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SOME FIXED POINT THEOREMS IN METRIC SPACES
BY ALTERING DISTANCES

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Abstract. A generalization is obtained for some of the fixed point theorems of Khan, Swaleh and Sessa, Pathak and Rekha Sharma, and Sastry and Babu for a self-map on a metric space, which involve the idea of alteration of distances between points.

Keywords: fixed point, alteration of distances

MSC 2000: 47H10, 54H25

The famous Banach contraction principle has been generalized by several authors in several ways. A comprehensive literature on the generalizations of the same for self-maps on a metric space can be found in Rhoades [4] and Tasković [9]. Khan, Swaleh and Sessa [1] obtained generalizations of the same for a self-map on a metric space by altering distances between points through the use of certain control functions. Sastry and Babu [5], [6] and [7] continued the study in this direction. It was further pursued by Naidu [2]. In an attempt to unify Theorem 2 of Khan, Swaleh and Sessa [1] and that of Pathak and Rekha Sharma [3], Sastry and Babu obtained a partial generalization (Theorem 2.1 of [5]). Here our aim is to unify all the three results.

Throughout this paper, unless otherwise stated, \((X,d)\) is a metric space, \(f\) is a self-map on \(X\), \(\mathbb{N}\) is the set of all positive integers, \(\mathbb{R}^+\) is the set of all nonnegative real numbers, \(\varphi: \mathbb{R}^+ \to \mathbb{R}^+\) is a monotonically increasing function with \(\varphi(t^+) < t\) \(\forall t \in (0,\infty)\), \(\theta: \mathbb{R}^+ \to [0,1]\) is a monotonically decreasing function with \(\theta(t) < 1\) \(\forall t \in (0,\infty)\), \(\zeta: \mathbb{R}^+ \to [\frac{1}{2},1)\) is continuous at zero, \(\varrho\) is a nonnegative real valued function on \(X \times X\) with the following two properties:

(i) \(\{\varrho(x_n,y_n)\}_{n=1}^\infty\) is convergent whenever \(\{x_n\}_{n=1}^\infty\) and \(\{y_n\}_{n=1}^\infty\) are sequences in \(X\) such that \(\{d(x_n,y_n)\}_{n=1}^\infty\) is convergent,
(ii) for any sequences \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) in \( X \), the sequence \( \{\varrho(x_n, y_n)\}_{n=1}^{\infty} \) converges to zero iff the sequence \( \{d(x_n, y_n)\}_{n=1}^{\infty} \) converges to zero; \( K \) is a nonnegative real number, and for \( x, y \in X \) we have

\[
\alpha(x, y) = (\max\{\varrho(x, y), \varrho(x, fx), \varrho(y, fy)\}) + K[\varrho(x, fy)\varrho(fx, y)]^{1/2},
\]
\[
\beta(x, y) = (\max\{\varrho(x, y), [\varrho(x, fy)\varrho(fx, y)]^{1/2}\}) + \max\{\varrho(x, fx), \varrho(y, fy)\},
\]
\[
\beta_0(x, y) = (\max\{\varrho(x, y), [\varrho(x, fy)\varrho(fx, y)]^{1/2}\})
\]
\[
+ (\min\{\max\{\varrho(x, fx), \varrho(y, fy)\}, \zeta(d(x, y))[\varrho(x, fx) + \varrho(y, fy)]\}),
\]
\[
\gamma(x, y) = \min\{\alpha(x, y), \beta(x, y)\} \quad \text{and} \quad \gamma_0(x, y) = \min\{\alpha(x, y), \beta_0(x, y)\}.
\]

From property (i) of \( \varrho \) we note that \( \varrho \) is symmetric and that \( \{\varrho(x_n, y_n)\}_{n=1}^{\infty} \) converges to \( \varrho(x, y) \) whenever \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) are sequences in \( X \) such that \( \{d(x_n, y_n)\}_{n=1}^{\infty} \) converges to \( d(x, y) \). From property (ii) of \( \varrho \) we note that \( \varrho(x, y) = 0 \) iff \( x = y \).

**Theorem 1.** Suppose that

\[
(1) \quad \varrho(fx, fy) \leq \max\{\varphi(\gamma(x, y)), \theta(d(x, y))\gamma(x, y)\}
\]

for all \( x, y \in X \). Then \( f \) has at most one fixed point in \( X \) and for any \( x \in X \), \( \{f^n x\} \) is Cauchy.

