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A NOTE ON g-METRIZABLE SPACES

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Abstract. In this paper, the relationships between metric spaces and g-metrizable spaces are established in terms of certain quotient mappings, which is an answer to Alexandroff's problems.

Keywords: metric spaces, g-metrizable spaces, 1-sequence-covering mappings, σ-mappings, quotient mappings

MSC 2000: 54E99, 54C10, 54D55

1. Introduction

A central question of Alexandroff's problem is that the relationships between various topological spaces and metric spaces are established by means of various mappings [1]. The concept of a g-metrizable space was first introduced by F. Siwiec in [2], as a generalization of metric spaces. How to characterize a g-metrizable space by a nice image of a metric space? S. Lin introduced the concept of 1-sequence-covering mappings in order to give characterizations for spaces with a point-countable weak base [3]. Recently, S. Lin introduced the concept of σ-mappings [4], and showed that a space X is a σ-space if and only if it is a σ-image of a metric space. This shows that 1-sequence-covering mappings and σ-mappings are very important in answering Alexandroff's problems. In this paper, the relationships between metric spaces and g-metrizable spaces are established by means of 1-sequence-covering and quotient σ-mappings, which is also an answer to Alexandroff's problems.

All spaces in this paper are assumed to be regular and T_1. Mappings are continuous and onto. $\mathbb{N}$ denotes the set of positive integers, $\omega = \{0\} \cup \mathbb{N}$.

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We recall some definitions.

Let X be a space, and $\mathscr{P}$ be a cover of X. $\mathscr{P}$ is a network for X if, whenever $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathscr{P}$. A subfamily $\mathscr{P}'$ of $\mathscr{P}$ is a network at $x \in X$ if $x \in \bigcap \mathscr{P}'$ and whenever $x \in U$ with U open in X, then $P \subset U$ for some $P \in \mathscr{P}'$. A space is a σ-space if it has a σ-locally finite network.

Definition 1.1. Let $f\colon X \to Y$ be a mapping.

(1) f is called a σ-mapping [4] if X has a base $\mathscr{B}$ such that $f(\mathscr{B}) = \{f(B)\colon B \in \mathscr{B}\}$ is σ-locally finite in Y.

(2) f is a 1-sequence-covering mapping [3] if for each $y \in Y$, there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with $x_n \in f^{-1}(y_n)$ for all n.

(3) f is sequentially quotient [5] if for each convergent sequence L of Y, there is a convergent sequence S of X such that $f(S)$ is a subsequence of L.

(4) f is quotient [6] if, whenever $f^{-1}(U)$ is open, then U is open in Y.

(5) f is pseudo-open [6] if, whenever $f^{-1}(y) \subset V$ with V open in X, then $y \in \operatorname{int}(f(V))$.

Obviously, every 1-sequence-covering mapping is sequentially quotient.

Let us recall some basic definitions. Let X be a space, and let $\mathscr{P}$ be a cover of X. A space X is determined by $\mathscr{P}$ if $U \subset X$ is open (closed) in X if and only if $U \cap P$ is open (closed) in P for every $P \in \mathscr{P}$. A space X is a k-space (a sequential space), if it is determined by the cover consisting of all compact (all compact metric) subsets of X. A space is a Fréchet if, whenever $x \in \overline{A}$, there is a sequence $\{a_n\colon n \in \mathbb{N}\}$ in A with $a_n \to x$.

Definition 1.2. Let X be a space, and let $\mathscr{P} = \bigcup\{\mathscr{P}_x\colon x \in X\}$ be a collection of subsets in X satisfying the following conditions:

(a) $\mathscr{P}_x$ is a network at $x \in X$.

(b) For any $U, V \in \mathscr{P}_x$, there is $W \in \mathscr{P}_x$ such that $W \subset U \cap V$.

Then,

(1) $\mathscr{P}$ is called a weak base for X [7] if for $G \subset X$, $x \in G$, there is $P \in \mathscr{P}_x$ with $x \in P \subset G$. Then G is open in X [13]. A space X is weakly first countable if X has a weak base $\mathscr{P} = \bigcup\{\mathscr{P}_x\colon x \in X\}$ such that each $\mathscr{P}_x$ is countable. A space X is a g-metrizable space [2] if it has a σ-locally finite weak base.

