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A CHARACTERIZATION OF 1-, 2-, 3-, 4-HOMOMORPHISMS OF ORDERED SETS

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Abstract. We characterize totally ordered sets within the class of all ordered sets containing at least four-element chains. We use a simple relationship between their isotone transformations and the so called 1-endomorphism which is introduced in the paper. Later we describe 1-, 2-, 3-, 4-homomorphisms of ordered sets in the language of super strong mappings.

Keywords: ordered sets, morphisms

MSC 2000: 06A10, 06A99

0. Introduction

In [4] new concepts of 2-, 3-, 4-endomorphisms of ordered sets were introduced. They appeared to be an efficient tool for the determination of chains in the class of all ordered sets satisfying a certain condition (the existence of a three-element chain). In this contribution we introduce a 1-endomorphism and demonstrate its conjunction with the above mentioned results. We declare that the requirement of a four-element chain is essential.

Let \((P, \leq)\) be an ordered set, \(\emptyset \neq X \subseteq P\). The symbol \(E_f(X)\) denotes \(f^{-}(f(X))\) where \(f^{-}(X)\) is the preimage of \(X\) under a mapping \(f\), i.e. \(f^{-}(X) = \{y \mid f(y) = x \text{ for some } x \in X\}\). By \([X]_{\leq} = \{y \in P \mid y \geq x \text{ for some } x \in X\}\) we denote the upper end of an ordered set \((P, \leq)\) generated by a subset \(X\). Let \((P, \leq), (Q, \leq)\) be ordered sets and let \(f : P \rightarrow Q\) be a mapping. The mapping \(f\) is isotone if for any pair of elements \(a, b \in P\) such that \(a \leq b\) we have \(f(a) \leq f(b)\). The mapping \(f\) is a strong homomorphism if \(f(z) \geq f(x)\) implies \(f(z) = f(u), f(x) = f(a)\) for some

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a, u \in P such that u \geq a. An isotone mapping of an ordered set into itself is called an \textit{endomorphism}. The set of all endomorphisms of \((P, \leq)\) endowed with a composition forms a monoid which is denoted by \text{End}(H, \leq).

\textbf{Remark.} There exists also another concept of a strong homomorphism. A mapping \(f : P \rightarrow Q\) between ordered sets \((P, \leq), (Q, \leq)\) is called a strong homomorphism if for any pair of elements \(x \in P, y \in Q\) we have \(f(x) \leq y\) if and only if there exists an element \(x' \in P\) such that \(x \leq x'\) and \(f(x') = y\) (L. L. Esakia: Heyting algebras I. Duality theory. Mecniereba, 1985, Tbilisi).

\textbf{Definition 1} ([4]). Let \((P, \leq), (Q, \leq)\) be ordered sets. A mapping \(f : P \rightarrow Q\) is called

(1) a 1\textit{-homomorphism} if it satisfies the condition

\[ f^{-}([f(x)]_{\leq}) = E_{f}([E_{f}(x)]_{\leq}) \quad \text{for any } x \in P, \]

(2) a 2\textit{-homomorphism} if it satisfies the condition

\[ f^{-}([f(x)]_{\leq}) = f^{-}(f([x]_{\leq})) \quad \text{for any } x \in P, \]

(3) a 3\textit{-homomorphism} if it satisfies the condition

\[ f^{-}([f(x)]_{\leq}) = [f^{-}(f(x))]_{\leq} \quad \text{for any } x \in P, \]

(4) a 4\textit{-homomorphism} if both the conditions for 2- and 3-homomorphisms are satisfied

\[ [f^{-}(f(x))]_{\leq} = f^{-}([f(x)]_{\leq}) = f^{-}(f([x]_{\leq})) \quad \text{for any } x \in P. \]

1. 1-ENDOMORPHISMS

\textbf{Proposition 1.} Let \((X, \leq)\) be an ordered set containing at least a four-element chain. Then for any ordered pair \((x, y)\) of \(\leq\)-incomparable elements \(x, y \in X\) there exists an isotone mapping \(f : (X, \leq) \rightarrow (X, \leq)\) such that

\[ f(x) < f(y) \quad \text{and} \quad \{x\} = E_{f}(x), \{y\} = E_{f}(y). \]

