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Moharram A. Khan

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COMMUTATIVITY OF RINGS WITH CONSTRAINTS INVOLVING A SUBSET

Moharram A. Khan, Jeddah

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Abstract. Suppose that R is an associative ring with identity 1, $J(R)$ the Jacobson radical of R, and $N(R)$ the set of nilpotent elements of R. Let $m \geqslant 1$ be a fixed positive integer and R an m-torsion-free ring with identity 1. The main result of the present paper asserts that R is commutative if R satisfies both the conditions

(i) $[x^m, y^m] = 0$ for all $x, y \in R \setminus J(R)$ and

(ii) $[(xy)^m + y^m x^m, x] = 0 = [(yx)^m + x^m y^m, x]$, for all $x, y \in R \setminus J(R)$.

This result is also valid if (i) and (ii) are replaced by (i)$'$ $[x^m, y^m] = 0$ for all $x, y \in R \setminus N(R)$ and (ii)$'$ $[(xy)^m + y^m x^m, x] = 0 = [(yx)^m + x^m y^m, x]$ for all $x, y \in R \setminus N(R)$.

Other similar commutativity theorems are also discussed.

Keywords: commutativity theorems, Jacobson radicals, nilpotent elements, periodic rings, torsion-free rings

MSC 2000: 16U80, 16U99

1. Introduction

Throughout, R will denote an associative ring, $Z(R)$ the centre of R, $U(R)$ the unit of R, $J(R)$ the Jacobson radical of R, $N(R)$ the set of nilpotent elements of R and $C(R)$ the commutator ideal of R. The symbol $[x, y] = xy - yx$ stands for the commutator in R where $x, y \in R$. Let $m \geqslant 1$ be a fixed positive integer and B a non-empty subset of R. For all $x, y \in B$ we consider the following ring properties.

$C_1(m, B)$ $\quad [x^m, y^m] = 0.$

$C_2(m, B)$ $\quad (xy)^m = x^m y^m.$

$C_3(m, B)$ $\quad (xy)^m - x^m y^m \in Z(R).$

$C_4(m,B)$	$(xy)^m = y^m x^m$.
$C_5(m,B)$	$(xy)^m - y^m x^m \in Z(R)$.
$C_6(m,B)$	$[(xy)^m + y^m x^m, x] = 0 = [(yx)^m + x^m y^m, x]$.
$C_7(m,B)$	$(yx)^m x^m - x^m (xy)^m \in Z(R)$.
$Q(m)$	For all $x, y \in R$, $m[x,y] = 0$ implies that $[x,y] = 0$.

A well-known theorem of Herstein [8] asserts that a ring R which possesses the property $C_2(m,R)$ must have a nil commutator ideal. In a recent paper [1], the author jointly with Abujabal, Bell and Khan proved that R is commutative if R satisfies $C_5(m,R)$. In their paper [4], Abu-Khuzam et al. established commutativity of the m-torsion-free ring R with identity 1 satisfying $C_1(m,R)$ and $C_3(m+1,R)$. Motivated by these observations, it is natural to ask a question: What can we say about the commutativity of R if the property $C_3(m+1,R)$ in the above result is replaced by $C_5(m+1,R)$?

The object of the present paper, in Section 2, is to establish that an m-torsion-free ring R with identity 1 satisfying $C_1(m, R\setminus J(R))$ and $C_6(m, R\setminus J(R))$ must be commutative. Further, it is shown that this result is also true for the case when the properties $C_1(m, R\setminus J(R))$ and $C_6(m, R\setminus J(R))$ are replaced by $C_1(m, R\setminus N(R))$ and $C_6(m, R\setminus N(R))$. In Section 3, commutativity of rings possessing one of the properties $C_7(m, R\setminus J(R))$ and $C_7(m, R\setminus N(R))$ has been studied. At the end of the sections counterexamples are given which show that the hypotheses are not altogether superfluous. Our theorems generalise the results obtained in [1], [3], [4], [6], [7], [10], [14].

