

Oleg Palumbíny

On oscillation criteria of fourth order linear differential equations

*Czechoslovak Mathematical Journal*, Vol. 53 (2003), No. 2, 467–477

Persistent URL: <http://dml.cz/dmlcz/127814>

## Terms of use:

© Institute of Mathematics AS CR, 2003

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

ON OSCILLATION CRITERIA OF FOURTH ORDER  
LINEAR DIFFERENTIAL EQUATIONS

OLEG PALUMBÍNÝ, Trnava

(Received June 20, 2000)

*Abstract.* The paper deals with oscillation criteria of fourth order linear differential equations with quasi-derivatives.

*Keywords:* linear differential equation, quasi-derivative, monotone solution, Kneser solution, oscillatory solution

*MSC 2000:* 34C10, 34C11, 34D05

1. INTRODUCTION

Consider the linear differential equation of the fourth order with quasi-derivatives

$$(L) \quad L(y) \equiv L_4y + P(t)L_2y + Q(t)y = 0,$$

where

$$\begin{aligned} L_0y(t) &= y(t), \\ L_1y(t) &= p_1(t)y'(t) = p_1(t)\frac{dy(t)}{dt}, \\ L_2y(t) &= p_2(t)(p_1(t)y'(t))' = p_2(t)(L_1y(t))', \end{aligned}$$

---

The author's research has been supported by the grant no. 1/9094/02 of the Scientific Grant Agency of the Ministry of Education of the Slovak Republic and the Slovak Academy of Sciences.

$$L_3y(t) = p_3(t)(p_2(t)(p_1(t)y'(t)))' = p_3(t)(L_2y(t))',$$

$$L_4y(t) = (p_3(t)(p_2(t)(p_1(t)y'(t))))' = (L_3y(t))',$$

$P(t)$ ,  $Q(t)$ ,  $p_i(t)$ ,  $i = 1, 2, 3$ , are real-valued continuous functions on an interval  $I_a = [a, \infty)$ ,  $-\infty < a < \infty$ . It is assumed throughout that

- (A)  $P(t) \leq 0$ ,  $Q(t) \leq 0$ ,  $p_i(t) > 0$ ,  $i = 1, 2, 3$ ,  $t \in I_a$  and  $Q(t)$  is not identically zero in any subinterval of  $I_a$ .

We note that in the whole paper we will use the notation  $I_b = [b, \infty)$ ,  $b$  is any real number.

In [4] sufficient conditions for (L) to be oscillatory have been stated. In this paper we will deal with other ones. We will describe two oscillation criteria for (L), which create the content of Theorem 3 and 4.

Theorem 3 asserts that (L) is oscillatory as a consequence of the fact that the binomial third order linear differential equation (see Eq. (L\*))

$$L_3^*x + \frac{\theta\mu(t)Q(t)}{p_1(t)}x = 0$$

is oscillatory.

Theorem 4 is a special case of Theorem 3 because it states a sufficient condition for the just mentioned third order differential equation to be oscillatory (and for (L), of course, too).

The paper is concluded by two examples illustrating the results mentioned above. In the end of the introduction we want to note that our theorems generalize some results which J. Regenda derived in [5] as well as in [6].

## 2. DEFINITIONS AND PRELIMINARY RESULTS

**Definition 1.** A solution  $y(t)$  of (L) on  $I_a$  is called positively (negatively) nonoscillatory iff there exists  $t_0 \geq a$  such that  $y(t) > 0$  ( $y(t) < 0$ ),  $t \geq t_0$ .

**Definition 2.** A solution  $y(t)$  of (L) on  $I_a$  is called nonoscillatory iff  $y(t)$  is positively or negatively nonoscillatory.

**Definition 3.** The equation (L) is called nonoscillatory iff every nontrivial solution of (L) on  $I_a$  is nonoscillatory.

