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## LASKERIAN LATTICES

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*Abstract.* In this paper we investigate prime divisors,  $B_w$ -primes and  $zs$ -primes in  $C$ -lattices. Using them some new characterizations are given for compactly packed lattices. Next, we study Noetherian lattices and Laskerian lattices and characterize Laskerian lattices in terms of compactly packed lattices.

*Keywords:* primary element, compactly packed lattice, Laskerian lattice

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By a  $C$ -lattice we mean a (not necessarily modular) complete multiplicative lattice, with a least element 0 and a compact greatest element 1 (a multiplicative identity), which is generated under joins by a multiplicatively closed subset  $C$  of compact elements. Throughout this paper  $L$  denotes a  $C$ -lattice and  $L_*$  denotes the set of all compact elements of  $L$ . For any prime element  $p$  of  $L$ ,  $L_p$  denotes the localization at  $F = \{x \in C \mid x \not\leq p\}$ . For details on  $C$ -lattices and their localization theory, the reader is referred to [10]. We note that in a  $C$ -lattice  $a = b$  if and only if  $a_m = b_m$  for all maximal elements  $m$  of  $L$ .

In this paper we study prime divisors,  $B_w$ -primes and  $zs$ -primes. Next we characterize compactly packed lattices. Also we establish some equivalent conditions for a  $C$ -lattice in which every prime element is locally compact to be a Noetherian lattice. Using these results we show that if  $L$  is generated by  $M$ -principal elements, then  $L$  is a Noetherian lattice if and only if the maximal elements of  $L$  are compact and every compact element of  $L$  has a normal primary decomposition. Finally, we introduce Laskerian lattices and characterize them in terms of compactly packed lattices.

Recall that an element  $e$  of  $L$  is said to be principal if it satisfies the dual identities (i)  $a \wedge be = ((a : e) \wedge b)e$  and (ii)  $a \vee (b : e) = (ae \vee b) : e$ . Principal elements were introduced into multiplicative lattices by R. P. Dilworth [6]. Elements satisfying (i)

are called meet principal and elements satisfying (ii) are called join principal. Elements satisfying the weaker identity (i')  $a \wedge e = (a : e)e$  obtained from (i) by setting  $b = 1$  are called weak meet principal, and elements satisfying the weaker identity (ii')  $a \vee (0 : e) = ae : e$  obtained from (ii) by setting  $b = 0$  are called weak join principal. Elements satisfying both (i') and (ii') are called weak principal. An element  $a \in L$  is said to be strong join principal if  $a$  is compact and join principal. An element  $a \in L$  is said to be a radical element if  $a = \sqrt{a}$ . Following [1], a prime element  $p$  of  $L$  is said to satisfy the condition  $\oplus$ , if for any collection  $\{p_\alpha\}$  of prime elements of  $L$ ,  $p \not\leq p_\alpha$  for all  $\alpha$  implies that there exists  $x \in L_*$  such that  $x \leq p$  and  $x \not\leq p_\alpha$  for all  $\alpha$ . The lattice  $L$  is said to be a *compactly packed lattice* if every prime element satisfies the condition  $\oplus$ .  $L$  is said to be a *Noetherian lattice* if  $L$  satisfies the ascending chain condition (a.c.c.). It is well known that  $L$  is a Noetherian lattice if and only if every element is a compact element. An  $r$ -lattice is a modular multiplicative lattice that is compactly generated, principally generated and has a compact greatest element 1. An  $r$ -lattice satisfying the ascending chain condition is called a *Noether lattice*.

For general background and terminology, the reader is referred to [2], [4], [10].

An element  $b \in L$  is said to be prime to  $a$  ( $a, b \in L$ ) if  $bc \leq a$  implies  $c \leq a$ . For any  $a \in L$  ( $a < 1$ ), let  $H_a = \{x \in L_* \mid x \text{ is prime to } a\}$  and  $\mathfrak{S}_a = \{x \in L \mid a \leq x \text{ and } H_a \cap [0, x] = \emptyset\}$ . Obviously  $H_a \cap [0, a] = \emptyset$  ( $[0, a] = \{x \in L \mid 0 \leq x \leq a\}$ ) and  $H_a$  is a multiplicative closed subset of  $L_*$ . So by Zorn's lemma,  $\mathfrak{S}_a$  contains maximal elements and every maximal element is a prime element.

**Definition 1.** A prime element  $p$  containing  $a$  ( $a, p \in L$ ) is called a maximal prime divisor if  $p \in \mathfrak{S}_a$  and  $p$  is a maximal element of  $\mathfrak{S}_a$ .

**Definition 2.** A prime element  $p$  containing  $a$  ( $a, p \in L$ ) is called a prime divisor if  $p \in \mathfrak{S}_{(a_p)} = \{x \in L \mid a_p \leq x \text{ and } H_{(a_p)} \cap [0, x] = \emptyset\}$  and  $p$  is a maximal element of  $\mathfrak{S}_{(a_p)}$ .

It is well known that a prime element  $p$  containing  $a$  is a minimal prime over  $a$  if and only if for any compact element  $x \leq p$ , there exists a compact element  $y \not\leq p$  such that  $x^n y \leq a$  for a positive integer  $n$  ([1], Lemma 3.5). Using this result, it can be easily shown that if  $p$  is a minimal prime over  $a$ , then  $p$  is a prime divisor of  $a$  and such prime elements are called minimal prime divisors of  $a$ .

We now prove several useful lemmas.

**Lemma 1.** Let  $L$  satisfy the ascending chain condition (a.c.c.) for prime elements and suppose that each compact element has only finitely many minimal prime divisors. Then  $L$  is a compactly packed lattice.