\textbf{Proof.} Suppose \((X, \leq)\) contains at least a four-element chain \(C\). Consider \(C_{0} \subseteq C\) such that \(C_{0} = \{a, b, c, d\}, a < b < c < d\), and \(x, y \in X\) are incomparable
elements. Now let $X^{xy}, X_{xy}$ be subsets of $X$ such that
\[
X^{xy} = \{z: z > x \text{ or } z > y\} = \{x, y\} \cup \{x, y\},
\]
\[
X_{xy} = \{z: z < x \text{ or } z < y\} = \{x, y\} \cup \{x, y\},
\]
\[
Y = X \setminus (X^{xy} \cup X_{xy} \cup \{x, y\}).
\]

Let $f(x) = b$ and $f(y) = c$, which means $f(x) < f(y)$. Furthermore let $f(t) = a$ for any $t \in X^{xy}$, $f(s) = d$ for any $s \in X^{xy}$ and $f(r) = d$ for any $r \in Y$ (cf. Fig. 1). Now $f(u) = f(v)$ for any pair $(u, v) \in X^{xy} \times X^{xy}$, $(u, v) \in X_{xy} \times X_{xy}$, $(u, v) \in Y \times Y$, and $f(u) < f(v)$ for any pair $(u, v) \in X_{xy} \times X^{xy}$, which implies $f$ is isotone, because $p \leq q$ implies $f(p) \leq f(q)$ for any $p, q \in X$ and $\{x\} = f^-(b) = E_f(x)$, $\{y\} = f^-(c) = E_f(y)$. Thus the proposition holds.

\[\square\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

**Lemma 1.** Let $f: X_1 \rightarrow X_2$ be a mapping of an ordered set $(X_1, \leq)$ into another one $(X_2, \leq)$. The following conditions are equivalent:

1. $f$ is isotone,
2. $E_f([E_f(x)]_{\leq}) \subseteq f^-(|[f(x)]_{\leq})$ for any $x \in X_1$.

**Proof.** (1) $\Rightarrow$ (2): Let $x \in X_1$ be an arbitrary element and in addition suppose $z \in E_f([E_f(x)]_{\leq})$, which means $f(z) \in f([E_f(x)]_{\leq})$. Then there exists $q \in [E_f(x)]_{\leq}$ such that $f(z) = f(q)$. It follows that there exists $r \in E_f(x)$, i.e. $f(r) = f(x)$ such that $r \leq q$. Since $f(r) \leq f(q)$ we have $f(x) \leq f(z)$, which implies $f(z) \in [f(x)]_{\leq}$ and consequently $z \in f^-(|[f(x)]_{\leq})$. We have $E_f([E_f(x)]_{\leq}) \subseteq f^-([f(x)]_{\leq})$.

(2) $\Rightarrow$ (1): Let $x, y$ be elements from $X_1$ such that $x \leq y$. Since $x \in E_f(x)$ we have $y \in [E_f(x)]_{\leq}$ and further $y \in E_f([E_f(x)]_{\leq})$. By the assumption $y \in f^-(|[f(x)]_{\leq})$, which implies $f(y) \in |f(x)]_{\leq}$ and thus $f(x) \leq f(y)$. Finally, the mapping $f$ is isotone. \[\square\]
Proposition 2. Let \((X, \leq)\) be an ordered set containing at least a four-element chain. Then \((X, \leq)\) is a chain if and only if any isotone selfmap \(f\) of the poset \((X, \leq)\) satisfies the following condition:

\[
E_f([E_f(x)]_{\leq}) = f^-(|[f(x)]_{\leq}) \quad \text{for any } x \in X.
\]

Proof. \(\Rightarrow:\) Let \((X, \leq)\) be a chain and \(f: (X, \leq) \rightarrow (X, \leq)\) an isotone mapping. Let \(x \in X\) be an arbitrary element and suppose \(z \in f^-(|[f(x)]_{\leq})\), which means \(f(z) \in [f(x)]_{\leq}\), i.e. \(f(x) \leq f(z)\). If \(f(x) = f(z)\) then \(z \in E_f(x)\) and as

\[
E_f(x) \subseteq [E_f(x)]_{\leq} \subseteq E_f([E_f(x)]_{\leq}),
\]

we have \(z \in E_f([E_f(x)]_{\leq})\). If \(f(x) < f(z)\) then \(x \leq z\) (since the mapping \(f\) is isotone and \((X, \leq)\) is a chain). Further, from \([E_f(x)]_{\leq} = \{t: \exists u \in X: f(u) = f(x), u \leq t\}\) we obtain \(z \in [E_f(x)]_{\leq}\), which implies \(f(z) \in f([E_f(x)]_{\leq})\) and consequently \(z \in E_f([E_f(x)]_{\leq})\). We have \(f^-(|[f(x)]_{\leq}) \subseteq E_f([E_f(x)]_{\leq})\). Since \(E_f([E_f(x)]_{\leq}) \subseteq f^-(|[f(x)]_{\leq})\) (Lemma 1) we have finally \(f^-(|[f(x)]_{\leq}) = E_f([E_f(x)]_{\leq})\).

