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CONTINUOUS EXTENDIBILITY OF SOLUTIONS  
OF THE NEUMANN PROBLEM  
FOR THE LAPLACE EQUATION

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*Abstract.* A necessary and sufficient condition for the continuous extendibility of a solution of the Neumann problem for the Laplace equation is given.

*Keywords:* Neumann problem, Laplace equation, continuous extendibility

*MSC 2000:* 35B65, 35J05, 35J25, 31B10

1. MAXIMUM AND REGULARITY PRINCIPLE

For  $x, y \in \mathbb{R}^m$ ,  $m \geq 2$ , denote

$$h_x(y) = \begin{cases} (m-2)^{-1} A^{-1} |x-y|^{2-m} & \text{for } x \neq y, \ m > 2, \\ A^{-1} \log |x-y|^{-1} & \text{for } x \neq y, \ m = 2, \\ \infty & \text{for } x = y, \end{cases}$$

where  $A$  is the area of the unit sphere in  $\mathbb{R}^m$ . For a finite real Borel measure  $\nu$  denote

$$\mathcal{U}\nu(x) = \int_{\mathbb{R}^m} h_x(y) \, d\nu(y),$$

the single layer potential corresponding to  $\nu$  for each  $x$  for which this integral has sense.

Let  $H$  be a bounded open set in  $\mathbb{R}^m$ ,  $g$  an arbitrary extended real-valued function defined on  $\partial H$ . We denote by  $\overline{U}_g^H$  the set of all hyperharmonic functions  $u$  on  $H$

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which are lower bounded on  $H$  and such that for any  $y \in \partial H$

$$\liminf_{x \rightarrow y} u(x) \geq g(y).$$

We put  $\underline{U}_g^H = -\overline{U}_{(-g)}^H$  and denote by  $\overline{H}_g^H$  (or  $\underline{H}_g^H$ ) the greatest lower (or least upper) bound of  $\overline{U}_g^H$  (or  $\underline{U}_g^H$ , respectively). (Compare [3], [14].)

A function  $g$  on  $\partial H$  is said to be *resolutive* (relative to  $H$ ), if  $\overline{H}_g^H = \underline{H}_g^H$  and  $|\overline{H}_g^H(x)| < \infty$  for any  $x \in H$ . We set  $H_g^H = \overline{H}_g^H$ , the generalized solution of the Dirichlet problem for the Laplace equation with the boundary condition  $g$ , provided  $g$  is resolutive. If  $g \in \mathcal{C}(\partial H)$  and  $u$  is a classical solution of the Dirichlet problem for the Laplace equation with the boundary condition  $g$  then  $g$  is resolutive and  $H_g^H = u$ . Any bounded Baire function on  $\partial H$  is resolutive ([3], Theorem 6 and the text on p. 94).

A set  $Z \subset \mathbb{R}^m$  is called a *polar set* if there is an open set  $U \supset Z$  and a function  $u$  superharmonic on  $U$  such that  $u = +\infty$  on  $Z$ .

For a compact  $K$  in  $\mathbb{R}^m$  denote by  $\mathcal{C}'(K)$  the Banach space of all finite real Borel measures with support in  $K$  with the total variation as a norm.

**Lemma 1.** *Let  $H \subset \mathbb{R}^m$  be a bounded regular set,  $\nu \in \mathcal{C}'(\partial H)$ . Then  $\mathcal{U}\nu$  is the generalized solution of the Dirichlet problem with the boundary condition  $\mathcal{U}\nu/\partial H$ . Let now  $f$  be a Borel measurable function on  $\partial H$  such that  $\{x \in \partial H; \mathcal{U}\nu(x) \neq f(x)\}$  is polar. Put  $f = \mathcal{U}\nu$  on  $H$ . If  $f$  is continuous and finite on  $\partial H$  then it is continuous on the closure of  $H$ . If  $f$  is bounded on  $\partial H$  then it is bounded on  $H$  and*

$$\inf_{x \in \partial H} f(x) \leq \inf_{x \in H} f(x) \leq \sup_{x \in H} f(x) \leq \sup_{x \in \partial H} f(x).$$

*Proof.* Suppose first that  $\nu$  is nonnegative. For  $z \in H$  denote by  $\mu_z$  the harmonic measure corresponding to  $H$  and  $z$ . If  $y \in \partial H$ ,  $z \in H$  then

$$\int_{\partial H} h_y(x) d\mu_z(x) = h_y(z)$$

by [19], pp. 299, 264. Using Fubini's theorem we get

$$\int \mathcal{U}\nu d\mu_z = \int_{\partial H} \int_{\partial H} h_y(x) d\mu_z(x) d\nu(y) = \int_{\partial H} h_y(z) d\nu(y) = \mathcal{U}\nu(z).$$

Thus  $\mathcal{U}\nu$  is a solution of the Dirichlet problem with the boundary condition  $\mathcal{U}\nu/\partial H$ .

Let  $\nu$  be general. Let  $\nu = \nu^+ - \nu^-$  be the Jordan decomposition of  $\nu$ . Then  $\mathcal{U}\nu = \mathcal{U}\nu^+ - \mathcal{U}\nu^-$  is a solution of the Dirichlet problem with the boundary condition

$\mathcal{U}\nu/\partial H$ . Since harmonic measures do not charge polar sets ([2], Lemma 4.4.5),  $\mathcal{U}\nu$  is a solution of the Dirichlet problem with the boundary condition  $f$ . If  $f$  is continuous on  $\partial H$  then  $f$  is continuous on the closure of  $H$ . If  $f$  is bounded on  $\partial H$  then  $f$  is bounded on  $H$  and since harmonic measures are probability measures we get the above inequalities.  $\square$

## 2. NEUMANN PROBLEM

Suppose that  $G \subset \mathbb{R}^m$  ( $m \geq 2$ ) is an open set with a non-void compact boundary  $\partial G$ . If  $h$  is a harmonic function on  $G$  such that

$$\int_H |\nabla h| d\mathcal{H}_m < \infty$$

for all bounded open subsets  $H$  of  $G$  we define the weak normal derivative  $N^G h$  of  $h$  as the distribution

$$\langle N^G h, \varphi \rangle = \int_G \nabla \varphi \cdot \nabla h d\mathcal{H}_m$$

for  $\varphi \in \mathcal{D}$  (= the space of all compactly supported infinitely differentiable functions in  $\mathbb{R}^m$ ). Here  $\mathcal{H}_k$  is the  $k$ -dimensional Hausdorff measure normalized so that  $\mathcal{H}_k$  is the Lebesgue measure in  $\mathbb{R}^k$ . We formulate the *Neumann problem for the Laplace equation with a boundary condition*  $\mu \in \mathcal{C}'(\partial G)$  in the sense of distributions as follows: determine a harmonic function  $h$  on  $G$  for which  $N^G h = \mu$ . It is usual to look for a solution  $h$  in the form of the single layer potential  $\mathcal{U}\nu$ , where  $\nu \in \mathcal{C}'(\partial G)$ . The single layer potential  $\mathcal{U}\nu$  is a harmonic function in  $G$  for which the weak normal derivative  $N^G \mathcal{U}\nu$  has sense. The operator  $N^G \mathcal{U}: \nu \mapsto N^G \mathcal{U}\nu$  is a bounded linear operator on  $\mathcal{C}'(\partial G)$  if and only if  $V^G < \infty$ , where

$$V^G = \sup_{x \in \partial G} v^G(x),$$

$$v^G(x) = \sup \left\{ \int_G \nabla \varphi \cdot \nabla h_x d\mathcal{H}_m; \varphi \in \mathcal{D}, |\varphi| \leq 1, \text{spt } \varphi \subset \mathbb{R}^m - \{x\} \right\}$$

(see [15]). There are more geometrical characterizations of  $v^G(x)$  in [15] which ensure  $V^G < \infty$  for  $G$  convex or for  $G$  with  $\partial G \subset \{\bigcup L_i; i = 1, \dots, k\}$ , where  $L_i$  are  $(m-1)$ -dimensional Ljapunov surfaces (i.e. of class  $C^{1+\alpha}$ ).

If  $z \in \mathbb{R}^m$  and  $\theta$  is a unit vector such that the symmetric difference of  $G$  and the half-space  $\{x \in \mathbb{R}^m; (x - z) \cdot \theta < 0\}$  has  $m$ -dimensional density zero at  $z$  then  $n^G(z) = \theta$  is termed *the exterior normal* of  $G$  at  $z$  in Federer's sense. If there is no exterior normal of  $G$  at  $z$  in this sense, we denote by  $n^G(z)$  the zero vector in  $\mathbb{R}^m$ .