**P r o o f.** By imitating the proof of Lemma 1 of [5], we can prove that for every prime element  $p$  of  $L$ , there exists  $x \in L_*$  such that  $p = \sqrt{x}$ . Now the result follows from the definition of a compactly packed lattice.  $\square$

**Lemma 2.** *If every prime element of  $L$  is locally compact, then  $L$  satisfies a.c.c. on prime elements.*

**P r o o f.** The proof of the lemma is similar to that of [5, Lemma 2].  $\square$

An element  $a \in L$  is said to have a primary decomposition, if there exist primary elements  $q_1, q_2, \dots, q_n$  in  $L$  such that  $a = q_1 \wedge \dots \wedge q_n$ . If  $q$  is a primary element of  $L$ , then  $\sqrt{q} = p$  is a prime element and it is called the prime associated with  $q$ . Note that if  $q_1$  and  $q_2$  are primary elements associated with the same prime, then  $q_1 \wedge q_2$  is also a primary element associated with  $p$ . An element  $a \in L$  is said to have a normal primary decomposition, if  $a = q_1 \wedge \dots \wedge q_n$  ( $q_i$ 's are primary elements with distinct radicals) and if no  $q_i$  contains the meet of the other primary elements. Note that if  $a$  has a primary decomposition, then this primary decomposition can be reduced to a normal primary decomposition.

**Lemma 3.** *Let  $a \in L$  have a normal primary decomposition  $a = q_1 \wedge \dots \wedge q_n$  and put  $p_i = \sqrt{q_i}$ . Then a compact element  $x$  of  $L$  is non prime to  $a$  if and only if  $x \leq p_i$  for some  $i$ .*

**P r o o f.** If  $x$  is non prime to  $a$ , then  $xy \leq a$  for a compact element  $y \not\leq a$ . So  $y \not\leq q_i$  for some  $i$ . Since  $xy \leq a \leq q_i$ ,  $y \not\leq q_i$  and  $q_i$  is primary, it follows that  $x \leq \sqrt{q_i} = p_i$ .

Conversely, assume that  $x \leq p_i$  for some  $i$ . Since  $\bigwedge_{i=1}^n q_i$  is a normal primary decomposition of  $a$ , it follows that  $a < \bigwedge_{j \neq i} q_j$ . Choose any compact element  $y \leq \bigwedge_{j \neq i} q_j$  such that  $y \not\leq a$ . As  $x \leq p_i = \sqrt{q_i}$ ,  $x^k \leq q_i$  for a positive integer  $k$  and so  $x^k y \leq a$ . Let  $i$  be the smallest integer such that  $x^i y \leq a$ . Then  $x(x^{i-1}y) \leq a$  and  $x^{i-1}y \not\leq a$  and hence  $x$  is non prime to  $a$ .  $\square$

**Lemma 4.** *Let  $a \in L$  have a normal primary decomposition  $a = q_1 \wedge \dots \wedge q_m$  and put  $p_i = \sqrt{q_i}$ . Let  $p$  be a prime element of  $a$ . Then  $a_p = \bigwedge \{q_i \mid p_i \leq p\}$ .*

**P r o o f.** The proof of the lemma follows from [10, Properties 0.7 and 0.8].  $\square$

**Lemma 5.** *Let  $a \in L$  have a normal primary decomposition  $a = q_1 \wedge \dots \wedge q_m$  and put  $p_i = \sqrt{q_i}$ . If  $p$  is a prime element containing  $a$ , then  $p = p_i$  for some  $i$  if and only if  $p$  is a prime divisor of  $a$ .*

**Proof.** Suppose  $p = p_k$  for some  $k$  ( $1 \leq k \leq m$ ). Then by Lemma 4,  $a_p = \bigwedge \{q_i \mid p_i \leq p_k\}$ . As  $\bigwedge_{i=1}^m q_i$  is a normal primary decomposition of  $a$ , it follows that  $\bigwedge \{q_i \mid p_i \leq p_k\}$  is a normal primary decomposition of  $a_p$ . By Lemma 3,  $p \in \mathfrak{S}_{(a_p)}$  and it is not hard to show that  $p$  is a maximal element of  $\mathfrak{S}_{(a_p)}$ . Therefore  $p$  is a prime divisor of  $a$ .

Conversely, assume that  $p$  is a prime divisor of  $a$ . Since  $a \leq p$ , it follows that  $p_i \leq p$  for some  $i$ . Note that  $a_p = \bigwedge \{q_i \mid p_i \leq p\}$  is a normal primary decomposition of  $a_p$ . By Lemma 3, each  $p_i$  ( $p_i \leq p$ ) is an element of  $\mathfrak{S}_{(a_p)}$ . Since  $p \in \mathfrak{S}_{(a_p)}$  for any compact element  $x \leq p$ ,  $x$  is non prime to  $a_p$  and so by Lemma 3,  $x \leq p_i$  ( $p_i \leq p$ ) for some  $i$ . This shows that  $p = p_i$  for some  $i$ .  $\square$

**Definition 3.** A prime element  $p$  containing  $a$  is called a  $B_w$ -prime of  $a$  if  $p$  is a minimal prime divisor of  $(a : x)$  for some  $x \in L_*$ .

**Definition 4.** A prime element  $p$  containing  $a$  ( $a, p \in L$ ) is said to be a  $zs$ -prime of  $a$  if  $p = \sqrt{(a : x)}$  for some  $x \in L_*$ .

