim\varepsilon$ if and only if the homomorphisms f_δ and f_ε are M-equivalent. In this case we denote by $\pi_{\varepsilon\delta}\colon F_\delta\to F_\varepsilon$ the isomorphism for which $f_\varepsilon\pi_{\varepsilon\delta}=f_\delta$. For each $\delta\in D$ let $D_\delta=\{\varepsilon\in D\mid\varepsilon\sim\delta\}$ be the equivalence class containing δ and $D'_\delta=D_\delta\setminus\{\delta\}$. Now for every $\varepsilon\in D'_\delta$ we set $G_{\varepsilon\delta}=\{x-\pi_{\varepsilon\delta}(x)\mid x\in F_\delta\}$ and we are going to verify that

$$\bigoplus_{\varepsilon\in D_\delta}F_\varepsilon=F_\delta\oplus\Bigl(\bigoplus_{\varepsilon\in D'_\delta}G_{\varepsilon\delta}\Bigr).$$

In order to show that the sum on the right is direct, let $x+\sum\limits_{i=1}^{n}y_i\in F_\delta+\sum\limits_{\varepsilon\in D'_\delta}G_{\varepsilon\delta}$ be such that $x\in F_\delta$, $y_i\in G_{\varepsilon_i\delta}$, where $\varepsilon_1,\dots,\varepsilon_n\in D'_\delta$ are pairwise different and $x+\sum\limits_{i=1}^{n}y_i=0$. There are elements $x_i\in F_\delta$ such that $y_i=x_i-\pi_{\varepsilon_i\delta}(x_i)$, $i=1,\dots,n$, and so $x+\sum\limits_{i=1}^{n}y_i=x+\sum\limits_{i=1}^{n}x_i-\sum\limits_{i=1}^{n}\pi_{\varepsilon_i\delta}(x_i)=0$, which yields $x_i=0$ for every $i=1,\dots,n$ and consequently $x=0$. Now if $x+\sum\limits_{i=1}^{n}y_i$, $x\in F_\delta$, $y_i\in F_{\varepsilon_i}$, $\{\varepsilon_1,\dots,\varepsilon_n\}\subseteq D'_\delta$, are arbitrary, then $y_i=\pi_{\varepsilon_i\delta}(x_i)$ for suitable $x_i\in F_\delta$, $i=1,\dots,n$, hence $x+\sum\limits_{i=1}^{n}y_i=x+\sum\limits_{i=1}^{n}x_i-\Bigl(\sum\limits_{i=1}^{n}(x_i-\pi_{\delta\varepsilon_i}(x_i)\Bigr)\in F_\delta\oplus\Bigl(\bigoplus\limits_{\varepsilon\in D'_\delta}G_{\varepsilon\delta}\Bigr)$ and the above equality is proved.

Let $D'\subseteq D$ be any set of representatives of the equivalence classes under $\sim$. Setting $U=\bigoplus\limits_{\delta\in D'}F_\delta$ and $V=\bigoplus\limits_{\delta\in D'}(\bigoplus\limits_{\varepsilon\in D'_\delta}G_{\varepsilon\delta})$, we obviously have $F=U\oplus V$, where $V\subseteq\operatorname{Ker}f$ by the definition of the relation $\sim$. Finally, the M-equivalence of f_δ and f_ε means that $\delta\sim\varepsilon$, which is impossible for $\delta,\varepsilon\in D'$, $\delta\neq\varepsilon$. $\square$

Definition 1.2. Let $\mathscr{G}$ be an abstract class of modules and let $\mathscr{G}' = \{G_\alpha \mid \alpha \in A\}$ be a co-abstract subset of $\mathscr{G}$. If $M \in R$-mod is arbitrary, then for every $\alpha \in A$ and $g \in \operatorname{Hom}(G_\alpha, M)$ we denote by $G_{\alpha g}$ an isomorphic copy of G_α. For all subsets $B \subseteq A$ and $H_\alpha \subseteq \operatorname{Hom}(G_\alpha, M)$ we take the direct sum $Y = \bigoplus_{\alpha \in B} \Big(\bigoplus_{g \in H_\alpha} G_{\alpha g} \Big)$ and denote by $\{X_\gamma \mid \gamma \in C\}$ the set of all modules from $\mathscr{G}$ which lie between Y and $E(Y)$ for some Y, where $E(Y)$ is a fixed injective envelope of Y. Now for each $\gamma \in C$ and each $g \in \operatorname{Hom}(X_\gamma, M)$ we take an isomorphic copy $X_{\gamma g}$ of X_γ together with the isomorphism $\psi_{\gamma g}\colon X_\gamma \to X_{\gamma g}$ and we finally set

$$G = G_M = \bigoplus_{\gamma \in C} \Big(\bigoplus\nolimits_{g \in \operatorname{Hom}(X_\gamma, M)} X_{\gamma g} \Big). \tag{1}$$

Moreover, $\varphi = \varphi_M\colon G_M \to M$ will denote the natural evaluation homomorphism induced by the maps $g\psi_{\gamma g}^{-1}\colon X_{\gamma g} \to M$.