The set  $\{y \in \mathbb{R}^m; |n^G(y)| > 0\}$  is called the reduced boundary of  $G$  and will be denoted by  $\widehat{\partial}G$ .

If  $G$  has a finite perimeter (which is fulfilled if  $V^G < \infty$ ) then  $\mathcal{H}_{m-1}(\widehat{\partial}G) < \infty$  and

$$v^G(x) = \int_{\widehat{\partial}G} |n^G(y) \cdot \nabla h_x(y)| d\mathcal{H}_{m-1}(y)$$

for each  $x \in \mathbb{R}^m$ . Throughout the paper we will assume that  $V^G < \infty$ . Then

$$N^G \mathcal{U}\nu(M) = \int_M d_G(x) d\nu(x) + \int_{\partial G} \int_{(\partial G \cap M)} n^G(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y) d\nu(x)$$

for each  $\nu \in \mathcal{C}'(\partial G)$  and a Borel set  $M$  (see [15]).

If  $L$  is a bounded linear operator on the Banach space  $X$  we denote by  $\|L\|_{\text{ess}}$  the essential norm of  $L$ , i.e. the distance of  $L$  from the space of all compact linear operators on  $X$ . The essential spectral radius of  $L$  is defined by

$$r_{\text{ess}}L = \lim_{n \rightarrow \infty} (\|L^n\|_{\text{ess}})^{1/n}.$$

If  $X$  is a complex Banach space then

$$\begin{aligned} r_{\text{ess}}L &= \sup\{|\lambda|; \lambda I - L \text{ is not a Fredholm operator}\} \\ &= \sup\{|\lambda|; \lambda I - L \text{ is not a Fredholm operator with index } 0\} \end{aligned}$$

(see [12], Satz 51.8, Theorem 51.1).

**Theorem** ([22]). *Let  $r_{\text{ess}}(N^G \mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$ , where  $I$  is the identity operator,  $\mu \in \mathcal{C}'(\partial G)$ . Then there is a harmonic function  $u$  on  $G$ , which is a solution of the Neumann problem*

$$N^G u = \mu,$$

*if and only if  $\mu \in \mathcal{C}'_0(\partial G)$  (= the space of such  $\nu \in \mathcal{C}'(\partial G)$  that  $\nu(\partial H) = 0$  for each bounded component  $H$  of  $\text{cl } G$ ). Moreover, if  $\mu \in \mathcal{C}'_0(\partial G)$  then there is a solution of this problem in the form of the single layer potential  $\mathcal{U}\nu$ , where  $\nu \in \mathcal{C}'(\partial G)$ .*

**Remark 1.** It is well-known that the condition  $r_{\text{ess}}(N^G \mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$  is fulfilled for sets with a smooth boundary (of class  $C^{1+\alpha}$ ) (see [16]) and for convex sets (see [23]). R. S. Angell, R. E. Kleinman, J. Král and W. L. Wendland proved that rectangular domains (i.e. formed from rectangular parallelepipeds) in  $\mathbb{R}^3$  have this property (see [1], [18]). A. Rathsfeld showed in [28], [29] that polyhedral cones in  $\mathbb{R}^3$  have this property. (By a polyhedral cone in  $\mathbb{R}^3$  we mean an open set  $\Omega$  whose boundary is locally a hypersurface (i.e. every point of  $\partial\Omega$  has a neighbourhood in  $\partial\Omega$

which is homeomorphic to  $\mathbb{R}^2$ ) and  $\partial\Omega$  is formed by a finite number of plane angles. By a polyhedral open set with bounded boundary in  $\mathbb{R}^3$  we mean an open set  $\Omega$  whose boundary is locally a hypersurface and  $\partial\Omega$  is formed by a finite number of polygons). N. V. Grachev and V. G. Maz'ya obtained independently an analogous result for polyhedral open sets with bounded boundary in  $\mathbb{R}^3$  (see [10]). (Let us note that there is a polyhedral set in  $\mathbb{R}^3$  which has not a locally Lipschitz boundary.) In [20] it was shown that the condition  $r_{\text{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$  has a local character. As a conclusion we obtain that this condition is fulfilled for  $G \subset \mathbb{R}^3$  such that for each  $x \in \partial G$  there are  $r(x) > 0$ , a domain  $D_x$  which is polyhedral or smooth or convex or a complement of a convex domain, and a diffeomorphism  $\psi_x: \mathcal{U}(x; r(x)) \rightarrow \mathbb{R}^3$  of class  $C^{1+\alpha}$ , where  $\alpha > 0$ , such that  $\psi_x(G \cap \mathcal{U}(x; r(x))) = D_x \cap \psi_x(\mathcal{U}(x; r(x)))$ . V. G. Maz'ya and N. V. Grachev proved this condition for several types of sets with "piecewise-smooth" boundary in the general Euclidean space (see [8], [9], [11]).

In the rest of the paper we will suppose that  $r_{\text{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$ . Denote by  $\mathcal{H}$  the restriction of  $\mathcal{H}_{m-1}$  onto  $\partial G$ . Then  $\mathcal{H}(\mathbb{R}^m) < \infty$  (see [22], Lemma 2).

**Notation.**  $\mathcal{C}'_c(\partial G)$  will stand for the subspace of those  $\mu \in \mathcal{C}'(\partial G)$  for which there exists a continuous function  $\mathcal{U}_c\mu$  on  $\mathbb{R}^m$  coinciding with  $\mathcal{U}\mu$  on  $\mathbb{R}^m \setminus \partial G$ . It was shown in [27] that if  $\nu \in \mathcal{C}'(\partial G)$  and the restriction of  $\mathcal{U}\nu$  onto  $\partial G$  is finite and continuous then  $\mathcal{U}\nu$  is finite and continuous in  $\mathbb{R}^m$  and  $\nu \in \mathcal{C}'_c(\partial G)$ . If  $\mu = f\mathcal{H}$ , where  $f \in L_p(\mathcal{H})$ ,  $p > m - 1$  then  $\mu \in \mathcal{C}'_c(\partial G)$  (see [21], Remark 6).

**Notation.** Denote by  $\mathcal{I}$  the set of all isolated points of  $\partial G$ ,  $\tilde{G} = G \cup \mathcal{I}$ . Then the set  $\mathcal{I}$  is finite by [22], Lemma 1. Therefore  $V^{\tilde{G}} = V^G < \infty$ ,  $N^{\tilde{G}}\mathcal{U}\nu = N^G\mathcal{U}\nu$  for  $\nu \in \mathcal{C}'(\partial\tilde{G})$  and  $r_{\text{ess}}(N^{\tilde{G}}\mathcal{U} - \frac{1}{2}I) = r_{\text{ess}}(N^G\mathcal{U} - \frac{1}{2}I)$ , because  $\mathcal{C}'(\partial\tilde{G})$  is a subspace of  $\mathcal{C}'(\partial G)$  of a finite codimension.

Denote by  $\Omega_R(x)$  the open ball with a centre  $x$  and a radius  $R$ .

**Lemma 2.** *Let  $R > 0$  be such that  $\partial G \subset \Omega_R(0)$ . Then  $\tilde{G} \cap \Omega_R(0)$ ,  $\Omega_R(0) \setminus \text{cl } \tilde{G}$  are regular sets.*

**Proof.** Since the density of  $\tilde{G} \cap \Omega_R(0)$  and the density of  $\Omega_R(0) \setminus \text{cl } \tilde{G}$  are positive at each point of the boundary of  $\tilde{G}$  by [22], Lemma 1, the sets  $\tilde{G} \cap \Omega_R(0)$ ,  $\Omega_R(0) \setminus \text{cl } \tilde{G}$  are regular (see [4], Chap. VII, §§ 2, 6, 19, Theorem 5.11, Theorem 5.10). □

**Lemma 3.**  *$G$  has finitely many components  $G_1, \dots, G_n$  and  $\text{cl } G_j \cap \text{cl } G_k = \emptyset$  for  $j \neq k$ .*

Proof. If we define for  $f \in L_\infty(\mathcal{H})$ ,  $x \in \partial G$

$$W^G f(x) = d_G(x)f(x) + \int_{\partial G} f(y)n^G(y) \cdot \nabla h_x(y) \, d\mathcal{H}(y),$$

then  $W^G$  is a bounded linear operator on  $L_\infty(\mathcal{H})$ , because  $V^G < \infty$ . If we define for  $f \in L_1(\mathcal{H})$ ,  $x \in \partial G$

$$(N^G \mathcal{U}\mathcal{H})f(x) = d_G(x)f(x) - \int_{\partial G} f(y)n^G(x) \cdot \nabla h_x(y) \, d\mathcal{H}(y),$$

then  $(N^G \mathcal{U}\mathcal{H})$  is a bounded linear operator on  $L_1(\mathcal{H})$  (compare [17], Theorem 1). Since  $N^G \mathcal{U}(f\mathcal{H}) = [(N^G \mathcal{U}\mathcal{H})f]\mathcal{H}$  for each  $f \in L_1(\mathcal{H})$  and  $\{f\mathcal{H}; f \in L_1(\mathcal{H})\}$  is a closed subspace of  $\mathcal{C}'(\partial G)$ , we have  $r_{\text{ess}}((N^G \mathcal{U}\mathcal{H}) - \frac{1}{2}I) \leq r_{\text{ess}}(N^G \mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$  by [20], Lemma 1.3 or [13], Lemma 15.