\(\Leftarrow:\) Let \((X, \leq)\) be a poset containing at least a four-element chain, let \(x, y \in X\) be incomparable \((x \parallel y)\) and suppose \(f^-(|[f(x)]_{\leq}) = E_f([E_f(x)]_{\leq})\) for any isotone mapping \(f: (X, \leq) \rightarrow (X, \leq)\). Let \(f_0\) be a mapping from Proposition 1, i.e. \(f_0(x) < f_0(y)\) and \(\{x\} = E_{f_0}(x)\), \(\{y\} = E_{f_0}(y)\). Since \(f_0(x) < f_0(y)\), then \(f_0(y) \in [f_0(x)]_{\leq}\), which implies \(y \in f_0^-([f_0(x)]_{\leq})\). Now \(y \in E_{f_0}([E_{f_0}(x)]_{\leq})\) by the assumption \((*)\). We get \(y \in E_{f_0}([\{x\}]_{\leq})\), which implies \(f_0(y) \in f_0([\{x\}]_{\leq})\). Then there exists \(z \in [\{x\}]_{\leq}\) such that \(f_0(z) = f_0(y)\). We get

\[
z \in E_{f_0}(z) = E_{f_0}(y) = \{y\},
\]

which implies \(z = y\) and therefore \(y \in [\{x\}]_{\leq}\), which means \(x \leq y\). This is a contradiction to the assumption of incomparability of \(x\) and \(y\). Thus \((X, \leq)\) is a chain. \(\square\)

Remark. It can be easily proved that the condition \((*)\) can be replaced by the dual one:

\[
f^-(|[f(x)]_{\leq}) = E_f([E_f(x)]_{\leq}) \quad \text{for any } x \in X.
\]

In the proof it is useful again to consider such an isotone mapping that \(f(x) < f(y)\) and \(\{x\} = E_f(x)\), \(\{y\} = E_f(y)\) whose existence was stated in Proposition 1.

Theorem 1. Let \((X, \leq)\) be an ordered set containing at least a four-element chain. Then the following conditions are equivalent:

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(1) \((X, \leq)\) is a totally ordered set,
(2) \(\End(X, \leq) \subseteq \1End(X, \leq)\),
(3) \(\End(X, \leq) = \1End(X, \leq)\).

Proof. (1) \(\Rightarrow\) (2): It follows from Proposition 2.

(2) \(\Rightarrow\) (3): Let \(f \in \1End(X, \leq)\) be an arbitrary mapping and suppose \(x, y \in X\), \(x \leq y\) are arbitrary elements. Since \(x \leq y\) and \(x \in E_f(x)\) hence \(y \in [E_f(x)]_{\leq}\) and further \(y \in E_f([E_f(x)]_{\leq})\). Now \(y \in f^\sim([f(x)]_{\leq})\) by the assumption of \(1\)-endomorphism. This implies \(f(y) \in [f(x)]_{\leq}\) and we get \(f(x) \leq f(y)\), thus the mapping \(f: (X, \leq) \to (X, \leq)\) is isotone. Finally, \(\End(X, \leq) \supseteq \1End(X, \leq)\), which implies \(\End(X, \leq) = \1End(X, \leq)\).

(3) \(\Rightarrow\) (1): It follows from Proposition 2. \(\square\)

Proposition 3. Let \((P, \leq), (Q, \leq)\) be ordered sets and \(f: P \to Q\) a mapping. Then the following conditions are equivalent:

(1) \(f\) is a \(1\)-homomorphism,
(2) a) \(f\) is isotone,
    b) for any \(z, x \in P\) the inequality \(f(z) \geq f(x)\) implies \(f(z) = f(u), f(x) = f(a)\)
        for some \(a, u \in P\) such that \(u \geq a\),
    i.e. \(f\) is an isotone strong homomorphism.