Lemma 1.3. *Let $\mathscr{G}$ be an abstract class of modules and $\mathscr{G}'$ a co-abstract subset of $\mathscr{G}$. Further, let $M \in R$-mod be an arbitrary module, let $\varphi\colon G \to M$ be as in the preceding definition and let $f\colon F \to M$ with $F \in \mathscr{G}$ be an arbitrary homomorphism. If F contains an essential submodule $F' = \bigoplus_{\delta \in D'} F_\delta$, where each F_δ is isomorphic to a member of $\mathscr{G}'$, and for any $\delta, \varepsilon \in D'$, $\delta \neq \varepsilon$, the homomorphisms $f\big|F_\delta$ and $f\big|F_\varepsilon$ are not M-equivalent, then there is a homomorphism $g\colon F \to G$ such that $\varphi g = f$.*

Proof. As above, we will use the brief notation $f_\delta = f\big|F_\delta$ for each $\delta \in D'$. If $\{G_\alpha \mid \alpha \in A\}$ is any list of elements of $\mathscr{G}'$ then for each $\delta \in D'$ there is an isomorphism $\theta_{\alpha\delta}\colon F_\delta \to G_\alpha$ which induces isomorphism $\varphi_{\alpha\delta}\colon F_\delta \to G_{\alpha, f_\delta \theta_{\alpha\delta}^{-1}}$. Since the equality $f_\delta \theta_{\alpha\delta}^{-1} = f_\varepsilon \theta_{\beta\varepsilon}^{-1}$ for some $\delta \neq \varepsilon$ in D' yields a contradiction $f_\delta = f_\varepsilon \theta_{\beta\varepsilon}^{-1} \theta_{\alpha\delta}$, the isomorphisms $\varphi_{\alpha\delta}$ induce isomorphism $\psi'\colon F' \to Y$, $Y = \bigoplus_{\delta \in D'} G_{\alpha, f_\delta \theta_{\alpha\delta}^{-1}}$. This isomorphism extends to isomorphism $\psi\colon F \to X_\gamma$ for a suitable $X_\gamma \in \mathscr{G}$ lying between Y and its injective envelope $E(Y)$. Denoting $h = f\psi^{-1}\colon X_\gamma \to M$ and $\iota_{\gamma h}: X_{\gamma h} \to G$ the canonical embedding, we can take $g\colon F \to G$ as $g = \iota_{\gamma h}\psi_{\gamma h}\psi$. Then we have $\varphi g = \varphi\iota_{\gamma h}\psi_{\gamma h}\psi = h\psi_{\gamma h}^{-1}\psi_{\gamma h}\psi = h\psi = f$ and the proof is complete.

□

Let $\varphi\colon F \to M$ and $\psi\colon G \to M$ be homomorphisms. We define an *ordering* $\leqslant$ on the class of all pairs (F, φ) in such a way that we put $(F, \varphi) \leqslant (G, \psi)$ if and only if $F \leqslant G$ and $\psi\big|F = \varphi$.

Recall that for an abstract class $\mathscr{G}$ of modules a homomorphism $\varphi\colon G \to M$, $G \in \mathscr{G}$, is a *$\mathscr{G}$-precover* of the module M, if for each $F \in \mathscr{G}$ and each homomorphism $f\colon F \to M$ there is a homomorphism $g\colon F \to G$ such that $\varphi g = f$. A $\mathscr{G}$-precover φ

of M is called a $\mathscr{G}$-*cover*, if each endomorphism g of G with $\varphi g = \varphi$ is an automorphism of G.

Lemma 1.4. *Let $\mathscr{G}$ be an abstract class of modules and $\varphi\colon F \to M$ a $\mathscr{G}$-precover of the module M. If $f\colon F \to F$ is a non-surjective monomorphism such that $\varphi f = \varphi$, then there is a $\mathscr{G}$-precover $\varphi_0\colon F_0 \to M$ of M such that $(F, \varphi) < (F_0, \varphi_0)$ and an isomorphism $\sigma\colon F \to F_0$ such that $\varphi_0\sigma = \varphi$.*

Proof. Using "standard" arguments, we can replace $f(F)$ in F by F and we obtain $F_0 = F \cup Y$, where Y is a copy of $F \setminus f(F)$. Defining σ as the identity map on Y and as f^{-1} on $f(F)$ and φ_0 as φ on Y and as φf on F, one can easily verify all the properties stated. $\square$

Lemma 1.5. *Let $\mathscr{G}$ be an abstract class of modules and*

$$\begin{array}{ccc} G & \xrightarrow{\ \psi\ } & M \\ {\scriptstyle f}\big\downarrow & & \big\| \\ F & \xrightarrow{\ \varphi\ } & M \end{array}$$

be a commutative diagram with $F, G \in \mathscr{G}$. If ψ is a $\mathscr{G}$-precover of M, then so is φ.