Fix a bounded component  $H$  of  $G$ . Since  $\mathcal{H}_{m-1}(\partial H) \leq \mathcal{H}_{m-1}(\partial G) < \infty$ , the perimeter of  $H$  is finite. Since  $\mathcal{H}_{m-1}(\partial G \setminus \hat{\partial}G) = 0$  by [22], Lemma 2 and  $n^H(y) = n^G(y)$  for each  $y \in \hat{\partial}H \cap \hat{\partial}G$ , we have

$$v^H(y) = \int_{\hat{\partial}H} |n^H(x) \cdot \nabla h_y(x)| \, d\mathcal{H}_{m-1} = \int_{\hat{\partial}H \cap \hat{\partial}G} |n^G(x) \cdot \nabla h_y(x)| \, d\mathcal{H}_{m-1} \leq v^G(y)$$

for each  $y \in \partial H$ . Therefore  $V^H < \infty$  and  $d_H(y)$  has a good meaning for each  $y \in \partial H$  by [15], Lemma 2.9. Put

$$u_H(y) = \begin{cases} 1 & \text{for } y \in \hat{\partial}H \cap \hat{\partial}G, \\ 0 & \text{for } y \in \partial G \setminus \hat{\partial}H \cap \hat{\partial}G. \end{cases}$$

Since  $n^G(y) = n^H(y)$  for  $y \in \hat{\partial}H \cap \hat{\partial}G$  and  $\mathcal{H}_{m-1}(\partial G \setminus \hat{\partial}G) = 0$ , [15], Proposition 2.8 and Lemma 2.15 yield

$$\begin{aligned} W^G u_H(x) &= \frac{1}{2}u_H(x) + \int_{\hat{\partial}H \cap \hat{\partial}G} n^G(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y) \\ &= \frac{1}{2}u_H(x) + \int_{\hat{\partial}H} n^H(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y) \\ &= \frac{1}{2}u_H(x) - d_H(x). \end{aligned}$$

If  $x \in \hat{\partial}H \cap \hat{\partial}G$  then  $d_H(x) = \frac{1}{2}$  and thus  $W^G u_H(x) = 0$ . If  $d_H(x) = 0$  then  $u_H(x) = 0$ , therefore  $W^G u_H(x) = 0$ . Since  $\mathcal{H}_{m-1}(\{x \in \hat{\partial}G \setminus \hat{\partial}H; d_H(x) > 0\}) \leq \mathcal{H}_{m-1}(\{x \in \partial H \setminus \hat{\partial}H; 0 < d_H(x) \leq d_G(x) \leq \frac{1}{2}\}) = 0$  by [33], Lemma 5.9.5 and  $\mathcal{H}_{m-1}(\partial G \setminus \hat{\partial}G) = 0$  by [22], Lemma 2,  $W^G u_H(x) = 0$  for  $\mathcal{H}$ -a.a.  $x \in \partial G$ . Since the

perimeter of a nonempty open bounded set is positive (see [33], Theorem 5.4.3) and  $\mathcal{H}_{m-1}(\hat{\partial}H)$  is equal to the perimeter of  $H$  (see [33], Theorem 5.81, Theorem 5.6.5) and  $\mathcal{H}_{m-1}(\partial H \setminus \hat{\partial}G) = 0$ , the function  $u_H$  is positive on the set  $\hat{\partial}H \cap \hat{\partial}G$  of positive  $\mathcal{H}$  measure.

If  $H_1, H_2$  are different bounded components of  $G$  then  $\hat{\partial}G \cap \hat{\partial}H_1 \cap \hat{\partial}H_2 = \emptyset$ , because  $H_1, H_2$  are disjoint. The set  $\{u_H; H \text{ is a bounded component of } G\}$  contains linearly independent elements of the kernel of  $W^G$ . Since  $N^G(\mathcal{U}\mathcal{H})$  is a Fredholm operator and  $W^G$  is an adjoint operator of  $N^G\mathcal{U}\mathcal{H}$ , the operator  $W^G$  is a Fredholm operator as well (see [12], Satz 51.8, Theorem 27.1). Since the dimension of the kernel of  $W^G$  is greater than or equal to the number of bounded components of  $G$  and  $W^G$  is a Fredholm operator,  $G$  has only finitely many components. (Since  $\partial G$  is bounded, there is at most one unbounded component of  $G$ .) According to [22], Note 5 the codimension of the range of  $N^G(\mathcal{U}\mathcal{H})$  is equal to the number of bounded components of the closure of  $G$ . Since the dimension of the kernel of  $W^G$  is equal to the codimension of the range of  $N^G(\mathcal{U}\mathcal{H})$ , because  $W^G$  is the adjoint operator of  $N^G(\mathcal{U}\mathcal{H})$  (see [12], Theorem 27.1), the number of bounded components of  $G$  is smaller than or equal to the number of bounded components of the closure of  $G$ . Therefore the number of bounded components of  $G$  is equal to the number of bounded components of the closure of  $G$  and the closures of any two different components of  $G$  are disjoint.  $\square$

**Theorem 1.** *Let  $\nu, \mu \in C'(\partial G)$ ,  $N^G\mathcal{U}\nu = \mu$ . Then the following assertions are equivalent:*

- a)  $\nu \in C'_c(\partial G)$ .
- b)  $\mu \in C'_c(\partial G)$ .
- c) *There is a finite continuous extension of  $\mathcal{U}\nu$  from  $G$  onto the closure of  $G$ .*
- d) *There is a finite continuous extension of  $\mathcal{U}\mu$  from  $G$  onto the closure of  $G$ .*

*If  $\partial G = \partial(\mathbb{R}^m \setminus G)$  then these assertions are equivalent to the following ones*

- e) *There are a polar set  $K$  and a finite continuous function  $f$  on  $\partial G$  such that  $\mathcal{U}\nu = f$  on  $\partial G \setminus K$ .*
- f) *There are a polar set  $K$  and a finite continuous function  $f$  on  $\partial G$  such that  $\mathcal{U}\mu = f$  on  $\partial G \setminus K$ .*

**Proof.** Denote  $\mu_{\mathcal{I}} = \mu/\mathcal{I}$ ,  $\mu_{\tilde{G}} = \mu/(\partial G \setminus \mathcal{I})$ ,  $\nu_{\mathcal{I}} = \nu/\mathcal{I}$ ,  $\nu_{\tilde{G}} = \nu/(\partial G \setminus \mathcal{I})$ . Since the density of  $G$  at each point of  $\partial G \setminus \mathcal{I}$  is positive by [22], Lemma 1, we have  $\mu_{\mathcal{I}} = \nu_{\mathcal{I}}$  by [15], Observation on p. 25. If  $\mu_{\mathcal{I}} = \nu_{\mathcal{I}} \neq 0$  then none of the assertions a)–d) is true. So we can suppose that  $\mu_{\mathcal{I}} = \nu_{\mathcal{I}} = 0$  and coming to  $\tilde{G}$  we can suppose that  $\partial G = \partial(\mathbb{R}^m \setminus G)$ .