**Definition 4.** A nontrivial solution  $y(t)$  of (L) on  $I_a$  is called oscillatory on  $I_a$  iff the set of all its zeros on  $I_a$  is not bounded from above.

**Definition 5.** The equation (L) is called oscillatory iff there exists at least one oscillatory solution of (L) on  $I_a$ .

**Definition 6.** A positively nonoscillatory solution  $y(t)$  of (L) on  $I_a$  such that  $y(t) > 0$  for  $t \geq t_0 \geq a$  is called a monotone (Kneser) solution on  $[t_0, \infty)$  iff  $L_k y(t) > 0$  ( $(-1)^k L_k y(t) > 0$ ),  $k = 0, 1, 2, 3$ ,  $t \geq t_0$ .

**Lemma 1** [1, Lemma 2.2]. Let  $f(t)$  be a real valued function defined in  $[t_0, \infty)$  for some real number  $t_0 \geq 0$ . Suppose that  $f(t) > 0$  and that  $f'(t)$  and  $f''(t)$  exist for  $t \geq t_0$ . Suppose also that if  $f'(t) \geq 0$  eventually, then  $\lim_{t \rightarrow \infty} f(t) = A < \infty$ . Then

$$\liminf_{t \rightarrow \infty} |t^\alpha f''(t) - \alpha t^{\alpha-1} f'(t)| = 0$$

for any  $\alpha \leq 2$ .

**Lemma 2** [7, Lemma 3]. Let (A) and  $\int (1/p_1(t)) dt = \infty$  hold. Then for every nonoscillatory solution  $y(t)$  of (L) there exists a number  $t_0 \geq a$  such that either

$$(y(t)L_1 y(t) > 0, y(t)L_2 y(t) > 0) \text{ or } (y(t)L_1 y(t) < 0, y(t)L_2 y(t) > 0) \\ \text{or } (y(t)L_1 y(t) > 0, y(t)L_2 y(t) < 0) \text{ for all } t \geq t_0.$$

**Lemma 3** [4, Lemma 6]. Let (A) hold. If every positively nonoscillatory solution of (L) on  $I_a$  is either monotone or Kneser, then (L) is oscillatory.

**Lemma 4.** Consider a linear differential equation

$$(M) \quad M(y) \equiv a_1(t)y''' + a_2(t)y'' + a_3(t)y' + a_4(t)y = 0,$$

where the functions  $a_i(t)$ ,  $i = 1, 2, 3, 4$ , are continuous on  $[b, c]$ ,  $b < c$ . Then every nontrivial solution of (M) on  $[b, c]$  admits at most two zeros on this interval if and only if there exist functions  $z_1(t)$ ,  $z_2(t)$ , both from the class  $C^3([b, c])$ , such that

$$z_1(t) > 0, z_2(t) > 0, W(z_1, z_2) > 0, M(z_1) \geq 0, M(z_2) \leq 0, t \in [b, c],$$

where  $W(z_1, z_2)$  denotes Wronski's determinant of  $z_1(t)$ ,  $z_2(t)$ .

**P r o o f.** The lemma is the special case of [2, Theorem 4.1] for  $n = 3$ . □

**Lemma 5** [3, Theorem]. Consider the linear differential equation

$$(N) \quad N(y) \equiv (p_3(t)(p_2(t)y')')' + r(t)y = 0,$$

where  $p_i(t)$ ,  $i = 2, 3$  are positive and continuous on  $I_a$ ,  $r(t)$  is a function nonpositive and continuous on  $I_a$ ,  $r(t)$  is not identically zero in any subinterval of  $I_a$ . If

$$\int_a^\infty \frac{1}{p_2(s)} ds = \int_a^\infty \frac{1}{p_3(s)} ds = \int_a^\infty -r(s) ds = \infty,$$

then (N) is oscillatory.