**Proof.** From inequality (1) we have

\[
\varrho(fx, fy) \leq \max\{\varphi(\alpha(x, y)), \theta(d(x, y))\alpha(x, y)\}
\]

for all \( x, y \in X \). Hence

\[
(2) \quad \varrho(fx, f^2x) \leq \max\{\varphi(\max\{\varrho(x, fx), \varrho(fx, f^2x)\}), \theta(d(x, fx))\max\{\varrho(x, fx), \varrho(fx, f^2x)\}\}.
\]

Suppose that \( fx \neq x \). Then \( \theta(d(x, fx)) < 1 \) and \( \varrho(x, fx) > 0 \). Hence from inequality (2) and the fact that \( \varphi(t) \leq \varphi(t+) < t \forall t \in (0, \infty) \) it follows that

\[
(3) \quad \varrho(fx, f^2x) \leq \max\{\varphi(\varrho(x, fx)), \theta(d(x, fx))\varrho(x, fx)\}.
\]

We note that inequality (3) remains valid even if \( fx = x \). Replacing \( x \) with \( f^{n-1}x \) in inequality (3) we obtain

\[
(4) \quad \varrho(f^n x, f^{n+1}x) \leq \max\{\varphi(\varrho(f^{n-1}x, f^n x)), \theta(d(f^{n-1}x, f^n x))\varrho(f^{n-1}x, f^n x)\}
\]
for all \( n \in \mathbb{N} \). Since \( \varphi(t) \leq t \) and \( \theta(t) \leq 1 \) \( \forall t \in \mathbb{R}^+ \), from inequality (4) we have
\[
\varrho(f^n x, f^{n+1} x) \leq \varrho(f^{n-1} x, f^n x)
\]
for all \( n \in \mathbb{N} \). Consequently, \( \{\varrho(f^n x, f^{n+1} x)\}_{n=0}^{\infty} \) is a monotonically decreasing sequence of nonnegative real numbers. Hence it converges to a nonnegative real number \( s \). First, suppose that \( s > 0 \). Then from property (ii) of \( \varrho \) it follows that \( \{d(f^n x, f^{n+1} x)\}_{n=0}^{\infty} \) is a sequence of positive real numbers bounded below by a positive real number \( \delta \). Since \( \theta \) is a monotonically decreasing function on \( \mathbb{R}^+ \), it follows that \( \theta(d(f^n x, f^n x)) \leq \theta(\delta) \forall n \in \mathbb{N} \). Hence from inequality (4) we have
\[
\varrho(f^n x, f^{n+1} x) \leq \max\{\varrho(f^{n-1} x, f^n x), \theta(\delta)\varrho(f^{n-1} x, f^n x)\}
\]
for all \( n \in \mathbb{N} \). Taking limit superiors on both sides of the above inequality as \( n \to +\infty \), we obtain
\[
s \leq \max\{\varphi(s+), \theta(\delta)s\}.
\]
Since \( \varphi(t+) < t \) \( \forall t \in (0, \infty) \), \( s > 0 \) and \( \theta(\delta) < 1 \), from the above inequality we have \( s < s \), which is absurd. Hence \( s = 0 \). Hence property (ii) of \( \varrho \) yields that \( \{d(f^n x, f^{n+1} x)\}_{n=0}^{\infty} \) converges to zero.