(2) $\mathscr{P}$ is called a sequential neighborhood network for X [3] if any $P \in \mathscr{P}_x$ is a sequential neighborhood of x in X (that is, if for each convergent sequence $x_n \to x$, there exists $m \in \mathbb{N}$ such that $\{x\} \cup \{x_n\colon n \geqslant m\} \subset P$). This $\mathscr{P}_x$ is called a sequential neighborhood network of x.

2. Results

Lemma 2.1. *The following conditions are equivalent for a space X:*

(1) *X is a 1-sequence-covering and σ-image of a metric space.*

(2) *X has a σ-locally finite sequential neighborhood network.*

Proof. (1) $\Rightarrow$ (2). Suppose that $f\colon M \to X$ is a 1-sequence-covering and σ-mapping, where M is a metric space, then there is a base $\mathscr{B}$ for M such that $f(\mathscr{B})$ is σ-locally finite in X. For each $x \in X$, there is $\beta_x \in f^{-1}(x)$ satisfying Definition 1.1 (2). Put $\mathscr{P}_x = \{f(B)\colon \beta_x \in B \in \mathscr{B}\}$, $\mathscr{P} = \bigcup\{\mathscr{P}_x\colon x \in X\}$, then it is easy to check that $\mathscr{P}$ is a σ-locally finite sequential neighborhood network for X.

(2) $\Rightarrow$ (1). Let X have a σ-locally finite sequential neighborhood network $\mathscr{P} = \bigcup\{\mathscr{P}_n\colon n \in \mathbb{N}\}$, where $\mathscr{P}$ is closed under finite intersections, and each $\mathscr{P}_n$ is locally finite. We can assume that $X \in \mathscr{P}_n \subset \mathscr{P}_{n+1}$. Let $\mathscr{P}_n = \{P_\alpha\colon \alpha \in A_n\}$. For each $n \in \mathbb{N}$, A_n is endowed with discrete topology. Put $M = \bigl\{\alpha = (\alpha_n) \in \prod\limits_{n\in\mathbb{N}} A_n\colon \{P_{\alpha_n}\colon n \in \mathbb{N}\}$ is a network at some point x_α in $X\bigr\}$, and equip with M the subspace topology induced by the product topology of the product space $\prod\limits_{n\in\mathbb{N}} A_n$. Then M is a metric space. The point x_α is unique in X because X is T_2. We define $f\colon M \to X$ by $f(\alpha) = x_\alpha$. Then

(a) f is surjective. For each $x \in X$, there is a subsequence $\{n_i\}$ of $\mathbb{N}$ such that $\alpha_{n_i} \in A_{n_i}$ and $\{P_{\alpha_{n_i}}\colon i \in \mathbb{N}\}$ is a network at x. For $n \in \mathbb{N} \setminus \{n_i\colon i \in \mathbb{N}\}$, take $\alpha_n \in A_n$ with $P_{\alpha_n} = X$. Let $\alpha = (\alpha_n)$. Then $\alpha \in M$ and $f(\alpha) = x$. Thus f is surjective.

(b) f is continuous. For each $\alpha = (\alpha_n) \in M$ we have $f(\alpha) = x_\alpha \in X$. If U is an open neighborhood of x_α in X, then there is $n \in \mathbb{N}$ such that $x_\alpha \in P_{\alpha_n} \subset U$ because $\{P_{\alpha_n}\colon n \in \mathbb{N}\}$ is a network at x_α in X. Put $W = \{\beta \in M\colon$ the n-th coordinate of β is $\alpha_n\}$, then W is an open neighborhood of α in M, and $f(W) \subset P_{\alpha_n} \subset U$. Hence f is continuous.