**Lemma 6.** Let the function  $L_3y(t)$  be continuous on an interval  $I_c$  and let  $y(t) > 0$ ,  $L_1y(t) > 0$ ,  $L_2y(t) > 0$ ,  $L_3y(t) < 0$ ,  $p_1'(t) \geq 0$ ,  $p_2'(t) \geq 0$  on  $I_c$ . Then

$$y(t) > \frac{t-c}{2p_1(t)} L_1y(t)$$

for all  $t \in I_c$ .

**Proof.** Let us consider the function  $f(t) = L_1y(t) - (t-c)(L_1y(t))'$  for  $t \in I_c$ . It is obvious that  $f(c) = L_1y(c) > 0$ . Then

$$\begin{aligned} f'(t) &= (L_1y(t))' - (L_1y(t))' - (t-c)(L_1y(t))'' \\ &= -(t-c) \left( \frac{L_2y(t)}{p_2(t)} \right)' \\ &= -(t-c) \frac{p_2(t)L_3y(t)/p_3(t) - L_2y(t)p_2'(t)}{p_2^2(t)} > 0 \quad \text{for } t > c. \end{aligned}$$

This implies that  $f(t)$  is nondecreasing on  $I_c$  and  $f(t) \geq f(c) > 0$  on  $I_c$ , i.e.

$$(1) \quad L_1y(t) > (t-c)(L_1y(t))' \quad \text{for } t \geq c.$$

The integration of (1) over  $[c, t]$ ,  $c < t$  yields

$$\begin{aligned} \int_c^t p_1(s)y'(s) ds &> \int_c^t (s-c)(L_1y(s))' ds \\ &> (t-c)L_1y(t) - \int_c^t L_1y(s) ds \\ &> (t-c)L_1y(t) - \int_c^t p_1(s)y'(s) ds. \end{aligned}$$

This implies that

$$\int_c^t p_1(s)y'(s) ds > \frac{(t-c)}{2}L_1y(t),$$

$$p_1(t)y(t) - p_1(c)y(c) + \int_c^t -p_1'(s)y(s) ds > \frac{(t-c)}{2}L_1y(t),$$

$$p_1(t)y(t) > \frac{(t-c)}{2}L_1y(t),$$

$$y(t) > \frac{(t-c)}{2p_1(t)}L_1y(t)$$

for all  $t \geq c$ . The assertion is proved.  $\square$

**Theorem 1.** Let (A),  $\int_a^\infty (1/p_1(t)) dt = \int_a^\infty (1/p_2(t)) dt = \infty$  hold. If (L) is nonoscillatory on  $I_a$ , then there exists  $t_0 \in I_a$  and a solution  $y(t)$  of (L) on  $I_{t_0}$  such that either  $(y(t) > 0, L_1y(t) > 0, L_2y(t) < 0)$  or  $(y(t) > 0, L_1y(t) > 0, L_2y(t) > 0, L_3y(t) < 0)$  on  $I_{t_0}$ .

*Proof.* According to Lemma 3 there exists a positively nonoscillatory solution  $y(t)$  of (L) on  $I_a$  such that  $y(t)$  is neither monotone nor Kneser. Lemma 2 yields that for this  $y(t)$  there exists  $b \in I_a$  such that

$$(2) \quad \begin{aligned} &(y(t) > 0, L_1y(t) > 0, L_2y(t) > 0) \\ &\text{or } (y(t) > 0, L_1y(t) < 0, L_2y(t) > 0) \\ &\text{or } (y(t) > 0, L_1y(t) > 0, L_2y(t) < 0) \end{aligned}$$

for  $t \geq b$ . Now (in accordance with (2)) let us assume that  $y(t) > 0, L_1y(t) > 0, L_2y(t) > 0$  on some  $I_{t_0}$ . Then  $L_4y(t) = -P(t)L_2y(t) - Q(t)y(t) \geq 0, t \geq t_0$  and  $L_4y(t) = 0$  holds at most at isolated points (according to (A)). This implies that  $L_3y(t)$  is increasing on  $I_{t_0}$ . Two cases may now occur:

$$(3) \quad L_3y(t) > 0 \text{ or } L_3y(t) < 0$$

on some  $I_{t_1}, t_1 \geq t_0$ . However, the assertion in the first expression of (3) cannot be true, because  $y(t)$  would be monotone on  $I_{t_1}$ , which is impossible.