Proof. (1) \(\Rightarrow\) (2): b) Suppose (1) is satisfied and \(f(z) \geq f(x)\) for some \(x, z \in P\). We have \(f(z) \in [f(x)]_{\leq}\) thus \(z \in f^\sim([f(x)]_{\leq}) = E_f([E_f(x)]_{\leq})\). Now \(f(z) \in f([E_f(x)]_{\leq})\), which means that there exists \(u \in P\) such that \(f(z) = f(u)\) and \(u \in [E_f(x)]_{\leq}\), therefore there exists \(a \in P\) such that \(a \leq u\) and \(a \in f^\sim(f(x))\), i.e. \(f(a) = f(x)\). The condition a) follows from Lemma 1.

(2) \(\Rightarrow\) (1): Assume (2) and \(z \in f^\sim([f(x)]_{\leq})\), i.e. \(f(z) \in [f(x)]_{\leq}\), which is \(f(z) \geq f(x)\). By (2) we have \(f(z) = f(u), f(a) = f(x)\) for some \(a, u \in P\) such that \(u \geq a\), which means \(f(u) \geq f(a)\). Consequently \(u \in [E_f(a)]_{\leq} = [E_f(x)]_{\leq}\) and \(f(z) = f(u) \in f([E_f(x)]_{\leq}), i.e. z \in E_f([E_f(x)]_{\leq})\). The converse inclusion \(E_f([E_f(x)]_{\leq}) \subseteq f^\sim([f(x)]_{\leq})\) follows from (2) a) by Lemma 1. \(\square\)

2. Super-strong mappings

Proposition 4. Let \((P, \leq), (Q, \leq)\) be ordered sets and \(f: P \to Q\) a mapping. Then the following conditions are equivalent:

(1) \(f\) is a \(2\)-homomorphism,
(2) a) \(f\) is isotone,
    b) for any \(z, x \in P\) the inequality \(f(z) \geq f(x)\) implies \(f(z) = f(u)\) for some \(u \geq x, u \in P\).
Proof. (1) ⇒ (2): b) Suppose (1) is satisfied and \( f(z) \geq f(x) \) for some \( x, z \in P \). Then \( f(z) \in [f(x)]_\leq \), which means \( z \in f^{-}([f(x)]_\leq) = f^{-}(f([x]_\leq)) \), i.e. \( f(z) \in f([x]_\leq) \). Thus there exists \( u \in [x]_\leq \), i.e. \( u \geq x \) such that \( f(u) = f(z) \).

a) Suppose \( x, y \in P, x \leq y \) are arbitrary elements. Then \( y \in [x]_\leq \), which implies \( f(y) \in f([x]_\leq) \) and \( y \in f^{-}(f(y)) \subseteq f^{-}(f([x]_\leq)) = f^{-}([f(x)]_\leq) \), thus \( f(y) \in [f(x)]_\leq \), which means \( f(x) \leq f(y) \).

(2) ⇒ (1): Suppose (2) holds and \( z \in f^{-}([(f(x)]_\leq) \), which is \( f(z) \in [(f(x)]_\leq \), i.e. \( f(z) \geq f(x) \). Applying (2) we have \( f(z) = f(u) \) for some \( u \geq x \) and consequently \( u \in [x]_\leq \), which implies \( f(u) \in f([x]_\leq) \). Finally \( f(z) \in f([x]_\leq) \) and \( z \in f^{-}(f([x]_\leq)) \). The converse inclusion follows from \( f([x]_\leq) \subseteq [(f(x)]_\leq \), which holds for any isotone mapping \( f \) (cf. [4], Lemma 2).

Proposition 5. Let \( (P, \leq), (Q, \leq) \) be ordered sets and \( f: P \to Q \) a mapping. Then the following conditions are equivalent:

(1) \( f \) is a 3-homomorphism,

(2) a) \( f \) is isotone,

b) for any \( y, x \in P \) the inequality \( f(y) \geq f(x) \) implies \( y \geq z \) for some \( z \in P \) such that \( f(z) = f(x) \).

Proof. (1) ⇒ (2): b) Suppose (1) and \( f(y) \geq f(x) \) for some \( x, y \in P \). Clearly \( f(y) \in [(f(x)]_\leq \) and thus \( y \in f^{-}([(f(x)]_\leq) = [f^{-}(f(x))]_\leq \), hence there exists \( z \in f^{-}(f(x)) \), i.e. \( f(z) = f(x) \) such that \( y \geq z \).

a) Suppose \( x, y \in P, x \leq y \). Since \( x \in f^{-}(f(x)) \) we have \( y \in [x]_\leq \subseteq [f^{-}(f(x))]_\leq = f^{-}([f(x)]_\leq) \), hence \( f(y) \in [f(x)]_\leq \). Consequently \( f(x) \leq f(y) \).