**Remark 1.** Clearly if  $p$  is a  $zs$ -prime of  $a$ , then  $p$  is a  $B_w$ -prime of  $a$  and it is not hard to show that every  $B_w$ -prime of  $a$  is a prime divisor of  $a$ . Also it should be mentioned that if  $R$  is a commutative ring with identity and  $L(R)$  is the lattice of all ideals of  $R$ , then a prime ideal  $P$  containing an ideal  $I$  of  $R$  is a  $B_w$ -prime ( $zs$ -prime) of  $I$  if and only if  $P$  is a  $B_w$ -prime ( $zs$ -prime) of  $I$  in the sense of [8].

**Theorem 1.** Let  $a \in L$  have a normal primary decomposition  $a = q_1 \wedge \dots \wedge q_m$  and put  $p_i = \sqrt{q_i}$ . Suppose  $p$  is a prime element containing  $a$ . Then the following statements are equivalent:

- (i)  $p = p_i$  for some  $i$  ( $1 \leq i \leq m$ ).
- (ii)  $p$  is a  $zs$ -prime of  $a$ .
- (iii)  $p$  is a  $B_w$ -prime of  $a$ .
- (iv)  $p$  is a prime divisor of  $a$ .

**Proof.** (i) $\Rightarrow$ (ii). Suppose (i) holds. Since  $\bigwedge_{i=1}^m q_i$  is a normal primary decomposition of  $a$ , it follows that  $\bigwedge_{j \neq i} \sqrt{q_j} \not\leq \sqrt{q_i}$ , so there exists  $x \in L_*$  such that  $x \not\leq \sqrt{q_i}$  and  $x \leq \bigwedge_{j \neq i} \sqrt{q_j}$ . Therefore  $x^k \leq \bigwedge_{j \neq i} q_j$  for a positive integer  $k$ . Consequently  $p_i = \sqrt{(a : x^k)}$ . Hence  $p$  is a  $zs$ -prime of  $a$  and (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follows from Remark 1 while (iv) $\Rightarrow$ (i) follows from Lemma 5. This completes the proof of the theorem.  $\square$

**Lemma 6.** Let  $p \leq q$  be prime elements of  $L$  and let  $a$  be an element of  $L$ . Then the following statements hold.

- (i)  $p$  is a minimal prime over  $a$  if and only if  $p_q$  is a minimal prime over  $a_q$  in  $L_q$ .
- (ii)  $p$  is a  $B_w$ -prime of  $a$  in  $L$  if and only if  $p_q$  is a  $B_w$ -prime of  $a_q$  in  $L_q$ .
- (iii) If  $p$  is the unique  $B_w$ -prime of  $a$ , then  $a$  is  $p$ -primary.
- (iv) If  $x \in L_*$ ,  $x_p = p_p$  and  $x_q$  is a  $p_q$ -primary element of  $L_q$ , then  $\bar{x}_q = p_q$ .
- (v) If  $\{\sqrt{(a : x)} \mid x \in L_*\}$  satisfies a.c.c., then every  $B_w$ -prime of  $a$  is also a  $zs$ -prime of  $a$ .
- (vi) Let  $a \in L$ . If  $a$  has only finitely many  $B_w$ -primes, then  $\{zs\text{-primes of } a\} = \{B_w\text{-primes of } a\} = \{\text{prime divisors of } a\}$ .

**Proof.** (i) and (iv) follow from [10, Properties 0.5, 0.7 and 0.8]. The proof of (ii) is a direct consequence of (i) and the proofs of (iii), (v) and (vi) are similar to those of [8, Lemma 1.1, Lemma 3.2 and Proposition 3.5]. □

**Theorem 2.** *The following statements on  $L$  are equivalent:*

- (i)  $L$  satisfies a.c.c. on radical elements.
- (ii) For every  $a \in L$ , there exists  $x \in L_*$  such that  $\sqrt{a} = \sqrt{x}$ .
- (iii)  $L$  is a compactly packed lattice.
- (iv) Every  $a \in L$  has only finitely many minimal prime divisors and  $L$  satisfies a.c.c. on prime elements.
- (v) Every compact element has only finitely many minimal prime divisors and  $L$  satisfies a.c.c. on prime elements.

**Proof.** (i) $\Rightarrow$ (ii). Suppose (i) holds and let  $a \in L$ . Then  $\{\sqrt{x} \mid x \in L_* \text{ and } x \leq a\}$  has a maximal element, say  $\sqrt{y}$ . Obviously  $\sqrt{a} = \sqrt{y}$ . (ii) $\Rightarrow$ (iii) follows from [1, Theorems 6.1, 6.2 and 6.5]. We show that (iii) $\Rightarrow$ (iv). Suppose (iii) holds. Note that by [1, Theorems 6.1, 6.2 and 6.5], if  $p$  is a prime element, then  $p = \sqrt{a}$  for some  $a \in L_*$ . Again by Zorn's lemma, for every  $a \in L$ ,  $\sqrt{a} = \sqrt{x}$  for some  $x \in L_*$ . Therefore by [1, Theorem 6.1], every element has only finitely many minimal prime divisors. Obviously,  $L$  has a.c.c. on prime elements. (iv) $\Rightarrow$ (v) is obvious. (v) $\Rightarrow$ (i) follows from Lemma 1 and the fact that if every prime element is the radical of some compact element, then every radical element is the radical of some compact element. □

**Remark.** If  $R$  is a commutative ring with identity, then  $L(R)$ , the lattice of all ideals of  $R$ , is a compactly packed lattice if and only if  $R$  has a Noetherian spectrum (in the sense of [11]).

**Theorem 3.** *Suppose every prime element of  $L$  is locally compact. If  $L$  satisfies any one of the following conditions:*

- (i) every compact element of  $L$  has a normal primary decomposition;
- (ii) every compact element of  $L$  has only finitely many  $B_w$ -primes;

- (iii) every compact element of  $L$  has only finitely many prime divisors;
- (iv) each  $x \in L_*$  has only finitely many minimal prime divisors and  $\sqrt{x}$  is compact, then every prime element is compact.

