Czechoslovak Mathematical Journal

Radomír Halaš; Daniel Hort

A characterization of 1-, 2-, 3-, 4-homomorphisms of ordered sets

Czechoslovak Mathematical Journal, Vol. 53 (2003), No. 1, 213–221

Persistent URL: http://dml.cz/dmlcz/127792

Czechoslovak Mathematical Journal, 53 (128) (2003), 213–221

A CHARACTERIZATION OF 1-, 2-, 3-, 4-HOMOMORPHISMS OF ORDERED SETS

Radomír Halaš and Daniel Hort, Olomouc

(Received March 31, 2000)

Abstract. We characterize totally ordered sets within the class of all ordered sets containing at least four-element chains. We use a simple relationship between their isotone transformations and the so called 1-endomorphism which is introduced in the paper. Later we describe 1-, 2-, 3-, 4-homomorphisms of ordered sets in the language of super strong mappings.

Keywords: ordered sets, morphisms

MSC 2000: 06A10, 06A99

0. Introduction

In [4] new concepts of 2-, 3-, 4-endomorphisms of ordered sets were introduced. They appeared to be an efficient tool for the determination of chains in the class of all ordered sets satisfying a certain condition (the existence of a three-element chain). In this contribution we introduce a 1-endomorphism and demonstrate its conjunction with the above mentioned results. We declare that the requirement of a four-element chain is essential.

Let $(P, \leqslant)$ be an ordered set, $\emptyset \neq X \subseteq P$. The symbol $E_f(X)$ denotes $f^-(f(X))$ where $f^-(X)$ is the preimage of X under a mapping f, i.e. $f^-(X) = \{y \mid f(y) = x$ for some $x \in X\}$. By $[X)_\leqslant = \{y \in P \colon y \geqslant x$ for some $x \in X\}$ we denote the upper end of an ordered set $(P, \leqslant)$ generated by a subset X. Let $(P, \leqslant)$, $(Q, \leqslant)$ be ordered sets and let $f\colon P \longrightarrow Q$ be a mapping. The mapping f is isotone if for any pair of elements $a, b \in P$ such that $a \leqslant b$ we have $f(a) \leqslant f(b)$. The mapping f is a *strong homomorphism* if $f(z) \geqslant f(x)$ implies $f(z) = f(u)$, $f(x) = f(a)$ for some

The paper was supported by the grant of the Czech Government Council J14/98.

$a, u \in P$ such that $u \geqslant a$. An isotone mapping of an ordered set into itself is called an *endomorphism.* The set of all endomorphisms of $(P, \leqslant)$ endowed with a composition forms a monoid which is denoted by $\mathrm{End}(H, \leqslant)$.

Remark. There exists also another concept of a strong homomorphism. A mapping $f\colon P \longrightarrow Q$ between ordered sets $(P, \leqslant)$, $(Q, \leqslant)$ is called a strong homomorphism if for any pair of elements $x \in P, y \in Q$ we have $f(x) \leqslant y$ if and only if there exists an element $x' \in P$ such that $x \leqslant x'$ and $f(x') = y$ (*L. L. Esakia*: Heyting algebras I. Duality theory. Mecniereba, 1985, Tbilisi).

Definition 1 ([4]). Let $(P, \leqslant)$, $(Q, \leqslant)$ be ordered sets. A mapping $f\colon P \to Q$ is called

(1) a 1-*homomorphism* if it satisfies the condition

$$f^{-}([f(x))_{\leqslant}) = E_f([E_f(x))_{\leqslant}) \quad \text{for any } x \in P,$$

(2) a 2-*homomorphism* if it satisfies the condition

$$f^{-}([f(x))_{\leqslant}) = f^{-}(f([x)_{\leqslant})) \quad \text{for any } x \in P,$$

(3) a 3-*homomorphism* if it satisfies the condition

$$f^{-}([f(x))_{\leqslant}) = [f^{-}(f(x)))_{\leqslant} \quad \text{for any } x \in P,$$

(4) a 4-*homomorphism* if both the conditions for 2- and 3-homomorphisms are satisfied

$$[f^{-}(f(x)))_{\leqslant} = f^{-}([f(x))_{\leqslant}) = f^{-}(f([x)_{\leqslant})) \quad \text{for any } x \in P.$$