If (in accordance with (2)) for this positively nonoscillatory solution  $y(t)$  the inequalities  $y(t) > 0, L_1y(t) < 0, L_2y(t) > 0$  held on some  $I_{t_0}$ , then (for the same reason) (3) would hold on some  $I_{t_2}, t_2 \geq t_0$ . However, the second inequality of (3) cannot be true, because  $y(t)$  would be Kneser on  $I_{t_2}$ , which is impossible, either. If the first inequality of (3) were true, then

$$L_2y(t) = L_2y(t_2) + \int_{t_2}^t \frac{L_3y(s)}{p_3(s)} ds \geq L_2y(t_2) > 0, \quad t \geq t_2$$

and

$$\begin{aligned} L_1y(t) &= L_1y(t_2) + \int_{t_2}^t \frac{L_2y(s)}{p_2(s)} ds \\ &\geq L_1y(t_2) + L_2y(t_2) \int_{t_2}^t \frac{ds}{p_2(s)} \rightarrow \infty \quad \text{for } t \rightarrow \infty, \end{aligned}$$

which would contradict  $L_1y(t) < 0$  on  $I_{t_2}$ . The theorem is established.  $\square$

**Remark 1.** A part (viz. the necessary condition for (L) to be nonoscillatory) of [5, Theorem 1.1] is a special case of Theorem 1, where  $p_i(t) \equiv 1$ ,  $i = 1, 2, 3$ ,  $t \in I_a$ .

**Theorem 2.** Let (A),  $p_2'(\infty) \geq 0$  on  $I_a$ ,  $p_2(\infty) < \infty$ ,  $p_3(\infty) < \infty$ , and  $t^2P(t) + (2tp_3(t))' \geq 0$  for  $t > t_0 \geq \max\{a, 0\}$  hold. Then the equation (L) does not admit a solution  $y(t)$  on  $I_a$  such that  $y(t) > 0$ ,  $L_1y(t) > 0$ ,  $L_2y(t) < 0$  for  $t > t_1 \geq t_0$ .

*Proof.* Let us assume for a while the existence of such a solution  $y(t)$  of (L) on some  $(t_1, \infty)$ . Then

$$(4) \quad L_4y(t) + P(t)L_2y(t) + Q(t)y(t) \equiv 0 \quad \text{on } [t_1, \infty).$$

Now we shall prove that there exists no  $t_2 \geq t_1$  such that  $L_3y(t) < 0$  on  $I_{t_2}$ . So, assume that such  $t_2$  exists. Then

$$\begin{aligned} L_1y(t) &= L_1y(t_2) + \int_{t_2}^t \frac{L_2y(s)}{p_2(s)} ds \\ &= L_1y(t_2) + \int_{t_2}^t \frac{1}{p_2(s)} \left( L_2y(t_2) + \int_{t_2}^s \frac{L_3y(r)}{p_3(r)} dr \right) ds \\ &\leq L_1y(t_2) + \int_{t_2}^t \frac{L_2y(t_2)}{p_2(s)} ds \rightarrow -\infty \quad \text{for } t \rightarrow \infty, \end{aligned}$$

which contradicts  $L_1y(t) > 0$  on  $(t_1, \infty)$  and implies the existence of  $t_3 > t_1$  such that  $L_3y(t_3) \geq 0$ . Multiplying (4) by  $t^2$  and integrating over  $[t_3, t]$  yields

$$(5) \quad \begin{aligned} &t^2L_3y(t) - t_3^2L_3y(t_3) - 2tp_3(t)L_2y(t) + 2t_3p_3(t_3)L_2y(t_3) \\ &+ \int_{t_3}^t (s^2P(s) + (2sp_3(s))')L_2y(s) ds + \int_{t_3}^t s^2Q(s)y(s) ds = 0. \end{aligned}$$