2. Commutativity theorems for rings with 1

We begin with

Lemma 2.1 [12, p. 221]**.** *If $[x,y]$ commutes with x, then $[x^n,y] = nx^{n-1}[x,y]$ for all positive integers $n \geqslant 1$.*

Lemma 2.2 [13, Theorem 1]**.** *Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, x_3, \ldots, x_n$ with integer coefficients. Then the following statements are equivalent:*

(i) *For any ring R satisfying the polynomial identity $f = 0$, $C(R)$ is nil.*

(ii) *For every prime p, $(G(F(p))_2$ fails to satisfy $f = 0$.*

Lemma 2.3 [9, Theorem]. *Let R be a ring in which for given $x, y \in R$ there exist integers $m = m(x, y) \geqslant 1$, $n = n(x, y) \geqslant 1$ such that $[x^m, y^n] = 0$. Then the commutator ideal of R is nil.*

Lemma 2.4 [7, Lemma 4]. *Let R be an m-torsion-free ring with unity 1 satisfying $C_1(m, R)$. Then*
(i) $a \in N(R)$, $x \in R$ *imply* $[a, x^m] = 0$;
(ii) $a \in N(R)$, $b \in N(R)$ *imply* $[a, b] = 0$.

Lemma 2.5 [14, Lemma]. *Let R be a ring with unity 1. If $dx^m[x, y] = 0$ and $d(x + 1)^m[x, y] = 0$ for some integers $m \geqslant 1$ and $d \geqslant 1$, then $d[x, y] = 0$ for all $x, y \in R$.*

Lemma 2.6 [11, Theorem 1]. *Let R be a ring without non-zero nil right ideal. Suppose that, given $x, y \in R$, there exist positive integers $s = s(x, y) \geqslant 1$, $m = m(x, y) \geqslant 1$ and $t = t(x, y) \geqslant 1$ such that $[x^s, [x^t, y^m]] = 0$. Then R is commutative.*

Now we prove the following results which are called steps.

Step 2.1. *Let R be a ring with identity 1 satisfying $C_1(m, R)$, $C_7(m, R)$ and $Q(m)$. Then R is commutative.*

Proof. First, we claim that $[a, x^m] = 0$ for all $x \in R$ and $a \in N(R)$. Since a is nilpotent, there exists a minimal positive integer t such that $[a^k, x^m] = 0$ for all integers $k \geqslant t$. Let $m = 2$. Then

$$0 = [(1 + a^{t-1})^m, x^m] = [1 + ma^{t-1} + \ldots + a^{(t-1)m}, x^m] = m[a^{t-1}, x^m].$$

By the property $Q(m)$, this gives $[a^{t-1}, x^m] = 0$, which contradicts the minimality of m. Hence $t = 1$, and $[a, x^m] = 0$.

In view of [10, Lemma 10], there exists a positive integer s such that $s[x^m, y] = 0$. Since $C(R) \subseteq N(R)$ by virtue of [9, Theorem], it follows from the above that $[x^m, [x^m, y]] = 0$. Thus by Lemma 2.1 we have

$$[x^{ms}, y] = sx^{m(s-1)}[x^m, y] = 0.$$

Further, let c, d be arbitrary elements of R. Then replacing x by c and y by $c^{ms-1}d$ in $C_5(m, R)$, and combining it with the above result, we get

$$[(c^{ms-1}dc)^m c^m - c^m(c^{ms})d^m, c] = 0$$

or

$$[(c^{ms-1+ms(m-1)}d^m c)c^m - c^m(c^{m^2 s}d^m), c] = 0,$$

that is

$$[(c^{m^2s-1}d^mc)c^m - c^m(c^{m^2s}d^m), c] = 0.$$

After a simplification, this gives

$$c^{ms-1}[c,[c^{m+1},d^m]] = 0.$$

Now, using the commutator identity $[xy,z] = x[y,z] + [x,z]y$ for all $x,y,z \in R$ and $C(m,R)$, we have

$$c^{m^2s-1}[c, c^m[c,d^m]] = 0$$

or

$$c^{m^2s-1+m}[c,[c,d^m]] = 0.$$

Therefore, by Lemma 2.5, $[c,[c,d^m]] = 0$, and then by Lemma 2.1 we obtain $0 = [c^m,d^m] = mc^{m-1}[c,d^m]$. Again by Lemma 2.5, $m[c,d^m] = 0$. Using the property $Q(m)$, we conclude that $[c,d^m] = 0$. Hence commutativity of R follows by [9, Theorem]. □