1. 1-ENDOMORPHISMS

Proposition 1. *Let $(X, \leqslant)$ be an ordered set containing at least a four-element chain. Then for any ordered pair (x, y) of $\leqslant$-incomparable elements $x, y \in X$ there exists an isotone mapping $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ such that*

$$f(x) < f(y) \quad \text{and} \quad \{x\} = E_f(x),\ \{y\} = E_f(y).$$

Proof. Suppose $(X, \leqslant)$ contains at least a four-element chain C. Consider $C_0 \subseteq C$ such that $C_0 = \{a, b, c, d\}$, $a < b < c < d$, and $x, y \in X$ are incomparable elements. Now let X^{xy}, X_{xy} be subsets of X such that

$$\begin{aligned} X^{xy} &= \{z\colon\ z > x \ \text{ or } \ z > y\} = [\{x,y\})_{\leqslant} \setminus \{x,y\}, \\ X_{xy} &= \{z\colon\ z < x \ \text{ or } \ z < y\} = (\{x,y\}]_{\leqslant} \setminus \{x,y\}, \\ Y &= X \setminus (X^{xy} \cup X_{xy} \cup \{x,y\}). \end{aligned}$$

Let $f(x) = b$ and $f(y) = c$, which means $f(x) < f(y)$. Furthermore let $f(t) = a$ for any $t \in X_{xy}$, $f(s) = d$ for any $s \in X^{xy}$ and $f(r) = d$ for any $r \in Y$ (cf. Fig. 1). Now $f(u) = f(v)$ for any pair $(u,v) \in X^{xy} \times X^{xy}$, $(u,v) \in X_{xy} \times X_{xy}$, $(u,v) \in Y \times Y$, and $f(u) < f(v)$ for any pair $(u,v) \in X_{xy} \times X^{xy}$, which implies f is isotone, because $p \leqslant q$ implies $f(p) \leqslant f(q)$ for any $p,q \in X$ and $\{x\} = f^{-}(b) = E_f(x)$, $\{y\} = f^{-}(c) = E_f(y)$. Thus the proposition holds. $\square$

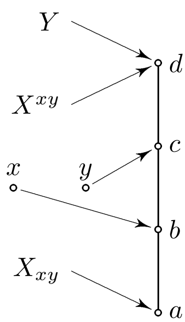

Figure 1

Lemma 1. *Let $f\colon X_1 \longrightarrow X_2$ be a mapping of an ordered set $(X_1, \leqslant)$ into another one $(X_2, \leqslant)$. The following conditions are equivalent:*

(1) *f is isotone,*

(2) *$E_f([E_f(x))_{\leqslant}) \subseteq f^{-}([f(x))_{\leqslant})$ for any $x \in X_1$.*

P r o o f. (1) $\Rightarrow$ (2): Let $x \in X_1$ be an arbitrary element and in addition suppose $z \in E_f([E_f(x))_{\leqslant})$, which means $f(z) \in f([E_f(x))_{\leqslant})$. Then there exists $q \in [E_f(x))_{\leqslant}$ such that $f(z) = f(q)$. It follows that there exists $r \in E_f(x)$, i.e. $f(r) = f(x)$ such that $r \leqslant q$. Since $f(r) \leqslant f(q)$ we have $f(x) \leqslant f(z)$, which implies $f(z) \in [f(x))_{\leqslant}$ and consequently $z \in f^{-}([f(x))_{\leqslant})$. We have $E_f([E_f(x))_{\leqslant}) \subseteq f^{-}([f(x))_{\leqslant})$.