Let us denote  $A(t) = t^2L_3y(t) - 2tp_3(t)L_2y(t) = p_3(t)(t^2(L_2y(t))' - 2tL_2y(t))$ . Then from (5) we obtain (6), where

$$(6) \quad \begin{aligned} &A(t) - t_3^2L_3y(t_3) + 2t_3p_3(t_3)L_2y(t_3) + \int_{t_3}^t (s^2P(s) + (2sp_3(s))')L_2y(s) ds \\ &+ \int_{t_3}^t s^2Q(s)y(s) ds = 0. \end{aligned}$$

Since the second, fourth and fifth terms on the left-hand side of (6) are nonpositive and the third is negative we have

$$(7) \quad |A(t)| \geq A(t) \geq -2t_3 p_3(t_3) L_2 y(t_3) > 0, \quad t \geq t_3.$$

In Lemma 1 let us put  $\alpha = 2$ ,  $f'(t) = \frac{df(t)}{dt} = L_2 y(t)$  on  $I_{t_3}$ . Then  $f'(t) < 0$ ,  $t \geq t_3$ , and on  $I_{t_3}$  we have

$$(8) \quad \begin{aligned} f(t) &= f(t_3) + \int_{t_3}^t p_2(s) (L_1 y(s))' ds \\ &\geq f(t_3) + p_2(\infty) \int_{t_3}^t (L_1 y(s))' ds \\ &\geq f(t_3) + p_2(\infty) \int_{t_3}^{\infty} (L_1 y(s))' ds \\ &\geq f(t_3) + p_2(\infty) L_1 y(\infty) - p_2(\infty) L_1 y(t_3) \\ &\geq f(t_3) - p_2(\infty) L_1 y(t_3) > -\infty. \end{aligned}$$

It follows from (8) that  $f(t_3)$  can be chosen such that  $f(t) > 0$  on  $I_{t_3}$ . Then Lemma 1 yields  $\liminf_{t \rightarrow \infty} |A(t)| = 0$ . However, (7) implies that  $\liminf_{t \rightarrow \infty} |A(t)| > 0$ . This contradiction proves the theorem.  $\square$

**Remark 2.** [5, Theorem 1.3] is a special case of Theorem 2, where  $p_i(t) \equiv 1$ ,  $i = 1, 2, 3$ ,  $t \in I_a$ .

### 3. OSCILLATION CRITERIA

**Theorem 3.** Let a function  $\mu(t)$  be positive and continuous in  $(T, \infty)$ ,  $T > \max\{a, 0\}$ , and such that  $\liminf_{t \rightarrow \infty} ((t - t_0)/\mu(t)) \geq 2$  for any  $t_0 \geq a$ . Let (A) hold and let  $p'_i(t) \geq 0$ ,  $i = 1, 2$ ,  $t^2 P(t) + (2tp_3(t))' \geq 0$  for  $t \geq T > \max\{a, 0\}$ ,  $p_j(\infty) < \infty$ ,  $j = 2, 3$ ,  $\int_a^\infty (1/p_1(t)) dt = \infty$ . For every  $\tau > T$  let there exist  $\tau_1 > \tau$  such that  $p_1(t)p'_2(t)p'_3(t) + p_1(t)p''_2(t)p_3(t) + \theta\mu(t)Q(t)(t - \tau) \leq 0$  for all  $t \geq \tau_1$ ,  $\theta \in (0, 1)$ . If the differential equation

$$(L^*) \quad L^*(x) \equiv L^*_3 x + \frac{\theta\mu(t)Q(t)}{p_1(t)} x \equiv (p_3(t)(p_2(t)x')')' + \frac{\theta\mu(t)Q(t)}{p_1(t)} x = 0$$

(where  $x = x(t)$ ,  $x' = dx/dt$ ) is oscillatory for some  $\theta \in (0, 1)$ , then (L) is oscillatory.