Now, suppose that \( \{f^n x\} \) is not Cauchy. Then there exists a positive real number \( \varepsilon \) such that for given \( N \in \mathbb{N} \) \( \exists m, n \in \mathbb{N} \) such that \( m > n > N \) and \( d(f^n x, f^m x) \geq \varepsilon \). Since \( \{d(f^n x, f^{n+1} x)\}_{n=0}^{\infty} \) converges to zero, it follows that there exist strictly increasing sequences \( \{n_k\}_{k=1}^{\infty} \) and \( \{m_k\}_{k=1}^{\infty} \) of positive integers such that \( 1 < n_k < m_k \), \( d(f^{n_k} x, f^{m_k-1} x) < \varepsilon \) and \( d(f^{n_k} x, f^{m_k} x) \geq \varepsilon \) \( \forall k \in \mathbb{N} \). Using the triangle inequality and the fact that \( \{d(f^n x, f^{n+1} x)\}_{n=0}^{\infty} \) converges to zero it can be shown that \( \{d(f^{n_k} x, f^{m_k} x)\}_{k=1}^{\infty}, \{d(f^{n_k} x, f^{m_k-1} x)\}_{k=1}^{\infty}, \{d(f^{n_k-1} x, f^{m_k} x)\}_{k=1}^{\infty} \) and \( \{d(f^{n_k-1} x, f^{m_k-1} x)\}_{k=1}^{\infty} \) all converge to \( \varepsilon \). Hence from property (i) of \( \varrho \) it follows that the sequences \( \{\varrho(f^{n_k} x, f^{m_k} x)\}_{k=1}^{\infty}, \{\varrho(f^{n_k} x, f^{m_k-1} x)\}_{k=1}^{\infty}, \{\varrho(f^{n_k-1} x, f^{m_k} x)\}_{k=1}^{\infty} \) and \( \{\varrho(f^{n_k-1} x, f^{m_k-1} x)\}_{k=1}^{\infty} \) all converge to the same limit \( b \) for some nonnegative real number \( b \). Since \( \varepsilon > 0 \), from property (ii) of \( \varrho \) it follows that \( b > 0 \).

We note that \( \{\beta(f^{n_k-1} x, f^{m_k-1} x)\}_{k=1}^{\infty} \) converges to \( b \). Since \( \varphi \) is monotonically increasing on \( \mathbb{R}^+ \), we have \( \liminf_{k \to +\infty} \varphi(\beta(f^{n_k-1} x, f^{m_k-1} x)) \leq \varphi(b+) \). Since \( \theta \) is monotonically decreasing on \( \mathbb{R}^+ \) and \( \{d(f^{n_k-1} x, f^{m_k-1} x)\}_{k=1}^{\infty} \) converges to \( \varepsilon \), \( \liminf_{k \to +\infty} \theta(d(f^{n_k-1} x, f^{m_k-1} x)) \leq \theta(\varepsilon-) \). Since \( \theta \) is monotonically decreasing on \( \mathbb{R}^+ \) and \( \theta(t) < 1 \) \( \forall t \in (0, \infty) \), it follows that \( \theta(t-) < 1 \) \( \forall t \in (0, \infty) \). From inequality (1) we have
\[
(5) \quad \varrho(f x, f y) \leq \max\{\varphi(\beta(x, y)), \theta(d(x, y))\beta(x, y)\}
\]
for all \(x, y \in X\). Taking \(f^{n_k-1}x\) and \(f^{m_k-1}x\) instead of \(x\) and \(y\) in the above inequality and then taking limit superiors on both sides as \(k \to +\infty\) we obtain
\[
b \leq \max\{\varphi(b+), \theta(\varepsilon-)b\}.
\]
Since \(\varphi(t+) < t\) and \(\theta(t-) < 1\) \(\forall t \in (0, \infty)\), \(b > 0\) and \(\varepsilon > 0\), from the above inequality we obtain \(b < b\) which is a contradiction. Hence \(\{f^n x\}\) is Cauchy.

If \(x, y\) are fixed points of \(f\) in \(X\), then \(\beta(x, y) = \varrho(x, y)\) and hence from inequality (5) we obtain
\[
\varrho(x, y) \leq \max\{\varphi(\varrho(x, y), \theta(d(x, y))\varrho(x, y)\}.
\]
Since \(\varphi(t) < t\) and \(\theta(t) < 1\) \(\forall t \in (0, \infty)\), from the above inequality we have \(\varrho(x, y) = 0\). Hence \(x = y\). Hence \(f\) has at most one fixed point in \(X\). \(\Box\)

**Remark 1.** Theorem 1 remains valid if inequality (1) is replaced with inequalities (3) and (5).

**Theorem 2.** Suppose that

\[
\varrho(f x, f y) \leq \max\{\varphi(\gamma(x, y)), \theta(d(x, y))\gamma_0(x, y)\}
\]

for all \(x, y \in X\). Then for any \(x \in X\), \(\{f^n x\}\) is Cauchy. For any \(x_0 \in X\), the limit of \(\{f^n x_0\}\), if it exists, is the unique fixed point of \(f\).