**Proof.** Note that (i) $\Rightarrow$ (ii) follows from Theorem 1. If  $L$  satisfies (iv), then by Lemma 1, every prime element is compact. Now by Remark 1 and Lemma 6 (vi), it suffices to show that if  $L$  satisfies the condition (iii), then every prime element is compact. Suppose every compact element has only finitely many prime divisors. Let  $p$  be a prime element of  $L$ . By Lemma 1 and Lemma 2,  $p = \sqrt{x}$  for some  $x \in L_*$ . By hypothesis  $p = p_p = a_p$  for some  $a \in L_*$ . Note that  $p = \sqrt{x \vee a}$  and  $(x \vee a)_p = p_p$ . Let  $x_1 = x \vee a$  and let  $p, p_1, \dots, p_n$  be the prime divisors of  $x_1$ . Without loss of generality assume that  $p < p_i$  for  $i = 1, 2, \dots, n$ . Again by hypothesis, there exist  $\gamma_i \in L_*$  ( $i = 1, 2, \dots, n$ ) such that  $(p)_{p_i} = (\gamma_i)_{p_i}$  for  $i = 1, 2, \dots, n$ . Let  $x_2 = x_1 \vee \gamma_1 \vee \gamma_2 \vee \dots \vee \gamma_n$ . Then  $p = \sqrt{x_2}$  and  $(x_2)_{p_i} = (p)_{p_i}$  for  $i = 1, 2, \dots, n$ . We show that for  $1 \leq i \leq n$ ,  $p_i$  is not a prime divisor of  $x_2$ . Choose any  $y_i \in L_*$  such that  $y_i \leq p_i$  and  $y_i \not\leq p$ . Then each  $y_i$  is prime to  $p$  and each  $y_i$  is prime to  $(p)_{p_i} = (x_2)_{p_i}$ . This shows that  $H_{(x_2)_{p_i}} \cap [0 p_i] \neq \emptyset$ . Consequently, no  $p_i$  is a prime divisor of  $x_2$ . Suppose that  $q$  ( $q \neq p$ ) is any prime which contains  $x_2$  and suppose that  $p_i \not\leq q$  for any  $i$ . Since  $x_1 \leq x_2 \leq q$ , we have  $p < q$ . Again since  $p = \sqrt{x_1}$ , it follows that  $p$  is the unique minimal prime divisor of  $x_1$ , so  $p_q$  is the unique minimal prime divisor of  $(x_1)_q$  (by Lemma 6 (i)) in  $L_q$ . So  $p_q$  is a  $B_w$ -prime of  $(x_1)_q$ . Again if  $q'_q$  is a  $B_w$ -prime of  $(x_1)_q$  in  $L_q$  ( $q'$  is a prime element and  $q' \leq q$ ), then by Lemma 6 (ii),  $q'$  is a  $B_w$ -prime of  $x_1$  in  $L$ , so  $q'$  is a prime divisor of  $x_1$  and hence  $q' = p$  (since  $p_i \not\leq q$  for any  $i$ ). Therefore  $p_q$  is the unique  $B_w$ -prime of  $(x_1)_q$ , so by Lemma 6 (iii),  $x_{1_q}$  is  $p_q$ -primary and again by Lemma 6 (iv),  $(x_2)_q = p_q$ . As  $p < q$ ,  $q$  is not a prime divisor of  $x_2$ . Therefore if  $p, p'_1, p'_2, \dots, p'_m$  are the prime divisors of  $x_2$ , then for  $1 \leq i \leq m$ ,  $p'_i > p_j$  for some  $j$ ,  $1 \leq j \leq n$ . As  $L$  satisfies a.c.c. for prime elements, a finite number of repetitions of the above procedure yields a compact element  $x_3 \in L_*$  such that  $(x_3)_p = p_p$  and  $p$  is the unique prime divisor of  $x_3$ . So by Lemma 6 (iii),  $x_3$  is  $p$ -primary and hence  $x_3 = p$ . Consequently,  $p$  is compact. Thus every prime element is compact and the proof is complete.  $\square$

**Definition 5.** An element  $x \in L$  is said to be a modular element (or an  $m$ -element) if for any  $a, b \in L$ ,  $a \geq b$  implies  $a \wedge (x \vee b) = (a \wedge x) \vee b$ .

**Definition 6.** An element  $x \in L$  is said to be an  $M$ -element if  $x^n$  is an  $m$ -element for every positive integer  $n$ .

Note that  $L$  is a modular lattice if and only if every element is an  $m$ -element. Also it is not hard to show that  $L$  is a modular lattice if and only if every compact element is a modular element.

A weak meet principal (meet principal, principal) element  $x$  is said to be  $m$ -weak meet principal ( $m$ -meet principal,  $m$ -principal) if  $x$  is a modular element.

**Theorem 4.** *Suppose  $L$  is generated by compact  $m$ -weak meet principal elements. If every prime element is compact, then every element is compact.*

*Proof.* Suppose every prime element is compact and let  $\Psi = \{x \in L \mid x \text{ is not compact}\}$  be a non empty set. By Zorn's lemma,  $\Psi$  has a maximal element, say  $p$ . By hypothesis  $p$  is not prime, so there exist compact  $m$ -weak meet principal elements  $x, y \in L$  such that  $xy \leq p$ ,  $x \not\leq p$  and  $y \not\leq p$ . So  $p < p \vee x$ ,  $p < p : x$  and hence  $p \vee x$  and  $p : x$  are compact elements. Since  $p \vee x$  is compact, it follows that  $p \vee x = p_1 \vee x$  for a compact element  $p_1 \leq p$ . Observe that  $p \leq p_1 \vee x$ , so  $p = p \wedge (x \vee p_1) = p_1 \vee (p \wedge x)$  (as  $x$  is an  $m$ -element)  $= p_1 \vee ((p : x)x)$  (as  $x$  is weak meet principal) and therefore  $p$  is compact as  $p_1, x, (p : x) \in L_*$ . This contradiction shows that every element is compact.  $\square$

An element  $a \in L$  is said to be meet irreducible if  $a = b \wedge c$  implies either  $a = b$  or  $a = c$ . It is well known that if  $L$  satisfies a.c.c, then every element is a finite meet of meet irreducible elements.