- a)  $\Rightarrow$  b).  $\mu \in C'_c(\partial G)$  by [15], Plemelj's exchange theorem 2.23.



b)  $\Rightarrow$  a). This assertion is true for  $m > 2$  by [21], Lemma 13. Let us suppose that  $m = 2$ . If we denote for  $f \in \mathcal{C}(\partial G)$  (= the space of all bounded continuous functions on  $\partial G$  equipped with the maximum norm) and  $x \in \partial G$

$$W^G f(x) = d_G(x)f(x) + \int_{\partial G} f(y)n^G(y) \cdot \nabla h_x(y) d\mathcal{H}(y),$$

then  $W^G$  is a bounded linear operator on  $\mathcal{C}(\partial G)$  and  $N^G\mathcal{U}$  is the dual operator of  $W^G$  (see [15], Proposition 2.5, Proposition 2.20). We shall show that  $\mathcal{U}_c\mu \in W^G(\mathcal{C}(\partial G))$ . Since  $\text{Ker}(I - N^G\mathcal{U}) \cap (I - N^G\mathcal{U})(\mathcal{C}'(\partial G)) = \{0\}$  by [22], Proposition 2 and  $\dim \text{Ker}(I - N^G\mathcal{U}) = \text{codim}(I - N^G\mathcal{U})(\mathcal{C}'(\partial G))$  because  $(I - N^G\mathcal{U})$  is a Fredholm operator with index 0, the space  $\mathcal{C}'(\partial G)$  is the direct sum of  $(I - N^G\mathcal{U})(\mathcal{C}'(\partial G))$  and  $\text{Ker}(I - N^G\mathcal{U})$ . Therefore  $\mu = \mu_1 + \mu_2$ , where  $\mu_1 \in \text{Ker}(I - N^G\mathcal{U})$  and  $\mu_2 \in (I - N^G\mathcal{U})(\mathcal{C}'(\partial G))$ . Since  $\mu_1 \in \mathcal{C}'_c(\partial G)$  by [22], Lemma 4, we get  $\mathcal{U}_c\mu_1 = \mathcal{U}_c(N^G\mathcal{U}\mu_1) = W^G(\mathcal{U}_c\mu_1)$  by [15], Plemelj's exchange theorem 2.23. Since  $\mu, \mu_1 \in \mathcal{C}'_c(\partial G)$ , we have  $\mu_2 \in \mathcal{C}'_c(\partial G)$ , too. Put  $\tilde{\nu} = \nu - \mu_1$ . Then  $N^G\mathcal{U}\tilde{\nu} = \mu_2$ . Put  $C = \mathbb{R}^m \setminus \text{cl}G$ . Since  $N^C\mathcal{U} = I - N^G\mathcal{U}$ , we have  $\mu_2 \in N^C\mathcal{U}(\mathcal{C}'(\partial G))$ . If  $G$  is bounded then we choose  $\varphi \in \mathcal{D}$  such that  $\varphi = 1$  in a neighbourhood of  $\text{cl}G$ . We get

$$\mu_2(\partial G) = \langle N^G\mathcal{U}\tilde{\nu}, \varphi \rangle = \int_G \nabla \varphi \cdot \nabla \mathcal{U}\tilde{\nu} = 0.$$

If  $G$  is unbounded we get  $\mu_2(\partial G) = 0$  in a similar way using the facts that  $\mu_2 \in N^C\mathcal{U}(\mathcal{C}'(\partial G))$  and  $C$  is bounded. Let  $\sigma \in \mathcal{C}'(\partial G)$ ,  $N^G\mathcal{U}\sigma = 0$ . Then  $\sigma \in \mathcal{C}'_c(\partial G)$  by [22], Lemma 4. Since the density of  $G$  is positive at each point of the boundary by [22], Lemma 1, we have  $\mathcal{H}(\partial G) > 0$  by Isoperimetric lemma ([15], p. 50). Put  $\sigma_1 = \sigma(\partial G)[\mathcal{H}(\partial G)]^{-1}\mathcal{H}$ ,  $\sigma_2 = \sigma - \sigma_1$ . Then  $\mathcal{U}\sigma_1$  is finite and continuous on  $\mathbb{R}^m$  by [22], Lemma 2, [15], Corollary 2.17, Lemma 2.18. Therefore  $\sigma_2 \in \mathcal{C}'_c(\partial G)$ . Using [22], Lemma 7 we get

$$\int_{\partial G} \mathcal{U}_c\mu_2 d\sigma_2 = \int_G \nabla \mathcal{U}\mu_2 \cdot \nabla \mathcal{U}\sigma_2 d\mathcal{H}_m = \int_{\partial G} \mathcal{U}_c\sigma_2 d\mu_2.$$

If  $x \in \partial G$ ,  $|\mu_2|(x) < \infty$  then  $\mathcal{U}_c\mu_2(x) = \mathcal{U}\mu_2(x)$ , because  $\mathcal{U}\mu_2$  is finely continuous at  $x$  (see [19], Chapter V, § 3) and  $\mathbb{R}^m \setminus G$  is not a fine neighbourhood of  $x$ , because  $d_G(x) > 0$  (see [4], Chap. VII, §§ 2, 6, 19, Theorem 5.11). Thus  $\mathcal{U}_c\mu_2 = \mathcal{U}\mu_2$  outside the polar set  $\{x; |\mu_2|(x) = \infty\}$ . Since  $\sigma_1$  does not charge polar sets (see [19], Theorem 3.1, Theorem 2.1) using Fubini's theorem we get

$$\int_{\partial G} \mathcal{U}_c\mu_2 d\sigma_1 = \int_{\partial G} \mathcal{U}\mu_2 d\sigma_1 = \int_{\partial G} \mathcal{U}\sigma_1 d\mu_2 = \int_{\partial G} \mathcal{U}_c\sigma_1 d\mu_2.$$

Denote by  $H_1, \dots, H_p$  the components of  $G$ . Then there are  $c_1, \dots, c_p \in \mathbb{R}$  such that  $\mathcal{U}_c \sigma = c_j$  on  $H_j$  for  $j = 1, \dots, p$  by [22], Lemma 12. Therefore

$$\int_{\partial G} \mathcal{U}_c \mu_2 \, d\sigma = \int_{\partial G} \mathcal{U}_c \sigma \, d\mu_2 = \sum_{j=1}^p c_j \mu_2(\partial H_j).$$

If  $H_j$  is bounded, choose  $\varphi \in \mathcal{D}$  such that  $\varphi = 1$  on  $H_j$  and  $\varphi = 0$  on  $\text{cl } G \setminus H_j$ . Then

$$\mu_2(\partial H_j) = \langle \mu_2, \varphi \rangle = \langle N^G \mathcal{U} \tilde{\nu}, \varphi \rangle = \int_G \nabla \mathcal{U} \tilde{\nu} \cdot \nabla \varphi \, d\mathcal{H}_m = 0.$$

If  $H_j$  is unbounded then we get  $\mu_2(\partial H_j) = 0$  from the facts that  $\mu_2(\partial G) = 0$  and  $\mu_2(\partial H_i) = 0$  for each bounded  $H_i$ . Therefore

$$(1) \quad \int_{\partial G} \mathcal{U}_c \mu_2 \, d\sigma = 0.$$

Since  $N^G \mathcal{U}$  is Fredholm, (1) yields that  $\mathcal{U}_c \mu_2 \in W^G(\mathcal{C}(\partial G))$  by [32], Chapter VII, Theorem 3.1. Since  $\mathcal{U}_c \mu_1 \in W^G(\mathcal{C}(\partial G))$ ,  $\mathcal{U}_c \mu_2 \in W^G(\mathcal{C}(\partial G))$  we have  $\mathcal{U}_c \mu \in W^G(\mathcal{C}(\partial G))$ .