**Proof.** Let us assume (L) to be nonoscillatory. It is clear that  $\int_a^\infty (dt/p_2(t)) = \infty$ . Then Theorem 1 and Theorem 2 yield the existence of a solution  $y(t)$  of (L) on



$[t_0, \infty)$ , where  $t_0 \geq T \geq a$ , such that  $y(t) > 0$ ,  $L_1y(t) > 0$ ,  $L_2y(t) > 0$ ,  $L_3y(t) < 0$ ,  $t \geq t_0$ . Thus, according to Lemma 6,  $y(t)$  satisfies

$$\begin{aligned}
 (9) \quad 0 &\equiv L_4y(t) + P(t)L_2y(t) + Q(t)y(t) \\
 &\leq L_4y(t) + P(t)L_2y(t) + \frac{(t-t_0)Q(t)L_1y(t)}{2p_1(t)} \\
 &\leq L_4y(t) + \frac{(t-t_0)Q(t)L_1y(t)}{2p_1(t)} \quad \text{on } I_{t_0}.
 \end{aligned}$$

Let us put  $z_1(t) \equiv L_1y(t)$ . Then

$$\begin{aligned}
 (10) \quad L_3^*z_1(t) &= L_4y(t) = (p_3(t)(p_2(t)z_1'(t))')' \\
 &= p_2(t)p_3(t)z_1'''(t) + (2p_3(t)p_2'(t) + p_3'(t)p_2(t))z_1''(t) \\
 &\quad + (p_2'(t)p_3'(t) + p_3(t)p_2''(t))z_1'(t) \quad \text{on } I_{t_0}.
 \end{aligned}$$

It follows from (9) and (10) that (11) holds, where

$$(11) \quad 0 \leq L_3^*z_1(t) + \frac{(t-t_0)Q(t)z_1(t)}{2p_1(t)} \quad \text{on } I_{t_0}.$$

The assumption  $\liminf_{t \rightarrow \infty} ((t-t_0)/\mu(t)) \geq 2$  implies the existence of  $\tau > t_0$  such that  $\frac{t-t_0}{\mu(t)} > 2\theta$  for all  $t \geq \tau$ , where  $\theta \in (0, 1)$  is an arbitrary fixed real number (the number  $\tau$  depends on  $\theta$ ). This result and (11) imply (12), where

$$(12) \quad 0 \leq L_3^*z_1(t) + \frac{\theta\mu(t)Q(t)z_1(t)}{p_1(t)} \quad \text{on } [\tau, \infty).$$

Let us put

$$\begin{aligned}
 a_1(t) &\equiv 1, \\
 a_2(t) &\equiv \frac{2p_2'(t)p_3(t) + p_2(t)p_3'(t)}{p_2(t)p_3(t)}, \\
 a_3(t) &\equiv \frac{p_2'(t)p_3'(t) + p_2''(t)p_3(t)}{p_2(t)p_3(t)}, \\
 a_4(t) &\equiv \frac{\theta\mu(t)Q(t)}{p_1(t)p_2(t)p_3(t)}
 \end{aligned}$$

in Lemma 4. Then owing to (10) and (12) we obtain

$$M(z_1) \equiv \frac{L_3^*z_1(t) + \frac{\theta\mu(t)Q(t)z_1(t)}{p_1(t)}}{p_2(t)p_3(t)} \geq 0 \quad \text{on } [\tau, \infty).$$