Proof is obvious. $\square$

2. Existence of precovers

If $\mathscr{G}$ is an abstract class of modules such that every left R-module has a $\mathscr{G}$-precover, then it is usual to say that $\mathscr{G}$ is a *precover class*. In other words this means that for each $M \in R$-mod there is a module $G \in \mathscr{G}$ and a homomorphism $f\colon F \to M$ such that every homomorphism $f\colon F \to M$, $F \in \mathscr{G}$, factors through φ, i.e. $f = \varphi g$ for some homomorphism $g\colon F \to G$. Rada and Saorín [5, Theorem 3.4] observed that to ensure that every module has a $\mathscr{G}$-precover it suffices to consider any (co-abstract) subset $\mathscr{G}' \subseteq \mathscr{G}$ having the property that every homomorphism $f\colon F \to M$, $F \in \mathscr{G}$, factors through a direct sum of members of $\mathscr{G}'$. We start this section with the simple proof of this fact, namely of [5, Corollary 3.7]. Anyway, this result show that "small" classes $\mathscr{G}$ of modules are precover classes in the sense that $\mathscr{G}$ consists of all direct sums of members of a (co-abstract) subset $\mathscr{G}'$ of $\mathscr{G}$ and their isomorphic copies. For such classes it is usual to use the notation $\mathscr{G} = \operatorname{Coprod}(\mathscr{G}')$. On the other hand, large classes, e.g. $\mathscr{G} = R$-mod, are also precover classes (the identity map 1_M for every module $M \in R$-mod). So, we shall continue in this section with some sufficient conditions for precover classes.

Proposition 2.1. *If $\mathscr{G}'$ is any (co-abstract) set of modules, then $\mathscr{G} = \operatorname{Coprod}(\mathscr{G}')$ is a precover class.*

Proof. We are going to verify that for every module M the homomorphism $\varphi\colon G \to M$ from Definition 1.2 is a $\operatorname{Coprod}(\mathscr{G}')$-precover of M. So, let $f\colon F \to M$, $F = \bigoplus_{\delta\in D} F_\delta$, where F_δ is an isomorphic copy of a member of $\mathscr{G}'$ for each $\delta \in D$, be arbitrary. By Lemma 1.1 there is a subset D' of D such that $F = U \oplus V$, where $U = \bigoplus_{\delta\in D'} F_\delta$ and $V \subseteq \operatorname{Ker} f$. By Lemma 1.3 there is $h\colon U \to G$ such that $\varphi h = f|U$ and consequently for $g = h \oplus 0\colon F \to G$ we obviously have $\varphi g = f$. □

We say that a class $\mathscr{G}$ of modules is *$\mathscr{G}$-cohereditary*, if it is closed under factor-modules by submodules lying in $\mathscr{G}$. Further, submodule N of a module M is said to be *$\mathscr{G}$-pure in M*, if the factor-module M/N lies in $\mathscr{G}$.

Theorem 2.2. *Let $\mathscr{G}$ be a $\mathscr{G}$-cohereditary class of modules closed under direct sums and such that the set of $\mathscr{G}$-pure submodules of any module lying in $\mathscr{G}$ is inductive. If $\mathscr{G}'$ is a co-abstract subset of the class $\mathscr{G}$ such that each $F \in \mathscr{G}$ contains an essential submodule isomorphic to a member of* $\operatorname{Coprod}(\mathscr{G}')$, *then $\mathscr{G}$ is a precover class.*

Proof. Let $M \in R$-mod be arbitrary and let $\varphi\colon G \to M$ be as in Definition 1.2. To show that φ is a $\mathscr{G}$-precover of the module M, let $f\colon F \to M$, $F \in \mathscr{G}$, be an arbitrary homomorphism. The hypothesis yields the existence of a maximal $\mathscr{G}$-pure submodule of F contained in $\operatorname{Ker} f$ and as can be easily verified, we may without loss of generality assume that $\operatorname{Ker} f$ contains no non-zero submodule which is $\mathscr{G}$-pure in F. By hypothesis and Lemma 1.1 the module F contains an essential submodule F' of the form $F' = U \oplus V$, where $U = \bigoplus_{\delta\in D'} F_\delta$ with no $f|F_\delta$, $f|F_\varepsilon$, $\delta, \varepsilon \in D'$, $\delta \neq \varepsilon$, M-equivalent and $V \subseteq \operatorname{Ker} f$. Further, $V \in \mathscr{G}$, $\mathscr{G}$ being abstract and closed under direct sums, and consequently V is $\mathscr{G}$-pure in F owing to the fact that $\mathscr{G}$ is $\mathscr{G}$-cohereditary. Thus $V = 0$, $F' = U$ is essential in F and it suffices to use Lemma 1.3. □

Recall that an abstract class $\mathscr{G}$ of modules is said to be *closed under extensions*, if $G \in \mathscr{G}$ whenever there is $H \leqslant G$ such that both H and G/H belong to $\mathscr{G}$.

Theorem 2.3. *Let $\mathscr{G}$ be an abstract, $\mathscr{G}$-cohereditary and inductive class of modules closed under direct sums and extensions. If $\mathscr{G}'$ is a co-abstract subset of the class $\mathscr{G}$ such that each $F \in \mathscr{G}$ contains an essential submodule isomorphic to a member of* $\operatorname{Coprod}(\mathscr{G}')$, *then $\mathscr{G}$ is a precover class.*