**Proof.** Since the validity of inequality (6) implies that of inequality (1), it follows from Theorem 1 that \(f\) has at most one fixed point in \(X\) and that for any \(x \in X\), \(\{f^n x\}\) is Cauchy. Let \(x_0 \in X\). Suppose that \(\{f^n x_0\}\) converges to an element \(z\) of \(X\). Since \(\varsigma\) is continuous at zero, \(\{\varsigma(d(f^n x_0, z))\}\) converges to \(\varsigma(0)\). From the properties of \(\varrho\) we note that the sequences \(\{\varrho(f^n x_0, f z)\}\), \(\{\varrho(f^{n+1} x_0, f z)\}\) converge to \(\varrho(z, f z)\) and that the sequences \(\{\varrho(f^n x_0, z)\}\), \(\{\varrho(f^{n+1} x_0, z)\}\) and \(\{\varrho(f^n x_0, f^{n+1} x_0)\}\) converge to zero. Hence \(\{\beta(f^n x_0, z)\}\) converges to \(\varrho(z, f z)\) and \(\{\beta_0(f^n x_0, z)\}\) converges to \(\varsigma(0)\varrho(z, f z)\). From inequality (6) we have

\[
\varrho(f x, f y) \leq \max\{\varphi(\beta(x, y)), \theta(d(x, y))\beta_0(x, y)\}
\]

for all \(x, y \in X\). Taking \(x = f^n x_0\) and \(y = z\) in inequality (7) and then taking limit superiors on both sides as \(n \to +\infty\) we obtain
\[
\varrho(z, f z) \leq \max\{\varphi(\varrho(z, f z)+), \theta(0)\varsigma(0)\varrho(z, f z)\}.
\]
Since \(\varphi(t+) < t\) \(\forall t \in (0, \infty)\), \(\theta(0) \leq 1\) and \(\varsigma(0) < 1\), from the above inequality we have \(\varrho(z, f z) = 0\). Hence \(f z = z\). \(\Box\)
Remark 2. Theorem 2 remains valid if inequality (6) is replaced with inequalities (3) and (7).

From Theorem 2 we have the following

**Corollary 1.** Suppose that
\[ \varrho(f(x), f(y)) \leq \theta(d(x, y)) \gamma_0(x, y) \]
for all \( x, y \in X \). Then for any \( x \in X \), \( \{f^n x\} \) is Cauchy. For any \( x_0 \in X \), the limit of \( \{f^n x_0\} \), if it exists, is the unique fixed point of \( f \).

From Corollary 1 we have

**Corollary 2.** Suppose that
\[ \varrho(f(x), f(y)) \leq \theta(d(x, y)) \max\{\varrho(x, y), \frac{1}{2}[\varrho(x, f(x)) + \varrho(y, f(y))], [\varrho(x, f(y))\varrho(f(x), y)]^{\frac{1}{2}}\} \]
for all \( x, y \in X \). Then for any \( x \in X \), \( \{f^n x\} \) is Cauchy. For any \( x_0 \in X \), the limit of \( \{f^n x_0\} \), if it exists, is the unique fixed point of \( f \).

Remark 3. In Corollary 2 the conclusion about the existence of a fixed point fails if the expression \( \frac{1}{2}[\varrho(x, f(x)) + \varrho(y, f(y))] \) in its governing inequality is replaced with \( \max\{\varrho(x, f(x)), \varrho(y, f(y))\} \). Example 1 shows that this is so even when \( (X, d) \) is a finite metric space and \( \varrho = d \). In particular, the hypothesis of Theorem 1 does not ensure the existence of a fixed point for \( f \).

**Example 1** (Example 4 of [8]). Let \( X = [0, 1] \) with the usual metric. Define \( f : X \rightarrow X \) as \( f(x) = x/2 \) if \( 0 < x \leq 1 \) and \( f(0) = 1 \). Define \( \theta : \mathbb{R}^+ \rightarrow [0, 1] \) as \( \theta(t) = 1 - t/2 \) if \( 0 \leq t \leq 1 \) and \( \theta(t) = \frac{1}{2} \) if \( 1 < t < +\infty \). Then \( \theta \) is a monotonically decreasing continuous function on \( \mathbb{R}^+ \), \( \theta(t) < 1 \) \( \forall t \in (0, \infty) \) and

\[ |f(x) - f(y)| \leq \theta|x - y| \max\{|x - y|, |x - f(x)|, |y - f(y)|\} \]

for all \( x, y \in X \). Nonetheless, \( f \) has no fixed point in \( X \).