Put

$$\nu_0 = \mu + \sum_{j=0}^{\infty} (I - 2N^G \mathcal{U})^j (2I - N^G \mathcal{U}) \mu.$$

Then  $N^G \mathcal{U} \nu_0 = \mu$  by [22], Theorem 1. Put

$$\mu_j = (I - 2N^G \mathcal{U})^j (2I - N^G \mathcal{U}) \mu$$

for  $j$  a nonnegative integer. According to [15], Plemelj's exchange theorem 2.23 we have  $\mu_j \in C'_c(\partial G)$  and

$$\mathcal{U}_c \mu_j = (I - 2W^G)^j (2I - W^G) \mathcal{U}_c \mu \text{ on } \partial G.$$

If  $\lambda$  is an eigenvalue of  $W^G$ ,  $|\lambda - \frac{1}{2}| \geq \frac{1}{2}$  then  $\lambda$  is an eigenvalue of  $N^G \mathcal{U}$ , because  $\lambda I - N^G \mathcal{U}$ ,  $\lambda I - W^G$  are Fredholm operators with index 0 and the kernels of these operators have the same dimension (see [32], Chapter IX, Theorem 2.1, Theorem 1.3, Chapter VII, Theorem 3.5, Chapter V, Theorem 4.1); therefore  $\lambda \in \{0; 1\}$  by [22], Proposition 1. Since  $\text{Ker}(\lambda I - N^G \mathcal{U})^2 = \text{Ker}(\lambda I - N^G \mathcal{U})$  by [22], Proposition 2 we have  $\text{Ker}(\lambda I - W^G)^2 = \text{Ker}(\lambda I - W^G)$  by [32], Chapter V, Theorem 2.3, Chapter V, Theorem 4.1. Now [22], Proposition 3 yields that there are constants  $q \in (0; 1)$ ,  $M > 0$  such that

$$\|(I - 2W^G)^j (2I - W^G) g\|_{C(\partial G)} \leq M q^j \|g\|_{C(\partial G)}$$

for all  $g \in W^G(\mathcal{C}(\partial G))$ . Since  $\mathcal{U}_c \mu \in W^G(\mathcal{C}(\partial G))$  we have

$$\sum_{j=0}^{\infty} \|\mathcal{U}_c \mu_j\|_{\mathcal{C}(\partial G)} = \sum_{j=0}^{\infty} \|(I - 2W^G)^j (2I - W^G) \mathcal{U}_c \mu\|_{\mathcal{C}(\partial G)} < \infty.$$

Since

$$\|\mu\|_{\mathcal{C}'(\partial G)} + \sum_{j=0}^{\infty} \|\mu_j\|_{\mathcal{C}'(\partial G)} < \infty, \quad \|\mathcal{U}_c \mu\|_{\mathcal{C}(\partial G)} + \sum_{j=0}^{\infty} \|\mathcal{U}_c \mu_j\|_{\mathcal{C}(\partial G)} < \infty,$$

[15], Lemma 4.5 yields that  $\nu_0 \in \mathcal{C}'_c(\partial G)$ .

Since  $N^G \mathcal{U}(\nu - \nu_0) = 0$ , we have  $\nu - \nu_0 \in \mathcal{C}'_c(\partial G)$  by [22], Lemma 4 and thus  $\nu \in \mathcal{C}'_c(\partial G)$ .

c)  $\Rightarrow$  e). Let  $f$  denote a finite continuous extension of  $\mathcal{U}\nu$  from  $G$  onto the closure of  $G$ . Because  $\mathcal{U}\nu^+$ ,  $\mathcal{U}\nu^-$  are superharmonic functions they are continuous with respect to the fine topology (see [19], Chapter V, § 3). Denote  $K = \{x \in \partial G; \mathcal{U}|\nu|(x) = \infty\}$ . Then  $K$  is polar and  $\mathcal{U}\nu(x)$  is the fine limit of  $\mathcal{U}\nu$  for each  $x \in \partial G \setminus K$ . Thus  $f(x) = \mathcal{U}\nu(x)$  for each  $x \in \partial G \setminus K$ , because every fine neighbourhood of  $x$  intersects  $G$  by Lemma 2, [19], Theorem 5.11, Theorem 5.10.

e)  $\Rightarrow$  a). Define  $f = \mathcal{U}\nu$  on  $\mathbb{R}^m \setminus \partial G$ . Fix  $R > 0$  such that  $\partial G \subset \Omega_R(0)$ . Using Lemma 1 and Lemma 2 for  $G \cap \Omega_R(0)$  and  $M = \Omega_R(0) \setminus \text{cl} G$  we get

$$f(x) = \lim_{y \rightarrow x, y \in \mathbb{R}^m \setminus \partial G} \mathcal{U}\nu(y) \text{ for } x \in \partial G.$$

Therefore  $\nu \in \mathcal{C}'_c(\partial G)$ . □

**Lemma 4.** Let  $H \subset \mathbb{R}^m$  be a bounded open set,  $\mathcal{H}_{m-1}(\partial H) < \infty$ ,  $\mu \in \mathcal{C}'(\partial H)$ , let  $u$  be a solution of the Neumann problem  $N^H u = \mu$ , finite and continuous up to the boundary of  $H$ . Then for each  $x \in H$

$$u(x) = \mathcal{U}\mu(x) - \mathcal{D}u(x),$$

where

$$\mathcal{D}u(x) = \int_{\partial H} u(y) n^H(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y)$$

is the double layer potential corresponding to the density  $u$ .

*Proof.* Fix  $x \in H$ ,  $r > 0$  such that  $\text{cl} \Omega_r(x) \subset H$ . Put  $H(r) = H \setminus \Omega_r(x)$ . Choose  $\varphi \in \mathcal{D}$  such that  $\varphi = 1$  in the neighbourhood of  $\text{cl} H(r)$  and  $\varphi = 0$  in the neighbourhood of  $x$ . Green's formula yields

$$\begin{aligned} \mathcal{U}\mu(x) &= \langle N^H u, h_x \varphi \rangle \\ &= \int_{H(r)} \nabla h_x \cdot \nabla u d\mathcal{H}_m + \int_{\partial \Omega_r(x)} h_x(y) n^{\Omega_r(x)}(y) \cdot \nabla u(y) d\mathcal{H}_{m-1}(y). \end{aligned}$$

Since  $\mathcal{H}_{m-1}(\partial H) < \infty$  there is a positive konstant  $K$  such that for each positive integer  $k$  there are balls  $\Omega_{r_1}(x_1), \dots, \Omega_{r_j}(x_j)$  such that  $\partial H \subset (\Omega_{r_1}(x_1) \cup \dots \cup \Omega_{r_j}(x_j))$ ,  $r_1^{m-1} + \dots + r_j^{m-1} \leq K$ ,  $\max(r_1, \dots, r_j) \leq \frac{1}{k}$ ,  $\text{dist}(x_i, \partial H) \leq \frac{1}{k}$  for  $i = 1, \dots, j$ ; put  $H_k(r) = H(r) \setminus (\Omega_{r_1}(x_1) \cup \dots \cup \Omega_{r_j}(x_j))$ . Then  $\mathcal{H}_m(H(r) \setminus H_k(r)) \rightarrow 0$  as  $k \rightarrow \infty$ ,  $\mathcal{H}_{m-1}(H_k(r)) \leq L \equiv (K + r^{m-1})\mathcal{H}_{m-1}(\partial\Omega_1(0))$ .

Fix  $\varepsilon > 0$ . Since  $\text{cl } H$  is compact, there is a polynomial  $p$  such that  $|u - p| \leq \varepsilon$  on  $\text{cl } H$ . Using Green's formula we get

$$\begin{aligned}
& \left| \int_{H(r)} \nabla h_x(y) \cdot \nabla u(y) \, d\mathcal{H}_m(y) - \int_{\partial H(r)} u(y) n^{H(r)}(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y) \right| \\
&= \left| \lim_{k \rightarrow \infty} \int_{H_k(r)} \nabla h_x(y) \cdot \nabla u(y) \, d\mathcal{H}_m(y) \right. \\
&\quad \left. - \int_{\partial H(r)} u(y) n^{H(r)}(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y) \right| \\
&= \left| \lim_{k \rightarrow \infty} \int_{\partial H_k(r)} u n^{H_k(r)} \cdot \nabla h_x \, d\mathcal{H}_{m-1} - \int_{\partial H(r)} u n^{H(r)} \cdot \nabla h_x \, d\mathcal{H}_{m-1} \right| \\
&\leq \left| \lim_{k \rightarrow \infty} \int_{\partial H_k(r)} p(y) n^{H_k(r)}(y) \cdot h_x(y) \, d\mathcal{H}_{m-1}(y) \right. \\
&\quad \left. - \int_{\partial H(r)} p(y) n^{H(r)}(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y) \right| + \frac{\varepsilon 2L}{r^{m-1}A} \\
&= \left| \lim_{k \rightarrow \infty} \int_{H_k(r)} \nabla p \cdot \nabla h_x \mathcal{H}_m - \int_{H(r)} \nabla p \cdot \nabla h_x \mathcal{H}_m \right| \\
&\quad + \frac{\varepsilon 2L}{r^{m-1}A} = \frac{\varepsilon 2L}{r^{m-1}A}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int_{H(r)} \nabla h_x \cdot \nabla u \, d\mathcal{H}_m = \int_{\partial H(r)} u(y) n^{H(r)} \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y), \\
\mathcal{U}\mu(x) &= \int_{\partial H(r)} u n^{H(r)} \cdot \nabla h_x \, d\mathcal{H}_{m-1} + \int_{\partial\Omega_r(x)} h_x n^{\Omega_r(x)} \cdot \nabla u \, d\mathcal{H}_{m-1}.
\end{aligned}$$

If  $r \rightarrow 0$  we get

$$\mathcal{U}\mu(x) = \int_{\partial H} u(y) n^H(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y) + u(x).$$

□

**Lemma 5.** *Let  $\mu \in \mathcal{C}'(\partial G)$ , let  $u$  be a solution of the Neumann problem  $N^G u = \mu$ , finite and continuous up to the boundary of  $G$ . Then  $\mu \in \mathcal{C}'_c(\partial G)$ .*

**Proof.** Let  $G$  be bounded. Then  $u = \mathcal{U}\mu - \mathcal{D}u$ . Since  $u$  is continuous and finite on  $\partial G$ , the double layer potential  $\mathcal{D}u$  is continuously extendible to the closure of  $G$  (see [15], Chapter 2). Therefore  $\mathcal{U}\mu = \mathcal{D}u + u$  is continuously extendible to the closure of  $G$ . Hence  $\mu \in \mathcal{C}'_c(\partial G)$  by Theorem 1.