Proof. Let $M \in R$-mod be arbitrary, $\varphi\colon G \to M$ as in Definition 1.2 and let $f\colon F \to M$, $F \in \mathscr{G}$, be an arbitrary homomorphism. By hypothesis there is an essential submodule $F' \in \mathscr{G}$ of F which can be by virtue of Lemma 1.1 written in the form $F' = U \oplus V$, $V \subseteq \operatorname{Ker} f$, $V \in \mathscr{G}$. The class $\mathscr{G}$ is inductive and so there is a submodule $V' \subseteq \operatorname{Ker} f$ maximal with respect to $V \subseteq V'$ and $V' \in \mathscr{G}$. By hypothesis, the factor-module $\overline{F} = F/V'$ belongs to $\mathscr{G}$ and f induces $\overline{f}\colon \overline{F} \to M$ naturally in such a way that $\overline{f}\pi = f$, π being the canonical projection $F \to \overline{F}$. Similarly to the case of F there is an essential submodule $\overline{F}' = \overline{U} \oplus \overline{V}$ of $\overline{F}$ with $\overline{V} \subseteq \operatorname{Ker} \overline{f}$. Then $\overline{V} = \tilde{V}/V'$, where $\tilde{V} \subseteq \operatorname{Ker} f$ and $\tilde{V} \in \mathscr{G}$ owing to the fact that $\mathscr{G}$ is closed under extensions. Now the maximality of V' yields $\overline{V} = 0$ and an application of Lemma 1.3 gives the existence of a homomorphism $\bar{g}\colon \overline{F} \to G$ with $\varphi\bar{g} = \overline{f}$, from which the assertion follows easily. □

Proposition 2.4. *Let $\mathscr{G}$ be an abstract class of modules closed under injective hulls. If M is an injective module, then a homomorphism $\varphi\colon G \to M$ is a $\mathscr{G}$-precover of M if and only if for every $H \in \mathscr{G}$, H injective, and every homomorphism $f\colon H \to M$ there is a homomorphism $g : H \to G$ such that $\varphi g = f$.*

Proof. Only the sufficiency requires verification. So, let $F \in \mathscr{G}$ and $h\colon F \to M$ be arbitrary. If $i\colon F \to E(F) = H$ is the canonical embedding, then there is $f\colon H \to M$ with $fi = h$, M being injective. By hypothesis, there is $g\colon H \to G$ such that $\varphi g = f$. Thus $\varphi gi = fi = h$ and we are through. □

We say that a co-abstract set $\mathscr{G}'$ is *closed under injective hulls*, if $\mathscr{G}'$ with each its element contains a copy of its injective hull.

Theorem 2.5. *Let $\mathscr{G}$ be a hereditary class of modules closed under direct sums and injective hulls and let $\mathscr{G}'$ be a co-abstract subset of $\mathscr{G}$ closed under injective hulls. If, for each $F \in \mathscr{G}$, F injective, the set of $\mathscr{G}$-pure submodules is inductive and F contains an essential submodule isomorphic to a member of* $\operatorname{Coprod}(\mathscr{G}')$*, then every injective module has a $\mathscr{G}$-precover.*

Proof. Let $M \in R$-mod be injective and let $\varphi\colon G \to M$ be as in Definition 1.2. By Proposition 2.4 it suffices to test the homomorphism φ by the injective elements of $\mathscr{G}$ only. So, let $f\colon F \to M$, $F \in \mathscr{G}$ injective, be an arbitrary homomorphism. By hypothesis and Lemma 1.1 there is an essential submodule $F' = U \oplus V$ of F such that $V \subseteq \operatorname{Ker} f$ and $U = \bigoplus\limits_{\delta \in D'} F_\delta$ where no different $f\big|F_\delta$, $f\big|F_\varepsilon$, $\delta, \varepsilon \in D'$, are M-equivalent.

If $V = 0$ then an application of Lemma 1.3 finishes the proof. Assuming $V \neq 0$ we shall adopt the notation of Lemma 1.1 and its proof. So, there are $\delta \neq \varepsilon$ in D with $\delta \sim \varepsilon$ and consequently $G_{\varepsilon\delta} \cong F_\delta$ is isomorphic to a member of $\mathscr{G}'$. Moreover, since F is injective, we may assume that F_δ and consequently $G_{\varepsilon\delta}$ are also injective. Then $F = G_{\varepsilon\delta} \oplus L$ where L can be taken as the injective hull of the direct sum of all remaining $F_{\delta'}$ and $G_{\varepsilon'\delta'}$ and so $L \in \mathcal{G}$ by the hypotheses. Thus the isomorphism $F/G_{\varepsilon\delta} \cong L$ shows that $G_{\varepsilon\delta}$ is a $\mathcal{G}$-pure submodule of F contained in $\operatorname{Ker} f$. So, the hypothesis yields the existence of a maximal $\mathcal{G}$-pure submodule K of F contained in $\operatorname{Ker} f$. Denoting $\overline{F} = F/K$ and $\pi\colon F \to \overline{F}$ the canonical projection, there is a natural homomorphism $\overline{f}\colon \overline{F} \to M$ with $\overline{f}\pi = f$. Now if i is the embedding of $\overline{F}$ into its injective envelope $H = E(\overline{F})$ then the injectivity of M yields the extension $f^*\colon H \to M$ of $\overline{f}$, $f^* i = \overline{f}$. By hypothesis and Lemma 1.1 the module H contains an essential submodule $H' = \overline{U} \oplus \overline{V}$ with $\overline{V} \cong \operatorname{Ker} f^*$. Now it remains to verify that $\overline{V} = 0$, since in that case Lemma 1.3 yields the existence of $\bar{g}\colon H \to G$ with $\varphi\bar{g} = f^*$ and consequently $\varphi\bar{g}i\pi = f^* i\pi = \overline{f}\pi = f$.