**Corollary 3** (Theorem 2 of [1]). Suppose that \( (X, d) \) is complete, \( \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a monotonically increasing continuous function with \( \psi(t) = 0 \) iff \( t = 0 \), \( a, b, c \) are monotonically decreasing functions from \( (0, \infty) \) into \( [0, 1) \) with \( a(t) + b(t) + c(t) < 1 \) \( \forall t \in (0, \infty) \), and

\[ \psi(d(f(x), f(y))) \leq a(d(x, y))\psi(d(x, y)) + \frac{1}{2}b(d(x, y))\psi(d(x, f(x)) + \psi(d(y, f(y))) \]

\[ + c(d(x, y))\min\{\psi(d(x, f(y))), \psi(d(f(x), y))\} \]

for all distinct \( x, y \in X \). Then \( f \) has a unique fixed point in \( X \).
Proof. Let \( \varrho = \psi \circ d \). Define \( \theta: \mathbb{R}^+ \to [0,1] \) as \( \theta(t) = a(t) + b(t) + c(t) \) if \( t \neq 0 \) and \( \theta(0) = 1 \). Then \( \varrho \) is a nonnegative real valued function on \( X \times X \) having properties (i) and (ii), \( \theta \) is a monotonically decreasing function on \( \mathbb{R}^+ \) with \( \theta(t) < 1 \) \( \forall t \in (0, \infty) \) and

\[
\varrho(fx, fy) \leq \theta(d(x,y)) \max \left\{ \varrho(x,y), \frac{1}{2} \varrho(x,fx) + \varrho(y,fy), \min \{ \varrho(x,y), \varrho(fx,y) \} \right\}
\]

for all \( x, y \in X \). Hence Corollary 3 follows from Corollary 2.

Corollary 4 (Theorem 2 of \([3]\)). Suppose that \((X,d)\) is complete, \( \psi: \mathbb{R}^+ \to \mathbb{R}^+ \) is a monotonically increasing continuous function with \( \psi(t) = 0 \) iff \( t = 0 \), \( a, b \) are monotonically decreasing functions from \((0, \infty)\) into \([0,1]\) with \( a(t) + b(t) < \frac{1}{2} \) \( \forall t \in (0, \infty) \), \( c \) is a constant in \([0,1]\) such that \( a(t)(1+c) < 1 \) \( \forall t \in (0, \infty) \), and

\[
\psi(d(fx, fy)) \leq a(d(x,y))\psi(d(x,y)) + c\psi(d(x,fx))\psi(d(fx,y)) + b(d(x,y))\psi(d(x,fx)) + \psi(d(y,fy))
\]

for all distinct \( x, y \in X \). Then \( f \) has a unique fixed point in \( X \).

Proof. Let \( \varrho = \psi \circ d \). Define \( \theta: \mathbb{R}^+ \to [0,1] \) as \( \theta(t) = 2[a(t) + b(t)] \) if \( t \neq 0 \) and \( \theta(0) = 1 \). Then \( \varrho \) is a nonnegative real valued function on \( X \times X \) having properties (i) and (ii), \( \theta \) is a monotonically decreasing function on \( \mathbb{R}^+ \) with \( \theta(t) < 1 \) \( \forall t \in (0, \infty) \) and

\[
\varrho(fx, fy) \leq \theta(d(x,y)) \max \left\{ \frac{1}{2} \varrho(x,y) + c\varrho(x,fx)\varrho(fx,y), \frac{1}{2} \varrho(x,fx) + \varrho(y,fy) \right\}
\]

for all \( x, y \in X \). Hence Corollary 4 follows from Corollary 2.

Remark 4. As observed by Sastry and Babu \([5]\), in Theorem 2 of Pathak and Rekha Sharma \([3]\) the condition \( 'a(t)(1+c) < 1 \forall t \in (0, \infty)' \) is redundant in view of the hypothesis on the functions \( a \) and \( b \), and the condition \( 'c \leq 1' \).

From Theorem 2 we have

Corollary 5. Suppose that

\[
\varrho(fx, fy) \leq \varphi(\gamma(x,y))
\]
for all $x, y \in X$. Then for any $x \in X$, $\{f^n x\}$ is Cauchy. For any $x_0 \in X$, the limit of $\{f^n x_0\}$, if it exists, is the unique fixed point of $f$.