If  $G$  is unbounded, fix  $R > 0$  such that  $\partial G \subset \Omega_R(0)$ . Put  $H = G \cap \Omega_R(0)$ . Then  $V^H < \infty$ ,  $r_{\text{ess}}(N^H\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$ . If we put

$$\tilde{\mu}(M) = \int_{\partial\Omega_R(0) \cap M} \frac{x}{|x|} \cdot \nabla u(x) \, d\mathcal{H}_{m-1}(x)$$

for a Borel measurable set  $M$  then  $N^H u = \mu + \tilde{\mu}$ . Since  $u$  is finite and continuous on  $\text{cl } H$ ,  $\mu + \tilde{\mu} \in \mathcal{C}'_c(\partial H)$ . Since  $\mathcal{U}\tilde{\mu}$  is continuous in a neighbourhood of  $\partial G$  by [15], Lemma 2.18, we have  $\mu \in \mathcal{C}'_c(\partial G)$ .  $\square$

**Lemma 6.** *Let  $G$  be unbounded, let  $w$  be a solution of the Neumann problem in the sense of distributions with the null boundary condition. Suppose that there are  $q \geq 1$ ,  $R > 0$  such that  $|\nabla w| \in L_q(G \setminus \Omega_R(0))$ . Then there is a real number  $a$  such that  $w - a = O(|x|^{1-m})$ ,  $|\nabla w| = O(|x|^{-m})$  as  $|x| \rightarrow \infty$ .*

**Proof.** Fix  $x_0 \in \mathbb{R}^m \setminus \text{cl } G$ . Then [31], Chapter I, Theorem 3.5 yields that there are real numbers  $a, b$  and a harmonic function  $v$  on a neighbourhood of 0 with  $v(0) = 0$  such that

$$w(x) = a + bh_{x_0} + |x - x_0|^{2-m} v\left(\frac{x - x_0}{|x - x_0|^2}\right).$$

Fix  $R > 0$  such that  $\partial G \subset \Omega_R(x_0)$ . If  $\varphi \in \mathcal{D}$ ,  $\varphi = 1$  on  $\Omega_R(x_0)$  then

$$\begin{aligned} 0 &= \langle N^G w, \varphi \rangle = \langle N^{G \cap \Omega_R(x_0)} w, \varphi \rangle + \langle N^{G \setminus \Omega_R(x_0)} w, \varphi \rangle \\ &= - \int_{\partial\Omega_R(x_0)} n^{\Omega_R(x_0)} \cdot \nabla w \, d\mathcal{H}_{m-1} \\ &= b - \int_{\partial\Omega_R(x_0)} n^{\Omega_R(x_0)}(x) \cdot \nabla \left[ |x - x_0|^{2-m} v\left(\frac{x - x_0}{|x - x_0|^2}\right) \right] \, d\mathcal{H}_{m-1}(x). \end{aligned}$$

Since  $|\nabla[|x - x_0|^{2-m} v((x - x_0)/|x - x_0|^2)]| = O(|x|^{-m})$  as  $|x| \rightarrow \infty$  by [31], Chapter I, Corollary and Remark 3.6, we get  $b = 0$  taking  $R \rightarrow \infty$ . Therefore  $|\nabla w(x)| = O(|x|^{-m})$ ,  $|w(x) - a| = O(|x|^{1-m})$  as  $|x| \rightarrow \infty$ .  $\square$

**Theorem 2.** *Denote by  $G_1, \dots, G_k$  all components of  $G$ . If  $\mu \in \mathcal{C}'_0(\partial G)$  then there is a solution of the Neumann problem in the sense of distributions with the boundary*

condition  $\mu$ , which is continuous up to the boundary, if and only if  $\mu \in C'_c(\partial G)$ . If  $G$  is bounded then the general form of this solution is

$$(2) \quad u = \mathcal{U}\nu + \sum_{j=1}^k c_j \chi_{G_j},$$

where

$$(3) \quad \nu = \mu + 2 \sum_{j=0}^{\infty} (I - 2N^G \mathcal{U})^j (I - N^G \mathcal{U})\mu,$$

$\chi_{G_j}$  are characteristic functions of  $G_j$  and  $c_j$  are arbitrary constants. If  $G$  is unbounded then (2) is a general form of solutions continuously extendible to the boundary of  $G$  for which there are  $R > 0$ ,  $p \geq 1$  such that  $|\nabla u| \in L_p(G \setminus \Omega_R(0))$ .

**Proof.** If  $\mu \in C'_c(\partial G)$ , then  $u$  given by (2) is a solution of the Neumann problem with the boundary condition  $\mu$ , which is continuous up to the boundary (see Theorem 1 and [22], Theorem 1).

If  $u$  is a continuous (up to the boundary) solution of the Neumann problem with the boundary condition  $\mu$ , then  $\mu \in C'_c(\partial G)$  by Lemma 5. Put  $w = u - \mathcal{U}\nu$ . Then  $w$  is a solution of the Neumann problem in the sense of distributions with the zero boundary condition, continuous up to the boundary.

Suppose that  $G$  is bounded. Then  $w = -\mathcal{D}w$  on  $G$  by Lemma 4. Since  $V^G < \infty$ ,  $-\mathcal{D}w$  has a limit  $W^{\mathbb{R}^m \setminus G} w$  on the boundary, where

$$(4) \quad W^{\mathbb{R}^m \setminus G} w(x) = d_{\mathbb{R}^m \setminus G}(x)w(x) - \int_{\partial G} w(y)n^G(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y)$$

by [15], Remark 2.24. If we denote for  $f \in C(\partial G)$  and  $x \in \partial G$

$$(5) \quad W^G f(x) = d_G(x)f(x) + \int_{\partial G} f(y)n^G(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y),$$

then  $W^G w = 0$ . Since  $N^G \mathcal{U}$  is a Fredholm operator, the codimension of the range of  $N^G \mathcal{U}$  is equal to  $k$  by [22], Theorem 1 and  $N^G \mathcal{U}$  is the adjoint operator of  $W^G$  by [15], Proposition 2.20, the dimension of the kernel of  $W^G$  is equal to  $k$  by [12], Theorem 27.1. In a similar way as for  $w$  we get that  $W^G \chi_{\partial G_j} = W^G \chi_{\text{cl}G_j} = 0$ . Since  $\chi_{G_1}, \dots, \chi_{G_k}$  form a base of the kernel of  $W^G$  and  $W^G w = 0$ ,  $w$  is constant on  $\partial G_j$  for each  $j = 1, \dots, k$ . Since  $w$  is harmonic and continuous, it is constant on  $G_j$  for each  $j = 1, \dots, k$ . So,  $u$  has the form (2).