Let us put  $z_2(t) \equiv t - \tau$ . Then by the assumptions of the theorem we have

$$\begin{aligned} M(z_2) &\equiv \frac{p_2'(t)p_3'(t) + p_2''(t)p_3(t)}{p_2(t)p_3(t)} \cdot 1 + \frac{\theta\mu(t)Q(t)}{p_1(t)p_2(t)p_3(t)} \cdot (t - \tau) \\ &= \frac{p_1(t)p_2'(t)p_3'(t) + p_1(t)p_2''(t)p_3(t) + \theta\mu(t)Q(t)(t - \tau)}{p_1(t)p_2(t)p_3(t)} \\ &\leq 0 \quad \text{for } t \geq \tau_1 > \tau. \end{aligned}$$

It is obvious that  $z_1(t) \equiv L_1y(t) > 0$ ,  $z_2(t) \equiv t - \tau > 0$  for  $t \geq \tau_1$ . Similarly

$$\begin{aligned} W(z_1, z_2) &= \begin{vmatrix} z_1(t) & z_2(t) \\ z_1'(t) & z_2'(t) \end{vmatrix} = \begin{vmatrix} z_1(t) & t - \tau \\ z_1'(t) & 1 \end{vmatrix} \\ &= z_1(t) - (t - \tau)z_1'(t) \\ &= z_1(\tau) + (t - \tau)z_1'(\xi) - (t - \tau)z_1'(t) \\ &= z_1(\tau) + (t - \tau)(z_1'(\xi) - z_1'(t)) \\ &= z_1(\tau) + \frac{(t - \tau)}{p_2(t)}(p_2(t)z_1'(\xi) - p_2(t)z_1'(t)) \\ &\geq z_1(\tau) + \frac{(t - \tau)}{p_2(t)}(p_2(\xi)z_1'(\xi) - p_2(t)z_1'(t)) > 0 \quad \text{for } t \geq \tau_1, \quad \xi \in (\tau, t), \end{aligned}$$

where we have used Lagrange's mean value formula and the fact that the function  $p_2(t)z_1'(t)$  is positive and decreasing (because  $(p_2(t)z_1'(t))' = L_3y(t)/p_3(t) < 0$ ). Then Lemma 4 yields that  $(L^*)$  does not admit a nontrivial solution on  $I_a$  having more than two zeros on  $I_{\tau_1}$ , i.e.  $(L^*)$  is nonoscillatory. The theorem is proved.  $\square$

**Remark 3.** A part (with the condition (1)) of [6, Theorem 5] is a special case of Theorem 3, where  $p_i(t) \equiv 1$ ,  $i = 1, 2, 3$ ,  $t \in I_a$ .

By combining the previous theorem and Lemma 5 we obtain another oscillation criterion.

**Theorem 4.** Let a function  $\mu(t)$  be positive and continuous in  $(T, \infty)$ ,  $T > \max\{a, 0\}$ , such that  $\liminf_{t \rightarrow \infty} ((t - t_0)/\mu(t)) \geq 2$  for any  $t_0 \geq a$ . Let (A) hold and let  $p_i'(t) \geq 0$ ,  $i = 1, 2$ ,  $t^2P(t) + (2tp_3(t))' \geq 0$  for  $t \geq T > \max\{a, 0\}$ ,  $p_j(\infty) < \infty$ ,  $j = 2, 3$ ,  $\int_T^\infty (-\mu(t)Q(t)/p_1(t)) dt = \int_a^\infty (1/p_i(t)) dt = \infty$ ,  $i = 1, 3$ . For every  $\tau > T$  let there exist  $\tau_1 > \tau$  such that  $p_1(t)p_2'(t)p_3'(t) + p_1(t)p_2''(t)p_3(t) + \theta\mu(t)Q(t)(t - \tau) \leq 0$  for all  $t \geq \tau_1$ ,  $\theta \in (0, 1)$ . Then  $(L)$  is oscillatory.