Proving indirectly let us assume that $\overline{V} \neq 0$. As in the case $V \neq 0$ we can find a non-zero submodule $L \subseteq \operatorname{Ker} f^*$ which is $\mathcal{G}$-pure in H. Then $0 \neq L \cap \overline{F} \subseteq \operatorname{Ker} \overline{f}$ and $\frac{\overline{F}}{L\cap\overline{F}} \cong \frac{\overline{F}+L}{L} \leqslant \frac{H}{L}$ yields that $L \cap \overline{F}$ is $\mathcal{G}$-pure in $\overline{F}$, $\mathcal{G}$ being a hereditary class of modules. Thus we have obtained a $\mathcal{G}$-pure submodule $0 \neq L \cap \overline{F} = S/K$ of $\overline{F} = F/K$ contained in $\operatorname{Ker} \overline{f}$. Hence $S \subseteq \operatorname{Ker} f$, $K \subset S$ and S is $\mathcal{G}$-pure in F since $F/S \cong F/K/S/K \in \mathcal{G}$. This contradicts the maximality of K and completes the proof of the theorem. □

Theorem 2.6. *If $\mathcal{G}$ is a hereditary class of modules, then every module has a $\mathcal{G}$-precover if an only if every injective module has a $\mathcal{G}$-precover.*

P r o o f. Only the sufficiency requires verification. So, let $M \in R$-mod be arbitrary and let $\beta\colon M \to E(M)$ be its injective hull. If $\varphi\colon F \to E(M)$ is a $\mathcal{G}$-precover of $E(M)$, then it is easy to see that in the pullback diagram

$$\begin{array}{ccc} G & \xrightarrow{\psi} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \beta} \\ F & \xrightarrow{\varphi} & E(M) \end{array}$$

the homomorphism f is injective, hence $G \in \mathcal{G}$ by hypothesis and it is easy to see that the homomorphism $\psi\colon G \to M$ is a $\mathcal{G}$-precover of the module M. □

3. Precovers that are covers

Theorem 3.1. *Let $\mathcal{G}$ be an abstract class of modules. If $\varphi\colon F \to M$ is a $\mathcal{G}$-cover of the module M then in every commutative diagram*

$$(*)\qquad \begin{array}{ccc} F & \xrightarrow{\varphi} & M \\ {\scriptstyle f}\downarrow & & \| \\ G & \xrightarrow{\psi} & M \end{array}$$

where ψ is a $\mathscr{G}$-precover of M, the homomorphism f is injective. The converse holds if the class $\mathscr{G}$ is inductive.

P r o o f. Since φ is a $\mathscr{G}$-precover of the module M, there is a homomorphism $g\colon G \to F$ such that $\varphi h = \psi$. But then $\varphi hf = \psi f = \varphi$ yields that $h\varphi$ is an automorphism of F and consequently f is a monomorphism.

To prove the converse we will say, for the sake of brevity, that a $\mathscr{G}$-precover φ of the module M has the property $(*)$ if it satisfies the condition of the theorem, i.e. if for any $\mathscr{G}$-precover $\psi\colon G \to M$ of M, every homomorphism $f\colon F \to G$ making the diagram $(*)$ commmutative is injective.

Take any set X with $F \subseteq X$, $|F| < |X|$, and consider the family $\Sigma = \{(F_0, \varphi_0)\}$, where $F_0 \subseteq X$ and $\varphi_0\colon F_0 \to M$ is a $\mathscr{G}$-precover of the module M having the property $(*)$. Since $(F, \varphi) \in \Sigma$, Σ is non-empty and we can define the natural order $\leqslant$ on Σ in such a way that $\{(F_0, \varphi_0)\} \leqslant \{(F_1, \varphi_1)\}$ if and only if $F_0 \subseteq F_1$ and $\varphi_1 | F_0 = \varphi_0$.

Let us verify, that Zorn's lemma can be applied. If $\{(F_i, \varphi_i) \mid i \in I\} \subseteq \Sigma$ is any chain, set $F^* = \bigcup\limits_{i \in I} F_i$ and define $\varphi^*\colon f^* \to M$ via $\varphi^*(x) = \varphi_i(x)$ whenever $x \in F_i$. Obviously, $F^* \subseteq X$ and $F^* \in \mathscr{G}$ by the hypothesis. To show that φ^* is a $\mathscr{G}$-precover of M it suffices to apply Lemma 1.5 to the commutative diagram

$$\begin{array}{ccc} F_i & \xrightarrow{\varphi_i} & M \\ {\scriptstyle \iota_i}\downarrow & & \| \\ F^* & \xrightarrow{\varphi^*} & M \end{array}$$

with the inclusion map ι_i, $i \in I$. In order to verify the property $(*)$, consider the commutative diagram

$$\begin{array}{ccccc} F_i & \xrightarrow{\iota_i} & F^* & \xrightarrow{\varphi^*} & M \\ {\scriptstyle g_i}\downarrow & & {\scriptstyle g}\downarrow & & \| \\ G & = & G & \xrightarrow{\psi} & M \end{array}$$

and assume that $\operatorname{Ker} g \neq 0$. Then $\operatorname{Ker} g \cap F_i \neq 0$ for a suitable $i \in I$. Since $\varphi^* \iota_i = \varphi_i$ and $\psi g_i = \psi g \iota_i = \varphi^* \iota_i = \varphi_i$, the homomorphism g_i is injective by the property $(*)$, owing to the fact that $(F_i, \varphi_i) \in \Sigma$. On the other hand, $F_i \cap \operatorname{Ker} g_i = 0$, which contradicts the choice of $i \in I$, and consequently $(F^*, \varphi^*) \in \Sigma$.