(2) ⇒ (1): Suppose (2) and let \( y \in f^{-}([(f(x)]_\leq) \), which means \( f(y) \in [(f(x)]_\leq \), i.e. \( f(y) \geq f(x) \). We have \( y \geq z \) for some \( z \in P \) such that \( f(z) = f(x) \) by (2) and consequently \( y \geq z \in f^{-}(f(z)) = f^{-}(f(x)) \) and \( y \in [f^{-}(f(x))]_\leq \). The converse inclusion follows from \( [f^{-}(f(x))]_\leq \subseteq f^{-}([(f(x)]_\leq) \), which holds for any isotone mapping \( f \) (cf. [4], Lemma 2).

A mapping satisfying the condition (2) b) of Proposition 4 or 5 is called \text{u-super strong} or \text{l-super strong}, respectively. If it satisfies both the conditions, it is called a \text{super strong} mapping.

There is a natural question whether 2-, 3-endomorphisms are closed under composition. The answer is negative, which means that 2, 3-\text{End}(P, \leq) is not a subgroupoid of \text{End}(P, \leq) \). Let \( P = \{a, b, c\} \) and \( a \leq b, a \parallel c \parallel b \) (cf. Fig. 2). The mappings \( f, g: (P, \leq) \to (P, \leq) \) \((f, g: (P, \geq) \to (P, \geq)) \) are 2-endomorphisms (3-endomorphisms) but for \( h = g \circ f \) we have \( h \not\in 2\text{-End}(P, \leq) \) \((h \not\in 3\text{-End}(P, \geq)) \).

Now we can extend in a certain sense Theorem 1 from [4] to the case of 4-endomorphisms.

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Theorem 2. Let \((P, \leq)\) be a totally ordered set. Then

\[ \text{End}(P, \leq) = 4 \cdot \text{End}(P, \leq). \]

Proof. The inclusion \(\text{End}(P, \leq) \supseteq 4 \cdot \text{End}(P, \leq)\) has been proved in [4], Lemma 3.

Suppose \(f(z) \geq f(x)\). Since \((P, \leq)\) is a chain we have either \(z \geq x\), i.e. condition (2) a) from Proposition 4 is satisfied, or \(z < x\), which implies \(f(z) \leq f(x)\) and consequently \(f(z) = f(x)\), i.e. for \(u = x\) \(f\) is also a 2-homomorphism. Similarly we can prove condition (2) a) from Proposition 5. □

There is a natural question how to construct 2-, 3-, 4-homomorphisms.

Let \((P, \leq)\) be a poset, \(\theta \in \text{Eqv} P\). Further, let us define two relations \(<\), \(\preceq\) on \(P/\theta\) in the following way:

\[
\begin{align*}
[x]_\theta \prec [y]_\theta & \text{ iff for any } q \in [x]_\theta \text{ there exists } p \in [y]_\theta \text{ such that } q \leq p, \\
[x]_\theta \preceq [y]_\theta & \text{ iff for any } p \in [y]_\theta \text{ there exists } q \in [x]_\theta \text{ such that } q \leq p.
\end{align*}
\]

It is easy to see that they are both reflexive and transitive but not antisymmetric in general.

Lemma 2. If the equivalence blocks of \(P/\theta\) are convex then \(<\cap\preceq\) is an order relation on \(P/\theta\).

Proof. It has been proved in [2].

Corollary 1. Let \((P, \leq)\) be a poset, \(\theta \in \text{Eqv} P\) such that \(\preceq\) is an order relation on \(P/\theta\). Then the canonical mapping \(\psi: P \to P/\theta, x \mapsto [x]_\theta\) is a 2-homomorphism.

Proof. It is enough to verify the validity of conditions (2) a), b) from Proposition 4. The definition of the relation \(\preceq\) yields

(i) \([x]_\theta \preceq [y]_\theta\) implies \([y]_\theta = [z]_\theta\) for some \(x \leq z\),
(ii) \(z \leq y\) implies \([z]_\theta \preceq [y]_\theta\)

and the corollary holds. □
Corollary 2. Let \((P, \leq)\) be a poset, \(\theta \in \text{Eqv } P\) such that \(\triangleleft\) is an order relation on \(P/\theta\). Then the canonical mapping \(\psi: P \to P/\theta, x \mapsto [x]_{\theta}\) is a 3-homomorphism.