(2) $\Rightarrow$ (1): Let x, y be elements from X_1 such that $x \leqslant y$. Since $x \in E_f(x)$ we have $y \in [E_f(x))_{\leqslant}$ and further $y \in E_f([E_f(x))_{\leqslant})$. By the assumption $y \in f^{-}([f(x))_{\leqslant})$, which implies $f(y) \in [f(x))_{\leqslant}$ and thus $f(x) \leqslant f(y)$. Finally, the mapping f is isotone. $\square$

Proposition 2. *Let $(X, \leqslant)$ be an ordered set containing at least a four-element chain. Then $(X, \leqslant)$ is a chain if and only if any isotone selfmap f of the poset $(X, \leqslant)$ satisfies the following condition:*

$$(*) \qquad E_f([E_f(x))_{\leqslant}) = f^{-}([f(x))_{\leqslant}) \quad \text{for any } x \in X.$$

Proof. $\Rightarrow$: Let $(X, \leqslant)$ be a chain and $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ an isotone mapping. Let $x \in X$ be an arbitrary element and suppose $z \in f^{-}([f(x))_{\leqslant})$, which means $f(z) \in [f(x))_{\leqslant}$, i.e. $f(x) \leqslant f(z)$. If $f(x) = f(z)$ then $z \in E_f(x)$ and as

$$E_f(x) \subseteq [E_f(x))_{\leqslant} \subseteq E_f([E_f(x))_{\leqslant}),$$

we have $z \in E_f([E_f(x))_{\leqslant})$. If $f(x) < f(z)$ then $x \leqslant z$ (since the mapping f is isotone and $(X, \leqslant)$ is a chain). Further, from $[E_f(x))_{\leqslant} = \{t\colon \exists u \in X\colon f(u) = f(x), u \leqslant t\}$ we obtain $z \in [E_f(x))_{\leqslant}$, which implies $f(z) \in f([E_f(x))_{\leqslant})$ and consequently $z \in E_f([E_f(x))_{\leqslant})$. We have $f^{-}([f(x))_{\leqslant}) \subseteq E_f([E_f(x))_{\leqslant})$. Since $E_f([E_f(x))_{\leqslant}) \subseteq f^{-}([f(x))_{\leqslant})$ (Lemma 1) we have finally $f^{-}([f(x))_{\leqslant}) = E_f([E_f(x))_{\leqslant})$.

$\Leftarrow$: Let $(X, \leqslant)$ be a poset containing at least a four-element chain, let $x, y \in X$ be incomparable ($x \parallel y$) and suppose $f^{-}([f(x))_{\leqslant}) = E_f([E_f(x))_{\leqslant})$ for any isotone mapping $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$. Let f_0 be a mapping from Proposition 1, i.e. $f_0(x) < f_0(y)$ and $\{x\} = E_{f_0}(x)$, $\{y\} = E_{f_0}(y)$. Since $f_0(x) < f_0(y)$, then $f_0(y) \in [f_0(x))_{\leqslant}$, which implies $y \in f_0^{-}([f_0(x))_{\leqslant})$. Now $y \in E_{f_0}([E_{f_0}(x))_{\leqslant})$ by the assumption $(*)$. We get $y \in E_{f_0}([\{x\})_{\leqslant})$, which implies $f_0(y) \in f_0([\{x\})_{\leqslant})$. Then there exists $z \in [\{x\})_{\leqslant}$ such that $f_0(z) = f_0(y)$. We get

$$z \in E_{f_0}(z) = E_{f_0}(y) = \{y\},$$

which implies $z = y$ and therefore $y \in [\{x\})_{\leqslant}$, which means $x \leqslant y$. This is a contradiction to the assumption of incomparability of x and y. Thus $(X, \leqslant)$ is a chain. $\square$

Remark. It can be easily proved that the condition $(*)$ can be replaced by the dual one:

$$f^{-}((f(x)]_{\leqslant}) = E_f((E_f(x)]_{\leqslant}) \quad \text{for any } x \in X.$$

In the proof it is useful again to consider such an isotone mapping that $f(x) < f(y)$ and $\{x\} = E_f(x)$, $\{y\} = E_f(y)$ whose existence was stated in Proposition 1.

Theorem 1. *Let $(X, \leqslant)$ be an ordered set containing at least a four-element chain. Then the following conditions are equivalent:*

(1) $(X, \leqslant)$ *is a totally ordered set,*
(2) $\mathrm{End}(X, \leqslant) \subseteq 1\text{-}\mathrm{End}(X, \leqslant)$,
(3) $\mathrm{End}(X, \leqslant) = 1\text{-}\mathrm{End}(X, \leqslant)$.

Proof. (1) $\Rightarrow$ (2): It follows from Proposition 2.