(c) f is a σ-mapping. For $n \in \mathbb{N}$, $\alpha_n \in A_n$, put $V(\alpha_1, \ldots, \alpha_n) = \{\alpha \in M\colon$ if $i \leqslant n$, then the i-th coordinate of α is $\alpha_i\}$, $\mathscr{B} = \{V(\alpha_1, \ldots, \alpha_n)\colon \alpha_i \in A_i,\ i \leqslant n$ and $n \in \mathbb{N}\}$. Then $\mathscr{B}$ is a base for M. It suffices to show that $f(V(\alpha_1, \ldots, \alpha_n)) = \bigcap\limits_{i\leqslant n} P_{\alpha_i}$. In fact, by the definition of f, $f(V(\alpha_1, \ldots, \alpha_n)) \subset P_{\alpha_i}$ for each $i \leqslant n$, and thus $f(V(\alpha_1, \ldots, \alpha_n)) \subset \bigcap\limits_{i\leqslant n} P_{\alpha_i}$. Conversely, since f is surjective, for each $x \in \bigcap\limits_{i\leqslant n} P_{\alpha_i}$ there is $\beta = (\beta_j) \in M$ with $f(\beta) = x$. For $j \in \mathbb{N}$ we have $P_{\beta_j} \in \mathscr{P}_j \subset \mathscr{P}_{j+n}$, thus there is $\alpha_{j+n} \in A_{j+n}$ with $P_{\alpha_{j+n}} = P_{\beta_j}$. Put $\alpha = (\alpha_j)$. Then $\alpha \in V(\alpha_1, \ldots, \alpha_n)$ and $f(\alpha) = x$, and thus $f(V(\alpha_1, \ldots, \alpha_n)) = \bigcap\limits_{i\leqslant n} P_{\alpha_i}$. Hence f is a σ-mapping.

(d) f is 1-sequence-covering. For each $x \in X$, there is $\beta = (\alpha_i) \in M$ with $\beta \in f^{-1}(x)$. From the fact above, we have $f(V(\alpha_1, \ldots, \alpha_n)) = \bigcap_{i \leqslant n} P_{\alpha_i}$. For a convergent sequence $\{x_j\}$ of X with $x_j \to x$, since $f(V(\alpha_1, \ldots, \alpha_n))$ is a sequential neighborhood of x in X, there exists $i(n) \in \mathbb{N}$ such that if $i \geqslant i(n)$, then $x_i \in f(V(\alpha_1, \ldots, \alpha_n))$. Thus $f^{-1}(x_i) \cap V(\alpha_1, \ldots, \alpha_n) \neq \emptyset$. We may assume $1 < i(n) < i(n+1)$. For each $j \in \mathbb{N}$, if $j < i(1)$, we take $\beta_j \in f^{-1}(x_j)$; if $i(n) \leqslant j < i(n+1)$, we take $\beta_j \in f^{-1}(x_j) \cap V(\alpha_1, \ldots, \alpha_n)$ for $n \in \mathbb{N}$. Then it is easy to show that the sequence $\{\beta_j\}$ converges to β in M. Hence f is 1-sequence-covering.

From (a)–(d) above, X is a 1-sequence-covering and σ-image of a metric space. □

By virtue of Definition 1.2 it is easy to check the following lemma (or see [3]).

Lemma 2.2. *Assume $\mathscr{P}$ is a cover of X.*

(1) *If $\mathscr{P}$ is a weak base for X, then $\mathscr{P}$ is a sequential neighborhood network for X.*
(2) *If $\mathscr{P}$ is a sequential neighborhood network of a sequential space, then $\mathscr{P}$ is a weak base for X.*

Lemma 2.3 [2]. *Every g-first countable space is a sequential space.*

Lemma 2.3 [5]. *Assume $f\colon X \to Y$ is a sequentially quotient mapping. If Y is a sequential space, then f is a quotient mapping.*

Theorem 2.4. *The following conditions are equivalent for a space X.*

(1) *X is a g-metrizable space.*
(2) *X is a 1-sequence-covering and quotient σ-image of a metric space.*

P r o o f. (1) ⇒ (2). Suppose X is a g-metrizable space, then X has a σ-locally finite weak base. By Lemma 2.2 and Lemma 2.3, X is a sequential space, and X has a σ-locally finite sequential neighborhood network. By Lemma 2.1, X is a 1-sequence-covering and σ-image of a metric space. In view of Lemma 2.3, the 1-sequence-covering mapping is a quotient mapping.

(2) ⇒ (1). Suppose X is a 1-sequence-covering and quotient σ-image of a metric space. Then X is a sequential space because f is a quotient mapping. By Lemma 2.1, X has a σ-locally finite sequential neighborhood network $\mathscr{P}$. In view of Lemma 2.2, $\mathscr{P}$ is σ-locally finite weak base for X. Hence X is a g-metrizable space. □

By Theorem 14 in [8], we have

Corollary 2.5. *Every 1-sequence-covering and pseudo-open σ-mapping preserves metrizability.*

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