Proof. The definition of the relation \(\triangleleft\) yields
(i) \([x]_{\theta} \triangleleft [y]_{\theta}\) implies \(z \leq y\) for some \(z \in [x]_{\theta}\),
(ii) \(z \leq y\) implies \([z]_{\theta} \triangleleft [y]_{\theta}\)
and the corollary holds. \(\square\)

Corollary 3. Let \((P, \leq)\) be a poset, \(\theta \in \text{Eqv } P\) such that the equivalence blocks are convex. Let us order \(P/\theta\) by \(\triangleleft \cap \bowtie\). Then the canonical mapping \(\psi: P \to P/\theta, x \mapsto [x]_{\theta}\) is a 4-homomorphism.

Proof. It follows immediately from Corollary 1 and Corollary 2. \(\square\)

Theorem 3. Let \((P, \leq)\) be a poset. Then the following conditions are equivalent:

(1) a) \((P, \leq)\) is an antichain or
   b) there exists an element \(a \in P\) such that \((P, \leq) = X \oplus \{a\}\) where \(X \neq \emptyset\) is an antichain or
   c) \((P, \leq)\) is at least a three element chain,
(2) \(\text{End}(P, \leq) \subseteq 2\cdot\text{End}(P, \leq)\),
(3) \(\text{End}(P, \leq) = 2\cdot\text{End}(P, \leq)\).

Proof. Conditions (2) and (3) are equivalent due to [4] Lemma 3 (this also follows from Proposition 4). It is enough to demonstrate the equivalence of (1) and (2). It has been recently proved in [4] that if \(P\) has at least a three-element chain it has to be a chain, i.e. (1) c) holds. Thus we can study only the cases where \((P, \leq)\) is of length one, i.e. it contains two-element chains only.

(1) \(\Rightarrow\) (2): This follows immediately from Proposition 4.

(2) \(\Rightarrow\) (1): Suppose that any isotone mapping is a 2-homomorphism, i.e. condition (2) b) from Proposition 4 is satisfied. This is clear if \((P, \leq)\) is an antichain or \((P, \leq)\) is a two-element chain. Suppose \((P, \leq)\) contains at least one two-element chain \(b < a\) and incomparable elements. Then for any pair of incomparable elements \(x, y \in P\) we can construct an isotone mapping \(f\) such that \(f(x) > f(y), f(z) = a\) for any \(z \in X^{xy}\), \(f(z) = b\) for any \(z \in X_{xy}\) \((X^{xy}, X_{xy}\) were defined in the proof of Proposition 1) and \(f(z) = a\) otherwise. The mapping \(f\) has to be a 2-homomorphism, i.e there exists an element \(z > y\) such that \(f(z) = f(x)\). If \(x \parallel z\) then we can again construct a similar mapping but for elements \(x\) and \(z\). This leads to the existence of a three-element chain and consequently \((P, \leq)\) is a chain. Thus \(x \leq z\), which means that \(P\) is up directed and must be of the form \(X \oplus \{a\}\) for \(a \in P, X\) an antichain. \(\square\)

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Theorem 4. Let $(P, \leq)$ be a poset. Then the following conditions are equivalent:

1. a) $(P, \leq)$ is an antichain or
   b) there exists an element $a \in P$ such that $(P, \leq) = \{a\} \oplus X$ where $X \neq \emptyset$ is an antichain or
   c) $(P, \leq)$ is at least a three element chain,
2. $\text{End}(P, \leq) \subseteq 3\text{-End}(P, \leq)$,
3. $\text{End}(P, \leq) = 3\text{-End}(P, \leq)$.

Proof. Dually to the proof of the previous Theorem 3.

Theorem 5. Let $(P, \leq)$ be a poset. Then the following conditions are equivalent:

1. a) $(P, \leq)$ is an antichain or
   b) $(P, \leq)$ is at least a three element chain,
2. $\text{End}(P, \leq) \subseteq 4\text{-End}(P, \leq)$,
3. $\text{End}(P, \leq) = 4\text{-End}(P, \leq)$.

Proof. It follows from Theorem 3 and Theorem 4.

References


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