(2) $\Rightarrow$ (3): Let $f \in 1\text{-}\mathrm{End}(X, \leqslant)$ be an arbitrary mapping and suppose $x, y \in X$, $x \leqslant y$ are arbitrary elements. Since $x \leqslant y$ and $x \in E_f(x)$ hence $y \in [E_f(x))_{\leqslant}$ and further $y \in E_f([E_f(x))_{\leqslant})$. Now $y \in f^-([f(x))_{\leqslant})$ by the assumption of 1-endomorphism. This implies $f(y) \in [f(x))_{\leqslant}$ and we get $f(x) \leqslant f(y)$, thus the mapping $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ is isotone. Finally, $\mathrm{End}(X, \leqslant) \supseteq 1\text{-}\mathrm{End}(X, \leqslant)$, which implies $\mathrm{End}(X, \leqslant) = 1\text{-}\mathrm{End}(X, \leqslant)$.

(3) $\Rightarrow$ (1): It follows from Proposition 2. $\square$

Proposition 3. *Let $(P, \leqslant)$, $(Q, \leqslant)$ be ordered sets and $f\colon P \longrightarrow Q$ a mapping. Then the following conditions are equivalent:*
(1) *f is a 1-homomorphism,*
(2) a) *f is isotone,*
  b) *for any $z, x \in P$ the inequality $f(z) \geqslant f(x)$ implies $f(z) = f(u)$, $f(x) = f(a)$ for some $a, u \in P$ such that $u \geqslant a$,*
 i.e. f is an isotone strong homomorphism.

Proof. (1) $\Rightarrow$ (2): b) Suppose (1) is satisfied and $f(z) \geqslant f(x)$ for some $x, z \in P$. We have $f(z) \in [f(x))_{\leqslant}$ thus $z \in f^-([f(x))_{\leqslant}) = E_f([E_f(x))_{\leqslant})$. Now $f(z) \in f([E_f(x))_{\leqslant})$, which means that there exists $u \in P$ such that $f(z) = f(u)$ and $u \in [E_f(x))_{\leqslant}$, therefore there exists $a \in P$ such that $a \leqslant u$ and $a \in f^-(f(x))$, i.e. $f(a) = f(x)$. The condition a) follows from Lemma 1.

(2) $\Rightarrow$ (1): Assume (2) and $z \in f^-([f(x))_{\leqslant})$, i.e. $f(z) \in [f(x))_{\leqslant}$, which is $f(z) \geqslant f(x)$. By (2) we have $f(z) = f(u)$, $f(a) = f(x)$ for some $a, u \in P$ such that $u \geqslant a$, which means $f(u) \geqslant f(a)$. Consequently $u \in [E_f(a))_{\leqslant} = [E_f(x))_{\leqslant}$ and $f(z) = f(u) \in f([E_f(x))_{\leqslant}$, i.e. $z \in E_f([E_f(x))_{\leqslant})$. The converse inclusion $E_f([E_f(x))_{\leqslant}) \subseteq f^-([f(x))_{\leqslant})$ follows from (2) a) by Lemma 1. $\square$

2. Super-strong mappings

Proposition 4. *Let $(P, \leqslant)$, $(Q, \leqslant)$ be ordered sets and $f\colon P \longrightarrow Q$ a mapping. Then the following conditions are equivalent:*
(1) *f is a 2-homomorphism,*
(2) a) *f is isotone,*
  b) *for any $z, x \in P$ the inequality $f(z) \geqslant f(x)$ implies $f(z) = f(u)$ for some $u \geqslant x$, $u \in P$.*

P r o o f. (1) $\Rightarrow$ (2): b) Suppose (1) is satisfied and $f(z) \geqslant f(x)$ for some $x, z \in P$. Then $f(z) \in [f(x))_{\leqslant}$, which means $z \in f^{-}([f(x))_{\leqslant}) = f^{-}(f([x)_{\leqslant}))$, i.e. $f(z) \in f([x)_{\leqslant})$. Thus there exists $u \in [x)_{\leqslant}$, i.e. $u \geqslant x$ such that $f(u) = f(z)$.

a) Suppose $x, y \in P$, $x \leqslant y$ are arbitrary elements. Then $y \in [x)_{\leqslant}$, which implies $f(y) \in f([x)_{\leqslant})$ and $y \in f^{-}(f(y)) \subseteq f^{-}(f([x)_{\leqslant})) = f^{-}([f(x))_{\leqslant})$, thus $f(y) \in [f(x))_{\leqslant}$, which means $f(x) \leqslant f(y)$.