Now we are going to verify that φ is a $\mathscr{G}$-cover of the module M. Proving indirectly, let us assume that there exists an endomorphism f of the module F such that $\varphi f = \varphi$, f is injective, but not surjective. By Lemma 1.4 and Zorn's lemma there is a maximal element (F^*, φ^*) of Σ such that $(F, \varphi) < (F^*, \varphi^*)$. By the property $(*)$ there exists a monomorphism $f_1\colon F^* \to F$ with $f_1 \varphi = \varphi^*$. Now the composition of f_1 with the inclusion map $\iota\colon F \to F^*$ yields an injective non-surjective endomorphism ιf_1 of F^* such that $\varphi^* \iota f_1 = \varphi f_1 = \varphi^*$. To obtain the final contradiction with the maximality of (F^*, φ^*) it suffices now to apply Lemma 1.4. $\square$

As a consequence of this theorem we can easily derive the result [9, Theorem 2.2.8] on the existence of $\mathcal{G}$- covers.

Corollary 3.2. *Let $\mathcal{G}$ be an abstract class of modules closed under direct limits. If a module M has a $\mathcal{G}$-precover, then it has a $\mathcal{G}$-cover.*

Proof. Using [9, Lemma 2.2.10] we see that there exists a $\mathcal{G}$-precover of M having the property $(*)$ and Theorem 3.1 applies. $\square$

Theorem 3.3. *Let $\mathcal{G}$ be an abstract and inductive class of modules and let $\varphi\colon F \to M$ be a $\mathcal{G}$-precover of the module M. If each endomorphism f of F with $\varphi f = \varphi$ is injective and $f(F)$ is essential in F, then φ is a $\mathcal{G}$-cover of M.*

Proof. Similarly as in the preceding proof we shall consider a set X with $F \subseteq X$, $|F| < |X|$ and the family $\Sigma = \{(F_0, \varphi_0)\}$ with $F \subseteq F_0 \subseteq X$, F essential in F_0, and $\varphi_0\colon F_0 \to M$ a $\mathcal{G}$-precover of the module M. The collection Σ is non-empty since $(F, \varphi) \in \Sigma$, and it is ordered by the relation $\leqslant$ where $(F_0, \varphi_0) \leqslant (F_1, \varphi_1)$ if and only if $F_0 \subseteq F_1$ and $\varphi_1 \big| F_0 = \varphi_0$.

If $\{(F_i, \varphi_i) \mid i \in I\}$ is a chain in Σ, then we set $F^* = \bigcup\limits_{i \in I} F_i$ and $\varphi^*(x) = \varphi_i(x)$ whenever $x \in F_i$. Then $F^* \in \mathcal{G}$ by hypothesis, φ^* is a $\mathcal{G}$-precover of M by Lemma 1.5 and so $(F^*, \varphi^*) \in \Sigma$, F being obviously essential in F^*.

Consider the commutative diagram

$$\begin{array}{ccc}
F & \xrightarrow{\ \varphi\ } & M \\
{\scriptstyle \iota}\big\downarrow & & \big\| \\
F^* & \xrightarrow{\ \varphi^*\ } & M \\
{\scriptstyle f_1}\big\downarrow & & \big\| \\
F & \xrightarrow{\ \varphi\ } & M
\end{array}$$

where (F^*, φ^*) is a maximal element of Σ, ι is the embedding and f_1 is an arbitrary homomorphism making the bottom square commutative. Now $\varphi f_1 \iota = \varphi^* \iota = \varphi$ and consequently $f_1 \iota$ is injective by hypothesis. Further, for $y \in \operatorname{Ker} f_1 \cap \iota(F)$ we have $y = \iota(x)$ for some $x \in F$, and so $f_1(y) = f_1 \iota(x) = 0$ yields $y = 0$, which means that $\operatorname{Ker} f_1 \cap \iota(F) = 0$. Thus $\operatorname{Ker} f_1 = 0$, $\iota(F)$ being essential in F^*. Moreover, $\operatorname{Im}(f_1 \iota)$ is essential in F by hypothesis and so is $\operatorname{Im} f_1$ in view of the obvious inclusion $\operatorname{Im}(f_1 \iota) \subseteq \operatorname{Im} f_1$.

Our next step is to show that f_1 is an epimorphism. If not, then ιf_1 is a non-surjective monomorphism of F^* such that $\varphi^* \iota f_1 = \varphi f_1 = \varphi^*$ and consequently Lemma 1.4 yields a contradiction with the maximality of (F^*, φ^*).