**Proof.** According to Lemma 5 the equation  $(L^*)$  is oscillatory for all  $\theta \in (0, 1)$ . Then Theorem 3 yields that  $(L)$  is oscillatory. The criterion is established.  $\square$

**Example 1.** The equation

$$(\arctan t((2 - e^{-t})(ty')')')' - t^2(2 - e^{-t})(ty')' - 2t^3y = 0$$

is oscillatory according to Theorem 4, where  $a = T = 1$ ,  $\mu(t) = t/2$  because

$$\begin{aligned} & \lim_{t \rightarrow \infty} (p_1(t)p_2'(t)p_3'(t) + p_1(t)p_2''(t)p_3(t) + \theta\mu(t)Q(t)(t - \tau)) \\ &= \lim_{t \rightarrow \infty} \left( te^{-t} \frac{1}{1+t^2} - te^{-t} \arctan t - \theta t^4(t - \tau) \right) = -\infty, \end{aligned}$$

i.e. for every  $\tau > 1$  there exists  $\tau_1 > \tau$  such that  $p_1(t)p_2'(t)p_3'(t) + p_1(t)p_2''(t)p_3(t) + \theta\mu(t)Q(t)(t - \tau) \leq 0$  for all  $t \geq \tau_1$ ,  $\theta \in (0, 1)$ .

**Example 2.** Let us consider the equation

$$(13) \quad y^{(4)} - \frac{y''}{t^3} - \frac{y}{t^4} = 0, \quad t \in [1, \infty).$$

If the equation

$$(14) \quad x''' - \frac{91}{216t^3}x = 0$$

is oscillatory, then Theorem 3 (where  $\mu(t) = t/2$  and  $\theta = 2 \cdot \frac{91}{216} \in (0, 1)$ ) yields that (13) is oscillatory. If we test (13) to be oscillatory, we cannot use Theorem 4, because

$$\int_1^\infty \frac{-\mu(t)Q(t)}{p_1(t)} dt = \int_1^\infty \frac{1}{2t^3} dt < \infty.$$

However, (14) is an Euler equation; the substitution  $t = e^u$  transforms (14) into

$$(15) \quad \frac{d^3x}{du^3} - 3\frac{d^2x}{du^2} + 2\frac{dx}{du} - \frac{91}{216}x = 0,$$

which is oscillatory, because the characteristic equation for (15)

$$r^3 - 3r^2 + 2r - \frac{7 \cdot 13}{6^3} \equiv \left( \left( r - \frac{5}{12} \right)^2 + \frac{1}{48} \right) \left( r - \frac{13}{6} \right) = 0$$

admits complex zeros. Therefore (15), (14) as well as (13) are oscillatory.

### References

- [1] *J. W. Heidel*: Qualitative behavior of solutions of a third order nonlinear differential equation. *Pacific. J. Math.* 27 (1968), 507–526.
- [2] *A. Yu. Levin*: Nonoscillation of solutions of the equation  $x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = 0$ . *Uspekhi mat. nauk*, XXIV 2(146) (1969), 43–96. (In Russian.)
- [3] *O. Palumbíny*: On existence of oscillatory solutions of third-order linear differential equations. *Acta Fac. Paed. Univ. Tyrnaviensis*, Ser. B, No. 1 (1997), 105–112.
- [4] *O. Palumbíny*: On oscillatory solutions of fourth order ordinary differential equations. *Czechoslovak Math. J.* 49(124) (1999), 779–790.
- [5] *J. Regenda*: Oscillation criteria for fourth-order linear differential equations. *Math. Slovaca* 29 (1979), 3–16.
- [6] *J. Regenda*: A note on oscillation and nonoscillation criteria for fourth order linear differential equations. *Math. Slovaca* 33 (1983), 297–302.
- [7] *M. Tóthová and O. Palumbíny*: On monotone solutions of the fourth order ordinary differential equations. *Czechoslovak Math. J.* 45(120) (1995), 737–746.

*Author's address*: Department of Mathematics, Faculty of Materials Science and Technology, Slovak University of Technology in Bratislava, Paulínska 16, 917 24 Trnava, Slovakia, e-mail: palum@mtf.stuba.sk.