Suppose now that  $G$  is unbounded and there are  $R > 0$ ,  $p \geq 1$  such that  $|\nabla u| \in L_p(G \setminus \Omega_R(0))$ . According to Lemma 6 there is a real number  $a$  such that  $|\nabla w(x)| =$

$O(|x|^{-m})$ ,  $|w(x) - a| = O(|x|^{1-m})$  as  $|x| \rightarrow \infty$ . Fix  $x_0 \in \mathbb{R}^m \setminus \text{cl} G$ ,  $R > 0$  such that  $\partial G \subset \Omega_R(x_0)$ . According to Lemma 4 we have for  $x \in G \cap \Omega_R(x_0)$

$$w(x) - a = \int_{\partial\Omega_R(x_0)} h_x n^{\Omega_R(x_0)} \cdot \nabla w \, d\mathcal{H}_{m-1} - \int_{\partial\Omega_R(x_0)} (w - a) n^{\Omega_R(x_0)} \cdot \nabla h_x \, d\mathcal{H}_{m-1} - \mathcal{D}(w - a)(x).$$

Tending  $R \rightarrow \infty$  we get  $w(x) - a = -\mathcal{D}(w - a)(x)$  in  $G$ . Since  $V^G < \infty$ ,  $-\mathcal{D}(w - a)$  has the limit  $W^{\mathbb{R}^m \setminus G}(w - a)$  (given by (4)) on the boundary. Therefore  $W^G(w - a) = 0$  ( $W^G f$  is given by (5)). Since  $N^G \mathcal{U}$  is a Fredholm operator, the codimension of the range of  $N^G \mathcal{U}$  is equal to  $k - 1$  by [22], Theorem 1 and  $N^G \mathcal{U}$  is the adjoint operator of  $W^G$ , the dimension of the kernel of  $W^G$  is equal to  $k - 1$  by [12], Theorem 27.1. In a similar way as for  $w$  we get that  $W^G \chi_{\partial G_j} = W^G \chi_{\text{cl} G_j} = 0$  for each bounded component  $G_j$  of  $G$ . Since  $\{\chi_{G_j}; G_j \text{ bounded}\}$  form a base of the kernel of  $W^G$  and  $W^G(w - a) = 0$ ,  $w$  is constant on  $\partial G_j$  for each  $j = 1, \dots, k$  and  $(w - a) = 0$  on the boundary of the unbounded component of  $G$ . Since  $(w - a)$  is harmonic, continuous on  $\text{cl} G$  and  $(w(x) - a)$  tends to 0 as  $|x|$  tends to infinity,  $w$  is constant on  $G_j$  for each  $j = 1, \dots, k$ . So,  $u$  has the form (2).  $\square$

**Remark 2.** If  $G$  is unbounded then the space of all solutions of the Neumann problem in the sense of distributions with the zero boundary condition, which are continuously extendible onto the closure of  $G$ , has infinite dimension. For a positive integer  $j$  put

$$f_j(x_1, \dots, x_m) = \sum_{k=0}^j \binom{2j}{2k} (-1)^{j-k} x_1^{2k} x_2^{2j-2k}.$$

Then  $f_j$  are harmonic functions in  $\mathbb{R}^m$ . According to Theorem 2 there are  $\nu_j \in \mathcal{C}'_c(\partial G)$  such that  $\mathcal{U}\nu_j$  is a solution of the Neumann problem in the sense of distributions with the boundary condition  $\frac{\partial f_j}{\partial n} \mathcal{H}$ . Then  $u_j = f_j - \mathcal{U}\nu_j$  are solutions of the Neumann problem in the sense of distributions with the zero boundary condition, which are continuously extendible onto the closure of  $G$ . Since  $\lim u_j(x_1, \dots, x_m)/x_1^j \rightarrow 1$  as  $x_1 \rightarrow \infty$ , the functions  $u_j$  are linearly independent.

**Lemma 7.** Let  $\nu \in \mathcal{C}'_c(\partial G)$ . If  $m > 2$  then  $|\nabla \mathcal{U}\nu| \in L_2(\mathbb{R}^m)$ . If  $m = 2$  then  $|\nabla \mathcal{U}\nu| \in L_{2,\text{loc}}(\mathbb{R}^m)$  and  $|\nabla \mathcal{U}\nu| \in L_2(\mathbb{R}^m)$  if and only if  $\nu(\mathbb{R}^m) = 0$ .

**Proof.** If  $m > 0$  or  $m = 2$  and  $\nu(\mathbb{R}^m) = 0$  then  $|\nabla \mathcal{U}\nu| \in L_2(\mathbb{R}^m)$  by [22], Lemma 2, Lemma 6. Let now  $m = 2$ ,  $\nu(\mathbb{R}^m) \neq 0$ . Choose  $x \in G$ ,  $r > 0$  such that  $\Omega_{2r}(x) \subset G$ . Put  $H = G \setminus \text{cl} \Omega_r(x)$  and let  $\mu$  be the restriction of  $\mathcal{H}_1$  onto  $\partial\Omega_r(x)$ . Fix a constant  $c$  such that  $\nu(\mathbb{R}^m) - c\mu(\mathbb{R}^m) = 0$ . Since  $\nu - c\mu \in \mathcal{C}'_c(\partial H)$

by [15], Lemma 2.18, we have  $|\nabla\mathcal{U}\nu - c\nabla\mathcal{U}\mu| \in L_2(\mathbb{R}^m)$  (see [22], Lemma 6). Easy calculation yields that there are constants  $c_1, c_2$  such that  $\mathcal{U}\mu = c_1$  in  $\Omega_r(x)$  and  $\mathcal{U}\mu = c_1 + c_2 \log(|x|/r)$  on  $\mathbb{R}^m \setminus \Omega_r(x)$ . Since  $|\nabla\mathcal{U}\mu| \in L_{2,\text{loc}}(\mathbb{R}^m) \setminus L_2(\mathbb{R}^m)$  we have got the assertion of the lemma.  $\square$

**Notation.** Denote by  $W^{1,2}(G)$  the collection of all functions  $f \in L_2(G)$  the distributional gradient of which belongs to  $[L_2(G)]^m$ .

**Lemma 8.** *Let  $\nu \in C'_c(\partial G)$ . If  $G$  is bounded then  $\mathcal{U}\nu \in W^{1,2}(G)$ . If  $G$  is unbounded and  $m > 4$  then  $\mathcal{U}\nu \in W^{1,2}(G)$ ; if  $3 \leq m \leq 4$  then  $\mathcal{U}\nu \in W^{1,2}(G)$  if and only if  $\nu(\mathbb{R}^m) = 0$ .*

*Proof.*  $\mathcal{U}\nu \in W^{1,2}(G)$  for  $G$  bounded because  $|\nabla\mathcal{U}\nu| \in L_{2,\text{loc}}(\mathbb{R}^m)$  and  $\mathcal{U}\nu$  is continuously extendible to  $\text{cl}G$ . Let now  $G$  be unbounded,  $m > 2$ . The assertion follows from the facts that  $|\nabla\mathcal{U}\nu| \in L_2(\mathbb{R}^m)$ ,  $\mathcal{U}\nu$  is continuously extendible to  $\text{cl}G$  and  $\mathcal{U}\nu(x) = \nu(\mathbb{R}^m)|x|^{2-m} + O(|x|^{1-m})$  for  $|x| \rightarrow \infty$ .  $\square$

Throughout the rest of paper we will suppose that  $\mathcal{D}$  is dense in  $W^{1,2}(G)$ . According to [33], Theorem 2.3.2 this condition is fulfilled if  $\{f/G; f \in W^{1,2}(\mathbb{R}^m)\} = W^{1,2}(G)$ .

**Definition.** Let  $L$  be a bounded linear functional on  $W^{1,2}(G)$  such that  $L(\varphi) = 0$  for each  $\varphi \in \mathcal{D}(G) = \{\varphi \in \mathcal{D}; \text{spt } \varphi \subset G\}$ . We say that  $u \in W^{1,2}(G)$  is a weak solution of the Neumann problem for the Laplace equation with the boundary condition  $L$  if

$$\int_G \nabla u \cdot \nabla v \, d\mathcal{H}_m = L(v)$$

for each  $v \in W^{1,2}(G)$ .

**Lemma 9.** *Let  $\mu \in C'_c(\partial G)$ . If  $G$  is bounded suppose that  $\mu(\partial G) = 0$ . Then there is a unique bounded linear functional  $L_\mu$  on  $W^{1,2}(G)$  such that*

$$(6) \quad L_\mu(\varphi) = \int_{\partial G} \varphi \, d\mu$$

for each  $\varphi \in \mathcal{D}$ .