**Lemma 7.** *Suppose  $L$  is generated by  $M$ -meet principal elements and let  $a \in L$  be a meet irreducible element. If  $\{(a : x) \mid x \in L\}$  satisfies a.c.c., then  $a$  is primary.*

*Proof.* The proof of the lemma is similar to that of [6, Theorem 3.1].  $\square$

**Theorem 5.** *Suppose  $L$  is generated by compact  $M$ -meet principal elements. If  $L$  is a Noetherian lattice, then  $L$  satisfies the conditions (i)–(iv) of Theorem 3. Conversely, if every prime element is locally compact and  $L$  satisfies the conditions of Theorem 3, then  $L$  is a Noetherian lattice.*

*Proof.* The proof of the theorem follows from Theorems 1, 3, 4 and Lemma 7.  $\square$

**Theorem 6.** *Let  $L$  be a quasi-local lattice generated by  $M$ -principal elements. Suppose the maximal element  $m$  is compact. Then the following statements are equivalent:*

- (i)  $L$  is a Noetherian lattice.
- (ii) Every compact element of  $L$  has a normal primary decomposition.
- (iii) For any two compact elements  $a$  and  $b$  of  $L$ , there exists an integer  $n$  such that  $(a \vee b^\ell) \wedge (a : b^\ell) = a$  for  $\ell \geq n$ .
- (iv)  $\bigwedge_{n=1}^{\infty} (m^n \vee a) = a$  for all compact elements  $a$  of  $L$ .
- (v) If  $b = a \vee mb$  and  $a \in L_*$ , then  $a = b$ .



**Proof.** (i)  $\Rightarrow$  (ii) follows from Lemma 7 and by imitating the proof of [3, Theorem 4.1], it can be easily shown that (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv). (i)  $\Rightarrow$  (iv) and (i)  $\Rightarrow$  (v) follow from [1, Corollary 1.4 and Theorem 1.1]. Now we prove that (iv)  $\Rightarrow$  (i) and (v)  $\Rightarrow$  (i). Suppose  $L$  is not Noetherian. By the proof of Theorem 4, there exists a prime element  $p$  such that  $p$  is maximal among the set of all non compact elements. Clearly  $p \neq m$ . Choose any  $M$ -principal element  $x \leq m$  such that  $x \not\leq p$ . Then  $x^n \not\leq p$  for all  $n \in \mathbb{Z}^+$ . Let  $n \geq 1$ . Then  $p < p \vee x^n$ , so  $p \vee x^n$  is compact and hence  $p \vee x^n = p_1 \vee x^n$  for a compact element  $p_1 \leq p$ . If  $a \leq p$  is any principal element, then  $a \vee p_1 = (a \vee p_1) \wedge (p_1 \vee x^n) = p_1 \vee ((a \vee p_1) \wedge x^n) = p_1 \vee (((a \vee p_1) : x^n)x^n)$  as  $x^n$  is an  $m$ -principal element. Since  $(a \vee p_1) : x^n \leq p$ ,  $x^n \not\leq p$  and  $p$  is prime, it follows that  $(a \vee p_1) : x^n \leq p$ . So  $a \leq p_1 \vee x^n p \leq p_1 \vee m^n p$  and therefore  $p = p_1 \vee m^n p$  and this is true for all  $n \in \mathbb{Z}^+$ . Consequently, either (iv) or (v) implies that  $p = p_1$ , a contradiction. This shows that  $L$  is a Noetherian lattice and the proof is complete.  $\square$

**Theorem 7.** *Suppose  $L$  is generated by  $M$ -principal elements. Then the following statements are equivalent:*

- (i)  $L$  is a Noetherian lattice.
- (ii) *The maximal elements of  $L$  are compact and every compact element of  $L$  has a normal primary decomposition.*

**Proof.** (i) $\Rightarrow$ (ii) follows from Lemma 7. Suppose (ii) holds. By hypothesis and Lemma 4, every compact element of  $L_m$  ( $m$  is a maximal element) has a normal primary decomposition, so by Theorem 6,  $L$  is a locally Noetherian lattice. Again by Theorem 5,  $L$  is a Noetherian lattice. This completes the proof of the theorem.  $\square$

**Corollary 1.** *Suppose  $L$  is an  $r$ -lattice in which every compact element is a finite meet of primary elements. If  $p$  is a compact prime element minimal over a principal element, then  $\text{rank } p \leq 1$ .*

**Proof.** The proof of the theorem follows from Theorem 6 and [6, Theorem 6.4].  $\square$

**Corollary 2.** *Suppose  $L$  is an  $r$ -lattice in which every compact element has a normal primary decomposition. If the prime elements are comparable and the maximal element is compact, then  $\dim L \leq 1$ .*

**Definition 7.**  $L$  is said to be a Laskerian lattice if every element is a finite meet of primary elements.

Noether lattices [6] are Laskerian lattices. If  $R$  is a Laskerian ring (see [7], [9]), then the lattice  $I(R)$  of all ideals of  $R$  is a Laskerian  $r$ -lattice. If  $L$  is an idempotent (i.e.,  $a^2 = a$  for all  $a \in L$ ) distributive lattice satisfying the ascending chain condition, then  $L$  is a Laskerian lattice ([1, Theorem 6.1]).