(2) $\Rightarrow$ (1): Suppose (2) holds and $z \in f^{-}([(f(x))_{\leqslant})$, which is $f(z) \in [(f(x))_{\leqslant}$, i.e. $f(z) \geqslant f(x)$. Applying (2) we have $f(z) = f(u)$ for some $u \geqslant x$ and consequently $u \in [x)_{\leqslant}$, which implies $f(u) \in f([x)_{\leqslant}))$. Finally $f(z) \in f([x)_{\leqslant})$ and $z \in f^{-}(f([x)_{\leqslant}))$. The converse inclusion follows from $f([x)_{\leqslant}) \subseteq [(f(x))_{\leqslant}$, which holds for any isotone mapping f (cf. [4], Lemma 2). $\square$

Proposition 5. *Let $(P, \leqslant)$, $(Q, \leqslant)$ be ordered sets and $f\colon P \longrightarrow Q$ a mapping. Then the following conditions are equivalent:*

(1) *f is a* 3*-homomorphism,*

(2) a) *f is isotone,*

b) *for any $y, x \in P$ the inequality $f(y) \geqslant f(x)$ implies $y \geqslant z$ for some $z \in P$ such that $f(z) = f(x)$.*

P r o o f. (1) $\Rightarrow$ (2): b) Suppose (1) and $f(y) \geqslant f(x)$ for some $x, y \in P$. Clearly $f(y) \in [(f(x))_{\leqslant}$ and thus $y \in f^{-}([(f(x))_{\leqslant}) = [f^{-}(f(x)))_{\leqslant}$, hence there exists $z \in f^{-}(f(x))$, i.e. $f(z) = f(x)$ such that $y \geqslant z$.

a) Suppose $x, y \in P$, $x \leqslant y$. Since $x \in f^{-}(f(x))$ we have $y \in [x)_{\leqslant} \subseteq [f^{-}(f(x)))_{\leqslant} = f^{-}([f(x))_{\leqslant})$, hence $f(y) \in [f(x))_{\leqslant}$. Consequently $f(x) \leqslant f(y)$.

(2) $\Rightarrow$ (1): Suppose (2) and let $y \in f^{-}([(f(x))_{\leqslant})$, which means $f(y) \in [(f(x))_{\leqslant}$, i.e. $f(y) \geqslant f(x)$. We have $y \geqslant z$ for some $z \in P$ such that $f(z) = f(x)$ by (2) and consequently $y \geqslant z \in f^{-}(f(z)) = f^{-}(f(x))$ and $y \in [f^{-}(f(x)))_{\leqslant}$. The converse inclusion follows from $[f^{-}(f(x)))_{\leqslant} \subseteq f^{-}([(f(x))_{\leqslant})$, which holds for any isotone mapping f (cf. [4], Lemma 2). $\square$

A mapping satisfying the condition (2) b) of Proposition 4 or 5 is called *u-super strong* or *l-super strong*, respectively. If it satisfies both the conditions, it is called a *super strong* mapping.

There is a natural question whether 2-, 3-endomorphisms are closed under composition. The answer is negative, which means that 2, 3-End$(P, \leqslant)$ is not a subgroupoid of End$(P, \leqslant)$. Let $P = \{a, b, c\}$ and $a \leqslant b$, $a \parallel c \parallel b$ (cf. Fig. 2). The mappings $f, g\colon (P, \leqslant) \to (P, \leqslant)$ $(f, g\colon (P, \geqslant) \to (P, \geqslant))$ are 2-endomorphisms (3-endomorphisms) but for $h = g \circ f$ we have $h \notin$ 2-End$(P, \leqslant)$ $(h \notin$ 3-End$(P, \geqslant))$.

Now we can extend in a certain sense Theorem 1 from [4] to the case of 4-endomorphisms.