**Corollary 6** (Theorem 2.1 of [5]). Suppose that $(X, d)$ is complete, $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ is a monotonically increasing continuous function with $\psi(t) = 0$ iff $t = 0$, $a, b, c$ are nonnegative constants with $a + b < 1$ and $a + c < 1$, and

$$
\psi(d(f x, f y)) \leq a \psi(d(x, y)) + \frac{1}{2} b [\psi(d(x, f x)) + \psi(d(y, f y))] + c [\psi(d(x, f y)) \psi(d(f x, y))]^{\frac{1}{2}}
$$

for all $x, y \in X$. Then $f$ has a unique fixed point in $X$.

**Proof.** Let $\varphi = \psi \circ d$. Define $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = \mu t$, where $\mu = \max\{a + b, a + c\}$. Then $\varphi$ is a nonnegative real valued function on $X \times X$ having properties (i) and (ii), $\varphi$ is a monotonically increasing function on $\mathbb{R}^+$ with $\varphi(t) < t \forall t \in (0, \infty)$ and

$$
\varphi(f x, f y) \\
\leq a \varphi(x, y) + \frac{1}{2} b [\varphi(x, f x) + \varphi(y, f y)] + c [\varphi(x, f y) \varphi(f x, y)]^{\frac{1}{2}}
$$

$$
\leq \min\{(a + b)(\max\{\varphi(x, y), \varphi(x, f x), \varphi(y, f y)\}) + c [\varphi(x, f y) \varphi(f x, y)]^{\frac{1}{2}},
(a + c)(\max\{\varphi(x, y), [\varphi(x, f y) \varphi(f x, y)]^{\frac{1}{2}}\}) + \frac{1}{2} b [\varphi(x, f x) + \varphi(y, f y)]
$$

$$
\leq \min\{(\max\{a + b, c\})(\max\{\varphi(x, y), \varphi(x, f x), \varphi(y, f y)\}) + [\varphi(x, f y) \varphi(f x, y)]^{\frac{1}{2}},
(\max\{a + c, b\})(\max\{\varphi(x, y), [\varphi(x, f y) \varphi(f x, y)]^{\frac{1}{2}}\}) + \frac{1}{2} [\varphi(x, f x) + \varphi(y, f y)]
$$

$$
\leq \min\{(\max\{a + b, c\})\varphi(x, y), (\max\{a + c, b\})\beta_0(x, y)\}
$$

$$
\leq \min\{\mu \alpha(x, y), \mu \beta_0(x, y)\}
$$

$$
= \mu \gamma_0(x, y) \leq \mu \gamma(x, y) = \varphi(\gamma(x, y))
$$

for all $x, y \in X$, where we have taken $K = 1$ in the definition of $\alpha(x, y)$. Hence Corollary 6 follows from Corollary 5. \qed

**Corollary 7.** Suppose that $a, b, c$ are nonnegative monotonically decreasing functions on $(0, \infty)$ with $a(t) + b(t) < 1$ and $a(t) + c(t) < 1 \forall t \in (0, \infty)$, and

$$
\varphi(f x, f y) \leq a(d(x, y)) \varphi(x, y) + \frac{1}{2} b(d(x, y))[\varphi(x, f x) + \varphi(y, f y)]
$$

$$
+ c (d(x, y))[\varphi(x, f y) \varphi(f x, y)]^{\frac{1}{2}}
$$

for all distinct $x, y \in X$. Then for any $x \in X$, $\{f^n x\}$ is Cauchy. For any $x_0 \in X$, the limit of $\{f^n x_0\}$, if it exists, is the unique fixed point of $f$.
Define $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as $\theta(t) = \max\{a(t) + b(t), a(t) + c(t)\}$ if $t \neq 0$ and $\theta(0) = 1$. Then $\theta$ is a monotonically decreasing function on $\mathbb{R}^+$ with $\theta(t) < 1 \forall t \in (0, \infty)$. Proceeding as in the proof of Corollary 6 it can be shown that

$$\varrho(fx, fy) \leq \theta(d(x, y))\gamma_0(x, y)$$

for all $x, y \in X$, with $K = 1$ in the definition of $\alpha(x, y)$. Hence Corollary 7 follows from Corollary 1.

**Remark.** Corollary 7 is also a generalization of Corollaries 3, 4 and 6. Corollary 7 shows that in Theorem 2 of Pathak and Rekha Sharma [3] the condition ‘$a(t) + b(t) < \frac{1}{2} \forall t \in (0, \infty)$’ can be replaced by the weaker conditions ‘$2a(t) < 1$ and $a(t) + 2b(t) < 1 \forall t \in (0, \infty)$’.

**References**


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