To complete the proof it suffices to consider the commutative diagram

$$\begin{array}{ccc} F^* & \xrightarrow{\varphi^*} & M \\ {\scriptstyle f_1}\big\downarrow & & \big\| \\ F & \xrightarrow{\varphi} & M \\ {\scriptstyle f}\big\downarrow & & \big\| \\ F & \xrightarrow{\varphi} & M \end{array}$$

Since $\varphi f f_1 = \varphi f_1 = \varphi^*$, $f f_1$ is an epimorphism by the preceding part, and so is f, as we wished to show. □

Let $\mathscr{G}$ be an abstract class of modules. We say that a proper submodule N of a module M is *almost $\mathscr{G}$-pure in M*, if the factor-module M/N contains a non-zero submodule from $\mathscr{G}$. Furthermore, we say that a module U is *almost $\mathscr{G}$-hereditary*, if every non-zero submodule of U contains a non-zero submodule from $\mathscr{G}$. Finally, the class $\mathscr{G}$ is called *almost $\mathscr{G}$-hereditary*, if every module $U \in \mathscr{G}$ is almost $\mathscr{G}$-hereditary.

Theorem 3.4. *Let $\mathscr{G}$ be an abstract inductive class of modules and let $\varphi : F \to M$ be a $\mathscr{G}$-precover of the module M. If F is almost $\mathscr{G}$-hereditary and* $\operatorname{Ker} \varphi$ *contains no non-zero submodule almost $\mathscr{G}$-pure in F, then φ is a $\mathscr{G}$-cover of M.*

Proof. The idea of the proof is to verify that any endomorphism f of the module F such that $\varphi f = \varphi$ is injective with essential image and then apply the preceding theorem.

First, $F/\operatorname{Ker} f \cong \operatorname{Im} f \leqslant F$ yields that $\operatorname{Im} f$ contains a non-zero element from $\mathscr{G}$, hence $\operatorname{Ker} f$ is almost $\mathscr{G}$-pure in F and $\operatorname{Ker} f = 0$ by hypothesis, owing to the obvious inclusion $\operatorname{Ker} f \subseteq \operatorname{Ker} \varphi$. Continuing indirectly, let us suppose that $f(F)$ is not essential in F. Thus there is a non-zero submodule K of F with $f(F) \cap K = 0$ and we may without loss of generality assume that $K \in \mathscr{G}$, F being almost $\mathscr{G}$-hereditary. Setting $S = \{x - f(x) \mid x \in K\}$ we have $x - f(x) = 0$ if and only if $x = f(x) \in K \cap f(F) = 0$, and the mapping $g\colon K \to S$ given by $g(x) = x - f(x)$, $x \in K$, is an isomorphism. Thus $S \cong K$ lies in the class $\mathscr{G}$ and obviously $S \subseteq \operatorname{Ker} \varphi$. Further, $S \cap f(F) = 0$ since for $x - f(x) = f(y)$, $x \in K$, $y \in F$, we have $x = f(x + y) \in K \cap f(F) = 0$, and so $\operatorname{Im} f \cong \frac{f(F) \oplus S}{S} \leqslant \frac{F}{S}$. By hypothesis, $\operatorname{Im} f$ contains a non-zero submodule from $\mathscr{G}$, hence $0 \neq S \subseteq \operatorname{Ker} \varphi$ is almost $\mathscr{G}$-pure in F, which contradicts the hypothesis. □

4. Applications

Recall that a hereditary torsion theory $\sigma = (\mathscr{T}, \mathscr{F})$ for the category R-mod consists of two abstract classes $\mathscr{T}$ and $\mathscr{F}$, the *σ-torsion class* and the *σ-torsionfree class*, respectively, such that $\operatorname{Hom}(T, F) = 0$ whenever $T \in \mathscr{T}$ and $F \in \mathscr{F}$, the class $\mathscr{T}$ is closed under submodules, factor modules, extensions and direct sums, the class $\mathscr{F}$ is closed under submodules, extensions and direct product and for each module M there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathscr{T}$ and $F \in \mathscr{F}$. With each hereditary torsion theory we associate a Gabriel filter of left ideals $\mathscr{L} = \{I \leqslant R \mid R/I \in \mathscr{T}\}$ and the torsion part $\sigma(M) = T$ of the module M consists of all elements $a \in M$ with $(0 : a) \in \mathscr{L}$. The torsion theory σ is said to be of *finite type*, if the filter $\mathscr{L}$ contains a cofinal subset of finitely generated left ideals. For more details see e.g. [3] or [1]. The following two consequences of the above theory can be found in [7, Theorem] and [2, Theorem 1].