We need the following lemma.

**Lemma 8.** *Let  $L$  be a Laskerian lattice generated by strong join principal elements. If  $p$  is a prime element containing  $a$ , then  $a_p = \bigwedge\{q \mid a \leq q \text{ and } q \text{ is } p\text{-primary}\}$ .*

**Proof.** Let  $b = \bigwedge\{q \mid a \leq q \text{ and } q \text{ is } p\text{-primary}\}$ . Clearly  $a_p \leq b$ . Suppose  $a_p < b$ . Then there exists a strong join principal element  $x \leq b$  such that  $x \not\leq a_p$ . As  $L$  is Laskerian, it follows that  $a \vee xp$  has a normal primary decomposition, say  $a \vee xp = q_1 \wedge \dots \wedge q_n$ , and  $p_i = \sqrt{q_i}$  ( $q_i^s$  are  $p_i$ -primary). By Lemma 4,  $(a \vee xp) = \bigwedge\{q_i \mid p_i \leq p\}$ . By Theorem 1.4 of [2],  $x_p \not\leq (a \vee xp)_p$ , so  $x_p \not\leq q_i$  ( $p_i \leq p$ ) for some  $i$  and hence  $x \not\leq q_i$ . Again since  $xp \leq q_i$ , it follows that  $p \leq p_i$ , so  $q_i$  is  $p$ -primary. This contradiction shows that  $b = a_p$  and the proof is complete.  $\square$

**Theorem 8.** *Suppose  $L$  is generated by strong join principal elements. If  $L$  is Laskerian, then  $L$  is a compactly packed lattice.*

**Proof.** Suppose  $L$  is Laskerian. Then clearly  $L$  contains only finitely many minimal primes. So by Theorem 2, it is enough if we show that  $L$  satisfies a.c.c. on prime elements. Let  $p_0 < p_1 < p'_1 < p_2 < p'_2 < p_3 < p'_3 < \dots$  be a chain of prime elements. By Theorem 1, every element has only finitely many  $zs$ -primes. We show that there is an element  $a \in L$  such that  $a$  has infinitely many  $zs$ -primes. First we show by induction that for  $n \in \mathbb{Z}^+$  there exist  $q_1, \dots, q_n \in L$ ,  $a_n, b_n$  and strong join principal elements  $x_1, x_2, \dots, x_n$  in  $L$  such that

- (i)  $q_i$  is  $p_i$ -primary for  $i$  and  $a_n = q_1 \wedge \dots \wedge q_n$ ,
- (ii) for  $1 \leq i \leq n$  we have  $x_i \leq \bigwedge_{j \neq i} q_j$  and  $x_i \not\leq q_i$ ,
- (iii)  $x_1 \vee x_2 \vee \dots \vee x_n \leq b_n$ ,  $a_n \not\leq b_n$  and every  $zs$ -prime of  $b_n$  is contained in  $p'_n$ .

Suppose  $n = 1$ . Then take  $q_1 = p_1$ . Since  $p_1 < p'_1$  and  $p_1$  is nonminimal, it follows that  $0_{p'_1} < p_1$ , so by Lemma 8,  $p_1 \not\leq q'_1$  for some  $p'_1$ -primary element  $q'_1$ . Choose any strong join principal element  $x_1 \leq q'_1$  such that  $x_1 \not\leq q_1$ . Let  $b_1 = (x_1)_{p'_1}$ . Clearly  $q_1 = p_1 \not\leq b_1$ . As  $L$  is Laskerian,  $b_1$  has a normal primary decomposition, say  $b_1 = h_1 \wedge \dots \wedge h_n$ ,  $r_i = \sqrt{h_i}$  ( $h_i^s$  are  $r_i$ -primary elements). Since  $b_1 = (b_1)_{p'_1}$ , by Lemma 4 we have  $r_i \leq p'_1$  for  $i = 1, 2, \dots, n$ . Again by Theorem 1,  $r_i^s$  ( $i = 1, 2, \dots, n$ ) are the only  $zs$ -primes of  $b_1$ . Therefore each  $zs$ -prime of  $b_1$  is contained in  $p'_1$ . Thus the conditions (i), (ii) and (iii) are satisfied.