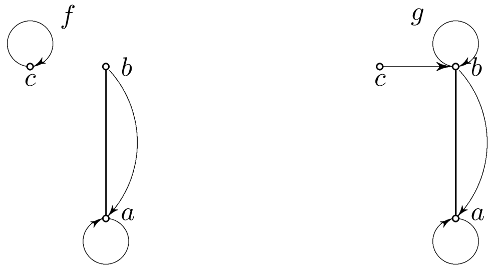

Figure 2

Theorem 2. *Let* $(P, \leqslant)$ *be a totally ordered set. Then*

$$\operatorname{End}(P, \leqslant) = 4\text{-}\operatorname{End}(P, \leqslant).$$

P r o o f. The inclusion $\operatorname{End}(P, \leqslant) \supseteq 4\text{-}\operatorname{End}(P, \leqslant)$ has been proved in [4], Lemma 3.

Suppose $f(z) \geqslant f(x)$. Since $(P, \leqslant)$ is a chain we have either $z \geqslant x$, i.e. condition (2) a) from Proposition 4 is satisfied, or $z < x$, which implies $f(z) \leqslant f(x)$ and consequently $f(z) = f(x)$, i.e. for $u = x$ f is also a 2-homomorphism. Similarly we can prove condition (2) a) from Proposition 5. $\square$

There is a natural question how to construct 2-, 3-, 4-homomorphisms.

Let $(P, \leqslant)$ be a poset, $\theta \in \operatorname{Eqv} P$. Further, let us define two relations $\lhd$, $\blacktriangleleft$ on P/θ in the following way:

$$[x]_\theta \blacktriangleleft [y]_\theta \quad \text{iff for any } q \in [x]_\theta \text{ there exists } p \in [y]_\theta \text{ such that } q \leqslant p,$$

$$[x]_\theta \lhd [y]_\theta \quad \text{iff for any } p \in [y]_\theta \text{ there exists } q \in [x]_\theta \text{ such that } q \leqslant p.$$

It is easy to see that they are both reflexive and transitive but not antisymmetric in general.

Lemma 2. *If the equivalence blocks of* P/θ *are convex then* $\lhd \cap \blacktriangleleft$ *is an order relation on* P/θ.

P r o o f. It has been proved in [2].

Corollary 1. *Let* $(P, \leqslant)$ *be a poset,* $\theta \in \operatorname{Eqv} P$ *such that* $\blacktriangleleft$ *is an order relation on* P/θ*. Then the canonical mapping* $\psi\colon P \to P/\theta$, $x \mapsto [x]_\theta$ *is a* 2*-homomorphism.*

P r o o f. It is enough to verify the validity of conditions (2) a), b) from Proposition 4. The definition of the relation $\blacktriangleleft$ yields

(i) $[x]_\theta \blacktriangleleft [y]_\theta$ implies $[y]_\theta = [z]_\theta$ for some $x \leqslant z$,

(ii) $z \leqslant y$ implies $[z]_\theta \blacktriangleleft [y]_\theta$

and the corollary holds. $\square$

Corollary 2. *Let $(P, \leqslant)$ be a poset, $\theta \in \operatorname{Eqv} P$ such that $\lhd$ is an order relation on P/θ. Then the canonical mapping $\psi\colon\ P \to P/\theta$, $x \mapsto [x]_\theta$ is a 3-homomorphism.*

P r o o f. The definition of the relation $\lhd$ yields

(i) $[x]_\theta \lhd [y]_\theta$ implies $z \leqslant y$ for some $z \in [x]_\theta$,

(ii) $z \leqslant y$ implies $[z]_\theta \lhd [y]_\theta$

and the corollary holds. □

Corollary 3. *Let $(P, \leqslant)$ be a poset, $\theta \in \operatorname{Eqv} P$ such that the equivalence blocks are convex. Let us order P/θ by $\lhd \cap \blacktriangleleft$. Then the canonical mapping $\psi\colon\ P \to P/\theta$, $x \mapsto [x]_\theta$ is a 4-homomorphism.*

P r o o f. It follows immediately from Corollary 1 and Corollary 2. □

Theorem 3. *Let $(P, \leqslant)$ be a poset. Then the following conditions are equivalent:*

(1) a) *$(P, \leqslant)$ is an antichain or*

b) *there exists an element $a \in P$ such that $(P, \leqslant) = X \oplus \{a\}$ where $X \neq \emptyset$ is an antichain or*

c) *$(P, \leqslant)$ is at least a three element chain,*

(2) $\operatorname{End}(P, \leqslant) \subseteq 2\text{-}\operatorname{End}(P, \leqslant)$,

(3) $\operatorname{End}(P, \leqslant) = 2\text{-}\operatorname{End}(P, \leqslant)$.