**Proof.** According to Theorem 2 and Theorem 1 there is  $\nu \in C'_c(\partial G)$  such that  $N^G \mathcal{U}\nu = \mu$ . Fix  $\psi \in \mathcal{D}$  such that  $\psi = 1$  in a neighbourhood of  $\partial G$ . If  $\varphi \in \mathcal{D}$  then

$$\begin{aligned} \int_{\partial G} \varphi \, d\mu &= \int_{\partial G} \psi \varphi \, dN^G \mathcal{U}\nu = \int_G \nabla(\psi \varphi) \cdot \nabla \mathcal{U}\nu \, d\mathcal{H}_m \\ &\leq \sup |\psi| \left( \int_{G \cap \text{spt } \psi} |\nabla \varphi|^2 \, d\mathcal{H}_m \right)^{1/2} \left( \int_{G \cap \text{spt } \psi} |\nabla \mathcal{U}\nu|^2 \, d\mathcal{H}_m \right)^{1/2} \\ &\quad + \sup |\nabla \psi| \left( \int_{G \cap \text{spt } \psi} |\varphi|^2 \, d\mathcal{H}_m \right)^{1/2} \left( \int_{G \cap \text{spt } \psi} |\nabla \mathcal{U}\nu|^2 \, d\mathcal{H}_m \right)^{1/2} \\ &\leq C \|\varphi\|_{W^{1,2}(G)}, \end{aligned}$$

where

$$C = 2(\sup |\psi| + \sup |\nabla \psi|) \left( \int_{G \cap \text{spt } \psi} |\nabla \mathcal{U}\nu|^2 \, d\mathcal{H}_m \right)^{1/2} < \infty$$

by Lemma 7. According to the Hahn-Banach theorem there is a bounded linear functional  $L_\mu$  on  $W^{1,2}(G)$  such that (6) holds. Since  $\mathcal{D}$  is dense in  $W^{1,2}(G)$ , the functional  $L_\mu$  is unique.  $\square$

**Theorem 3.** *Let  $\mu \in C'_0(\partial G) \cap C'_c(\partial G)$ . If  $G$  is unbounded suppose moreover that  $m > 2$  and  $\mu(\mathbb{R}^m) = 0$  for  $3 \leq m \leq 4$ . Then there is a weak solution  $u \in W^{1,2}(G)$  of the Neumann problem for the Laplace equation with the boundary condition  $L_\mu$ . If  $G_1, \dots, G_k$  are all components of  $G$  then the general solution of this problem has the form (2), where  $\nu$  is given by (3) and  $c_j = 0$  for  $G_j$  unbounded while  $c_j$  is arbitrary constant for  $G_j$  bounded.*

**Proof.** Let  $\nu$  be given by (3). Then  $N^G \mathcal{U}\nu = \mu$  and  $\nu \in C'_c(\partial G)$  by Theorem 1, Theorem 2. If  $\mu(\mathbb{R}^m) = 0$  then  $\nu(\mathbb{R}^m) = 0$ , because  $N^G \mathcal{U}\mu(\mathbb{R}^m) = 0$  by [22], Lemma 9. According to Lemma 8 we have  $\mathcal{U}\nu \in W^{1,2}(G)$ . For a fixed  $v \in W^{1,2}(G)$  choose  $\varphi_n \in \mathcal{D}$  such that  $\varphi_n \rightarrow v$  in  $W^{1,2}(G)$  as  $n \rightarrow \infty$ . Then

$$L_\mu(v) = \lim_{n \rightarrow \infty} \int \varphi_n \, d\mu = \lim_{n \rightarrow \infty} \int_G \nabla \varphi_n \cdot \nabla \mathcal{U}\nu \, d\mathcal{H}_m = \int_G \nabla v \cdot \nabla \mathcal{U}\nu \, d\mathcal{H}_m.$$

$\mathcal{U}\nu$  is a weak solution of the Neumann problem for the Laplace equation with the boundary condition  $L_\mu$ . If  $u$  has the form (2), where  $c_j = 0$  for  $G_j$  unbounded, then  $u$  is a weak solution of the Neumann problem for the Laplace equation with the boundary condition  $L_\mu$ .

Let  $u \in W^{1,2}(G)$  be a weak solution of the Neumann problem for the Laplace equation with the boundary condition  $L_\mu$ . Since  $u - \mathcal{U}\nu \in W^{1,2}(G)$  we have

$$0 = \int_G \nabla u \cdot \nabla(u - \mathcal{U}\nu) \, d\mathcal{H}_m - \int_G \nabla \mathcal{U}\nu \cdot \nabla(u - \mathcal{U}\nu) \, d\mathcal{H}_m = \int_G |\nabla(u - \mathcal{U}\nu)|^2 \, d\mathcal{H}_m.$$

Since  $(u - \mathcal{U}\nu)$  is locally constant on  $G$ ,  $u$  has the form (2).  $\square$

**Theorem 4.** Let  $L$  be a bounded linear functional on  $W^{1,2}(G)$  and let  $\mu \in \mathcal{C}'(\partial G)$  be such that  $L(\varphi) = \int \varphi \, d\mu$  for each  $\varphi \in \mathcal{D}$ . If  $u \in W^{1,2}(G)$  is a weak solution of the Neumann problem for the Laplace equation with the boundary condition  $L$ , then  $u$  is continuously extendible to the closure of  $G$  if and only if  $\mu \in \mathcal{C}'_c(\partial G)$ .

*Proof.* Since  $N^G u = \mu$ , [22], Theorem 1 yields that  $\mu \in \mathcal{C}'_0(\partial G)$ . If  $u$  is continuously extendible to the closure of  $G$  then  $\mu \in \mathcal{C}'_c(\partial G)$  by Theorem 2. Suppose now that  $\mu \in \mathcal{C}'_c(\partial G)$ . If  $G$  is bounded put  $\tilde{G} = G$ ,  $\tilde{\mu} = \mu$ . If  $G$  is unbounded fix  $R > 0$  such that  $\partial G \subset \Omega_R(0)$  and put  $\tilde{G} = G \cap \Omega_R(0)$ ,  $\tilde{\mu} = \mu + \frac{\partial u}{\partial n}(\mathcal{H}_{m-1}/\partial\Omega_R(0))$ . Since  $V^G < \infty$  we have  $V^{\tilde{G}} < \infty$ . Since  $r_{\text{ess}}(N^G \mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$  and  $(N^H \mathcal{U} - \frac{1}{2}I)$  is compact for each bounded open set  $H$  with a smooth boundary, [20], Theorem 2.3 yields that  $r_{\text{ess}}(N^{\tilde{G}} \mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$ . Since  $N^{\tilde{G}} u = \tilde{\mu}$ , [22], Theorem 1 yields that  $\mu \in \mathcal{C}'_0(\partial G)$ . If  $G$  is unbounded then  $\frac{\partial u}{\partial n}(\mathcal{H}_{m-1}/\partial\Omega_R(0)) \in \mathcal{C}'_c(\partial\tilde{G})$  by [21], Remark 6 and therefore  $\tilde{\mu} \in \mathcal{C}'_c(\partial\tilde{G})$ . Since  $u$  is a weak solution of the Neumann problem for the Laplace equation on  $\tilde{G}$  with the boundary condition  $L_{\tilde{\mu}}$ , Theorem 3 and Theorem 2 yield that  $u$  is continuously extendible to the closure of  $\tilde{G}$ .  $\square$

**Remark on the Boundary Element Method.** Let  $H$  be a bounded domain in  $\mathbb{R}^m$  ( $m = 2$  or  $3$ ) with a piecewise-smooth boundary, let  $f$  be a bounded measurable function on  $\partial H$ . We want to solve the Neumann problem for the Laplace equation with the boundary condition  $f$ . Denote by  $\mathcal{H}$  the surface measure on  $\partial H$ . Since  $\mathcal{U}(f\mathcal{H})$  is a continuous function in  $\mathbb{R}^m$  (see [15], Lemma 2.18), there is  $u \in \mathcal{C}(\partial H)$  which is a solution of the Neumann problem for the Laplace equation with the boundary condition  $f\mathcal{H}$  in the sense of distributions (see Theorem 2). According to Lemma 4

$$u(x) = \mathcal{U}(f\mathcal{H})(x) - \mathcal{D}u(x)$$

for each  $x \in H$ . Using the boundary behaviour of a double layer potential with a continuous density ([15], Chapter 2), we get for  $x \in \partial H$

$$u(x) = \mathcal{U}(f\mathcal{H})(x) - \mathcal{D}u(x) + d_{\mathbb{R}^m \setminus \text{cl} H}(x)u(x).$$

Therefore, the equation

$$d_H(x)u(x) + \mathcal{D}u(x) = \mathcal{U}(f\mathcal{H})(x),$$

which is the starting point of the boundary element method, holds and there is a continuous solution  $u$  of this equation.

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