Corollary 4.1. *If $\sigma = (\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory of finite type for* R-mod*, then every module has a σ-torsionfree cover.*

P r o o f. First we show that every R-module has a σ-torsionfree precover. Since the class $\mathscr{F}$ is hereditary, it suffices by virtue of Theorem 2.6 to show that every injective module has an $\mathscr{F}$-precover. For this reason we are going to verify the hypotheses of Theorem 2.5. Clearly, $\mathscr{F}$ is closed under direct sums and injective hulls. If $\mathscr{F}'$ is a co-abstract set consisting of injective hulls of cyclic modules from $\mathscr{F}$, then obviously every injective module $F \in \mathscr{F}$ contains an essential submodule isomorphic to a member of $\operatorname{Coprod}(\mathscr{F}')$. Since the set of $\mathscr{F}$-pure submodules of any module is inductive by [3, Proposition 6.18], the proof of this part is complete. Now if $\psi\colon G \to M$ is an $\mathscr{F}$-precover of the module M, then $\operatorname{Ker}\psi$ contains a maximal $\mathscr{F}$-pure submodule K of G. Denoting $F = G/K$ and $\varphi\colon F \to M$ the homomorphism naturally induced by ψ, Lemma 1.5 yields that φ is a σ-precover of M and Theorem 3.4 applies. □

Corollary 4.2. *Over any commutative domain every module has a torsionfree cover.*

P r o o f is obvious. □

Let $\sigma = (\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for R-mod and let $\mathscr{I}$ denote the class of *σ-torsionfree σ-injective modules.*

Lemma 4.3. *If σ is a hereditary torsion theory of finite type, then a σ-torsionfree module M is σ-injective if the induced map* $\operatorname{Hom}(R, M) \to \operatorname{Hom}(I, M)$ *is an epimorphism for every finitely generated left ideal I from $\mathscr{L}$.*

P r o o f. In view of the relative Baer's criterion, we can investigate the commutative diagram

$$\begin{array}{ccccc} I & \xrightarrow{\ i\ } & J & \xrightarrow{\ \iota\ } & R \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ M & = & M & = & M \end{array}$$

with $J \in \mathscr{L}$ and $f\colon J \to M$ given. By hypothesis there is a finitely generated left ideal $I \in \mathscr{L}$ with the inclusion map $i\colon I \to J$ and a homomorphism $g\colon R \to M$ such that $g\iota i = f'$, where $f' = f \mid I = fi$. For an arbitrary $j \in J$ we have $K = (I : j) \in \mathscr{L}$ and for each $k \in K$ we have $k(f - g\iota)(j) = (f - g\iota)(kj) = 0$ since $kj \in I$ and $f\big|I = g\iota i$. Hence $K(f - g\iota)(j) = 0$, which means $(f - g\iota)(j) \in \sigma(M) = 0$ and consequently $f = g\iota$, as desired. □

Lemma 4.4. *If $\sigma = (\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory of finite type, then the class $\mathscr{I}$ of all σ-torsionfree σ-injective modules is inductive.*

P r o o f. See [3, Proposition 42.9]. □

The following two corollaries partly generalize some results from [8, Corollary 2.10] and [4, Proposition 2], respectively. Recall that a hereditary torsion theory σ is called *exact* if $M \in \mathscr{I}$ implies $E(M)/M \in \mathscr{I}$ and that σ is called *perfect* if it is exact and of finite type.

Corollary 4.5. *If $\sigma = (\mathscr{T}, \mathscr{F})$ is a perfect torsion theory for R-*mod*, then every module has a σ-torsionfree σ-injective cover.*

P r o o f. Using Theorem 2.3 we first show that every module has an $\mathscr{I}$-precover. The class $\mathscr{I}$ of all σ-torsionfree σ-injective modules is $\mathscr{I}$-cohereditary by [3, Proposition 44.1], it is inductive by Lemma 4.4 and it is easy to see that it is closed under direct sums and extensions. Taking any co-abstract subset $\mathscr{I}'$ of $\mathscr{I}$ consisting of elements which are essential extensions of σ-torsionfree cyclic modules, then using [3, Proposition 10.11] it is a routine to check that each $F \in \mathscr{I}$ contains an essential submodule isomorphic to a member of $\operatorname{Coprod}(\mathscr{I}')$. Thus every module has an $\mathscr{I}$-precover and by virtue of inductivity and Lemma 1.5 we may assume that for an arbitrary module M there exists an $\mathscr{I}$-precover $\varphi\colon F \to M$ such that $\operatorname{Ker}\varphi$ contains no non-zero submodule $\mathscr{I}$-pure in F. Considering the diagram $(*)$ in Theorem 3.1, the isomorphism $F/\operatorname{Ker} f \cong \operatorname{Im} f$ yields that $\operatorname{Ker} f$ is σ-closed in F and consequently $\mathscr{I}$-pure in F by [3, Proposition 10.11]. In view of the obvious inclusion $\operatorname{Ker} f \subseteq \operatorname{Ker}\varphi$, $\operatorname{Ker} f = 0$ and φ is the $\mathscr{I}$-cover of M by Theorem 3.1. □

Corollary 4.6. *If $\sigma = (\mathscr{T}, \mathscr{F})$ is a centrally splitting torsion theory for R-mod, then every module has a σ-torsionfree σ-injective cover.*

P r o o f. The corresponding radical filter $\mathscr{L}$ has the smallest element $I = Re$, e being a central idempotent, and so σ is obviously of finite type. Let $M \in \mathscr{F}$ be arbitrary. Assuming $\sigma(E(M)/M) = K/M \neq 0$, for each $x \in K \setminus M$ we have $Ix \subseteq M$, i.e. $ex \in M$. Moreover, $e(x - ex) = 0$, so $I(x - ex) = 0$ and $x = ex$, M being σ-torsionfree. Hence $x \in M$, which is a contradiction proving that every σ-torsionfree module is σ-injective; an application of Corollary 4.1 completes the proof. □

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