P r o o f. Conditions (2) and (3) are equivalent due to [4] Lemma 3 (this also follows from Proposition 4). It is enough to demonstrate the equivalence of (1) and (2). It has been recently proved in [4] that if P has at least a three-element chain it has to be a chain, i.e. (1) c) holds. Thus we can study only the cases where $(P, \leqslant)$ is of length one, i.e. it contains two-element chains only.

(1) $\Rightarrow$ (2): This follows immediately from Proposition 4.

(2) $\Rightarrow$ (1): Suppose that any isotone mapping is a 2-homomorphism, i.e. condition (2) b) from Proposition 4 is satisfied. This is clear if $(P, \leqslant)$ is an antichain or $(P, \leqslant)$ is a two-element chain. Suppose $(P, \leqslant)$ contains at least one two-element chain $b < a$ and incomparable elements. Then for any pair of incomparable elements $x, y \in P$ we can construct an isotone mapping f such that $f(x) > f(y)$, $f(z) = a$ for any $z \in X^{xy}$, $f(z) = b$ for any $z \in X_{xy}$ (X^{xy}, X_{xy} were defined in the proof of Proposition 1) and $f(z) = a$ otherwise. The mapping f has to be a 2-homomorphism, i.e there exists an element $z > y$ such that $f(z) = f(x)$. If $x \parallel z$ then we can again construct a similar mapping but for elements x and z. This leads to the existence of a three-element chain and consequently $(P, \leqslant)$ is a chain. Thus $x \leqslant z$, which means that P is up directed and must be of the form $X \oplus \{a\}$ for $a \in P$, X an antichain. □

Theorem 4. *Let $(P, \leqslant)$ be a poset. Then the following conditions are equivalent:*

(1) a) *$(P, \leqslant)$ is an antichain or*

b) *there exists an element $a \in P$ such that $(P, \leqslant) = \{a\} \oplus X$ where $X \neq \emptyset$ is an antichain or*

c) *$(P, \leqslant)$ is at least a three element chain,*

(2) $\mathrm{End}(P, \leqslant) \subseteq 3\text{-}\mathrm{End}(P, \leqslant)$,

(3) $\mathrm{End}(P, \leqslant) = 3\text{-}\mathrm{End}(P, \leqslant)$.

Proof. Dually to the proof of the previous Theorem 3. □

Theorem 5. *Let $(P, \leqslant)$ be a poset. Then the following conditions are equivalent:*

(1) a) *$(P, \leqslant)$ is an antichain or*

b) *$(P, \leqslant)$ is at least a three element chain,*

(2) $\mathrm{End}(P, \leqslant) \subseteq 4\text{-}\mathrm{End}(P, \leqslant)$,

(3) $\mathrm{End}(P, \leqslant) = 4\text{-}\mathrm{End}(P, \leqslant)$.

Proof. It follows from Theorem 3 and Theorem 4. □

References

[1] *G. Grätzer*: General Lattice Theory. Akademie-Verlag, Berlin, 1978, Russian translation 1982.

[2] *R. Halaš*: Congruences on posets. Contributions to General Algebra No. 12. Verlag J. Heyn, Klagenfurt, 2000, pp. 195–210.

[3] *D. Hort*: A construction of hypergroups from ordered structures and their morphisms. Preprint (1998).

[4] *D. Hort, J. Chvalina and J. Moučka*: Characterizations of totally ordered sets by their various endomorphisms. Czechoslovak Math. J. *52* (2002), 23–32.

[5] *J. Jantosciak*: Homomorphisms, equivalences and reductions in hypergroups. Riv. Mat. Pura Appl. *9* (1991), 23–47.

Authors' address: Department of Algebra and Geometry, Přírodovědecká fakulta, Palacký University, Tomkova 40, 771 76 Olomouc, Czech Republic, e-mail: `{halas, hort}@risc.upol.cz`.