Suppose we have  $q_1, \dots, q_n, a_n, b_n$  and strong join principal elements satisfying (i)–(iii). Since  $a_n \not\leq b_n$ , there exists a strong join principal element  $y_{n+1}$  such that  $y_{n+1} \leq a_n$  and  $y_{n+1} \not\leq b_n$ . Since  $p'_n < p_{n+1}$  and  $b_n < p_{n+1}$ , by Lemma 8 there exists a  $p_{n+1}$ -primary element  $q_{n+1}$  such that  $b_n \leq q_{n+1}$  and  $y_{n+1} \not\leq q_{n+1}$ . Define  $a_{n+1} = a_n \wedge q_{n+1}$ . We show that  $a_{n+1} \not\leq b_n$ . As  $L$  is Laskerian,  $b_n$  has a normal primary decomposition, say  $b_n = h_1 \wedge \dots \wedge h_k$ ,  $r_i = \sqrt{h_i}$  ( $1 \leq i \leq k$ ) where  $r_i^{s}$  are  $zs$ -primes of  $b_n$ . By (iii), each  $r_i \leq p'_n$  and therefore  $q_{n+1} \not\leq r_i$  for  $i = 1, 2, \dots, k$ . If  $a_{n+1} \leq b_n$ , then  $a_n \wedge q_{n+1} \leq b_n \leq h_i$  for  $i = 1, 2, \dots, k$ . Since  $q_{n+1} \not\leq r_i$  for  $i = 1, 2, \dots, k$  and  $h_i^{s}$  are  $r_i$ -primary elements, it follows that  $a_n \leq \bigwedge_{i=1}^k h_i = b_n$ , a contradiction. This shows that  $a_{n+1} \not\leq b_n$ . Note that  $b_n = (b_n)_{p'_{n+1}}$  since each  $r_i \leq p'_n < p'_{n+1}$  and by Lemma 8,  $b_n = \bigwedge_{\lambda \in \Delta} \{c_\lambda \mid b_n \leq c_\lambda \text{ and } c_\lambda \text{ is a } p'_{n+1}\text{-primary element}\}$ . Since  $a_{n+1} \not\leq b_n$ , it follows that  $a_{n+1} \not\leq c_\lambda$  for some  $\lambda \in \Delta$ . Consequently,  $a_{n+1} \not\leq (b_n \vee y_{n+1} c_\lambda)_{p'_{n+1}}$  as  $(b_n \vee y_{n+1} c_\lambda)_{p'_{n+1}} \leq c_\lambda$ . As  $p_{n+1} < p'_{n+1}$ , we have  $c_\lambda \not\leq p_{n+1}$ , so there exists a strong join principal element  $r \leq c_\lambda$  such that  $r \not\leq p_{n+1}$ . Define  $x_{n+1} = y_{n+1} r$  and  $b_{n+1} = (b_n \vee x_{n+1})_{p'_{n+1}}$ . Observe that  $x_{n+1}$  is a strong join principal element. Since  $y_{n+1} \not\leq q_{n+1}$  and  $r \not\leq p_{n+1}$ , it follows that  $x_{n+1} \not\leq q_{n+1}$ . Thus (i) and (ii) are satisfied for  $q_1, q_2, \dots, q_{n+1}$  and  $x_1, x_2, \dots, x_{n+1}$ . Moreover, (iii) is satisfied for  $b_{n+1}$ , by the choice of  $x_{n+1}$  and  $b_{n+1}$ . Therefore, we conclude by induction that there exist infinite sequences  $\{q_i\}_{i=1}^\infty, \{a_n\}_{n=1}^\infty, \{x_i\}_{i=1}^\infty$  and  $\{b_n\}_{n=1}^\infty$  such that the conditions (i), (ii) and (iii) are satisfied for all  $n$ . Now let us define  $a = \bigwedge_{n=1}^\infty a_n$ . Since  $x_n \leq \bigwedge_{j \neq n} q_j$  and  $x_n \not\leq q_n$ , it follows that  $(a : x_n) = (a_n : x_n) = (q_n : x_n)$  is  $p_n$ -primary, so  $p_n$  is a  $zs$ -prime of  $a$  and this is true for all  $n$ . Therefore  $a$  has infinitely many  $zs$ -primes. This contradiction shows that  $L$  satisfies a.c.c. on prime elements and the proof is complete.  $\square$

**Theorem 9.** *Suppose  $L$  is generated by  $M$ -principal elements. Then  $L$  is Laskerian if and only if  $L$  satisfies the following conditions:*

- (i)  $L$  is a compactly packed lattice.
- (ii) For each  $a \in L$ , there is a prime element  $p$  minimal over  $a$  and an  $M$ -principal element  $x \not\leq p$  such that  $(a : x)$  is  $p$ -primary.

*Proof.* Suppose  $L$  is a Laskerian lattice. By Theorem 2 and Theorem 8,  $L$  is a compactly packed lattice. Again by imitating the proof of Theorem 1 ((i)  $\Rightarrow$  (ii)), it can be easily shown that  $L$  satisfies the condition (ii).

Conversely, assume that  $L$  satisfies (i) and (ii). Let  $a \in L$  and let  $p$  be a minimal prime over  $a$  such that  $(a : x)$  is  $p$ -primary for some  $M$ -principal element  $x \not\leq p$ . Then  $(a : x) \wedge (a \vee x) = ((a : x) \wedge a) \vee ((a : x) \wedge x)$  ( $x$  is a modular element)  $= a \vee ((a : x) \wedge x) = a \vee ((a : x^2) \wedge x)$  (as  $x$  is weak meet principal). Note that  $(a : x^2) \leq (a : x)$

since  $x \not\leq p$  and  $(a : x)$  is  $p$ -primary. Therefore  $(a : x^2)x \leq (a : x)x \leq a$  and hence  $a = (a : x) \wedge (a \vee x)$ . Put  $a_1 = (a \vee x)$  and  $q_1 = (a : x)$ . Then  $a = q_1 \wedge a_1$  where  $\sqrt{a} < \sqrt{a_1}$  since  $x \leq \sqrt{a_1}$ . Similarly  $a_1 = q_2 \wedge a_2$  where  $q_2 = (a_1 : y)$  is  $p_1$ -primary,  $p_1$  is a minimal prime over  $a_1$ ,  $y \not\leq p_1$  is an  $m$ -principal element and  $\sqrt{a_1} < \sqrt{a_2}$ . By continuing this process, we get sequences of elements  $q_1, q_2, \dots, q_n$  and  $a_1, a_2, \dots, a_n$  such that  $a_{i-1} = q_i \wedge a_i$ ,  $q_i$  is primary for  $i = 1, 2, \dots, n$  ( $a_0 = a$ ) and  $\sqrt{a_0} < \sqrt{a_1} < \sqrt{a_2} < \dots < \sqrt{a_n}$ . Since  $L$  satisfies a.c.c. on radical elements, it follows that  $\sqrt{a_0} < \sqrt{a_1} < \sqrt{a_2} < \dots < \sqrt{a_n}$  is a finite chain with  $\sqrt{a_n}$  as a maximal element. Then  $a_n = 1$  and hence  $a = q_1 \wedge \dots \wedge q_n$ . This shows that  $L$  is Laskerian and the proof is complete.  $\square$

**Lemma 9.** *Suppose  $L$  is a compactly packed lattice. Let  $a \in L$  and let  $p$  be a minimal prime over  $a$ . Then  $p = \sqrt{(a : x)}$  for a compact element  $x \not\leq p$ .*

*Proof.* Let  $a \in L$  and let  $p$  be a minimal prime over  $a$ . Since  $L$  satisfies a.c.c. on radical elements, it follows that  $\Gamma = \{\sqrt{(a : x)} \mid x \in L_*, x \not\leq p \text{ and } p \text{ is a minimal prime over } \sqrt{(a : x)}\}$  has a maximal element, say  $\sqrt{(a : x)}$ . Suppose  $p_0$  is any other minimal prime over  $\sqrt{(a : x)}$ . Choose any element  $y \leq p_0$  such that  $y \not\leq p$ . Since  $xy \not\leq p$  and  $\sqrt{(a : x)} \leq \sqrt{(a : xy)}$ , it follows by the maximality that  $\sqrt{(a : x)} = \sqrt{(a : x^n)} = \sqrt{(a : xy)} \leq \sqrt{(a : xy^m)}$  for all  $m, n \in \mathbb{Z}^+$ . Since  $y \leq p_0$  and  $p_0$  is any other minimal prime over  $\sqrt{(a : x)}$ , it follows that there exists  $z \not\leq p_0$  such that  $y^n z \leq \sqrt{(a : x)}$ , so  $z \leq \sqrt{(a : x)} \leq p_0$ , a contradiction. This shows that  $p$  is the unique minimal prime over  $\sqrt{(a : x)}$  and hence  $p = \sqrt{(a : x)}$ .  $\square$

**Lemma 10.** *Suppose  $L$  is a compactly packed lattice in which every primary element with non maximal prime radical is compact. Then for each  $a \in L$ , there is a prime element  $p$  minimal over  $a$  and a compact element  $x \not\leq p$  such that  $(a : x)$  is  $p$ -primary.*

*Proof.* Let  $a \in L$  and let  $p \in L$  be a minimal prime over  $a$ . By Lemma 9,  $p = \sqrt{(a : x)}$  for some  $x \not\leq p$ . If  $p$  is maximal, then  $(a : x)$  is  $p$ -primary. Suppose  $p$  is non maximal. Note that  $q = a_p$  is  $p$ -primary. Again by hypothesis,  $xq \leq a$  for a compact element  $x \not\leq p$ . As  $q$  is  $p$ -primary, it follows that  $q = (a : x)$ .  $\square$

**Theorem 10.** *Suppose  $L$  is a compactly packed lattice generated by  $M$ -principal elements. If every primary element with non maximal prime radical is compact, then  $L$  is a Laskerian lattice.*

*Proof.* Suppose every primary element with non maximal prime radical is compact. Let  $a \in L$  and let  $p$  be a minimal prime over  $a$ . Then by Lemma 9 and Lemma 10,  $(a : x)$  is  $p$ -primary for a compact element  $x \not\leq p$ . As  $L$  is generated

by  $M$ -principal elements, it follows that there is an  $M$ -principal element  $x_1 \leq x$  such that  $x_1 \not\leq p$ . Since  $(a : x) \leq (a : x_1)$  and  $(a : x)$  is  $p$ -primary, it follows that  $(a : x) = (a : x_1)$ . Now the result follows from Theorem 9.  $\square$

Let  $r^* = \bigwedge \{m \in L \mid m \text{ is a maximal element of } L\}$ . The element  $r^*$  is called the *Jacobson radical of  $L$* . The following theorem gives some of the properties of Laskerian lattices.

**Theorem 11.** *Suppose  $L$  is a Laskerian lattice generated by compact join principal elements. Let  $a, c \in L$  and let  $b = \bigwedge_{n=1}^{\infty} (a^n \vee c)$ . Then the following statements hold.*

- (i) *If  $a$  is compact and  $a \leq r^*$ , then  $b = c$ .*
- (ii)  *$0 = \bigwedge \{q \in L \mid q \text{ is } m\text{-primary for a maximal element } m \text{ of } L\}$ .*
- (iii) *If both  $a$  and  $b$  are compact elements of  $L$ , then  $b = \vee \{r \in L \mid r \text{ is join principal, } a \vee (c : r) = 1\}$ .*
- (iv) *If both  $a$  ( $a < 1$ ) and  $b' = \bigwedge_{n=1}^{\infty} a^n$  are compact elements of  $L$ , then  $\bigwedge_{n=1}^{\infty} a^n = 0$  if and only if there is no zero divisor  $r$  ( $\neq 0$ ) such that  $a \vee r = 1$ .*

**Proof.** (i) Suppose  $a$  is compact and let  $a \leq r^*$ . Let  $m$  be any maximal element of  $L$ . Note that for any  $m$ -primary element  $q$  of  $L$   $b \leq q$  if and only if  $c \leq q$ . Therefore by Lemma 8,  $b_m = c_m$  and hence  $b = c$ .

(ii) Let  $x$  be any compact join principal element such that  $x \leq \bigwedge \{q \in L \mid q \text{ is } m\text{-primary for a maximal element } m \text{ of } L\}$ . Then by Lemma 8,  $x_m = 0_m$  for every maximal element  $m$  of  $L$ . Consequently,  $x = 0$ .

(iii) By imitating the proof of [1, Theorem 1.2], we can get the result and (iv) directly follows from (iii). This completes the proof of the theorem.  $\square$

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