Step 2.2. *Let R be a ring. Suppose that $N(R)$ is commutative and assume that $a^2 = 0$ and $r \in R$ imply that $ra \in N(R)$. Then $N(R)$ is an ideal.*

Proof. Let $a \in N(R)$. Since $N(R)$ is commutative, $(N(R),+)$ is a subgroup of R. By induction on n we show that

$$\text{if } a^n = 0 \text{ and } r \in R, \text{ then } (ra)^n = (ar)^n = 0.$$

Let $a^2 = 0$. Then $ra \in N(R)$ and in view of the hypothesis we have $ara = ra^2 = 0$ and hence

$$(ar)^2 = (ra)^2 = 0.$$

Suppose that $b^t = 0$, $t < n$ implies that $(rb)^t = (br)^t = 0$ for all $r \in R$, and let $a^n = 0$, $n \geqslant 3$. Hence $a^2, \dots, a^{n-1}$ all have powers lower than the n-th power equal to zero, thus $ra^2, \dots, ra^{n-1}, a^2r, a^3r, \dots, a^{n-1}r \in N(R)$ for all $r \in R$. We have $(ara)^{n-1} = a(ra^2)^{n-2}ra = ra^3(ra^2)^{n-3}ra = r^2a^5(ra^2)^{n-4}ra = \ldots = r^{n-2}a^{2n-3}ra = r^{2n-2}a^{2n-2}r = 0$, because $2n-2 \geqslant n$. Hence $(ara)^{n-1} = 0$, so $rara \in N(R)$ by virtue of the induction hypothesis. Hence, $ra \in N(R)$. Since $N(R)$ is commutative, clearly

$$(ra)^n = (ar)^n = 0.$$

This implies that $ar = ra \in N(R)$, that is N(R) is an ideal. □

Step 2.3. *Let R be a ring with identity 1, and let $m \geqslant 1$ be a fixed positive integer. If R satisfies $C_1(m,R)$, $C_6(m,R)$ and $Q(m)$, then R is commutative.*

Proof. By hypothesis, we have $[(xy)^m + y^m x^m, x] = 0$ and $[(yx)^m + x^m y^m, x] = 0$ for all $x, y \in R$. The first property can be written as

$$x\{(xy)^m - (yx)^m\} = y^m x^{m+1} - xy^m x^m \text{ for all } x, y \in R, \tag{2.1}$$

while the latter gives

$$\{(xy)^m - (yx)^m\}x = x^m y^m x - x^{m+1} y^m \text{ for all } x, y \in R. \tag{2.2}$$

Multiplying (2.1) by x on the right and (2.2) by x on the left, and then subtracting we get

$$[x, [x^{m+1}, y^m]] = 0 \text{ for all } x, y \in R. \tag{2.3}$$

But $[x^{m+1}, y^m] = x^m[x, y^m] + [x^m, y^m]x$ in view of the property $C_1(m,R)$ and (2.3) yields that $x^m[x, [x, y^m]] = 0$. Now, replace x by $1 + x$ and use Lemma 2.5 to get

$$[x, [x, y^m]] = 0 \text{ for all } x, y \in R. \tag{2.4}$$

From the hypothesis $C_1(m,R)$ and by Lemma 2.3 the commutator ideal is nil. It follows that $N(R)$ forms an ideal. In view of Lemma 2.4 (ii), $N(R)$ is a commutative ideal. This implies that $(N(R))^2 \subseteq Z(R)$. Next, for any $a \in N(R)$, replace y by $1 + a$ in (2.4) and use $Q(m)$ to get

$$[x, [x, a]] = 0 \text{ for all } x \in R \text{ and } a \in N(R). \tag{2.5}$$

From Lemma 2.4 (i) we have

$$[a, x^m] = 0 \text{ for all } x \in R \text{ and } a \in N(R). \tag{2.6}$$

Using (2.5) and Lemma 2.1 together with (2.6), we get

$$mx^{m-1}[a, x] = 0.$$

Replacing x by $x + 1$ and using Lemma 2.5 together with $Q(m)$, we get $[a, x] = 0$ for all $x \in R$ and $a \in N(R)$. But then $C(R) \subseteq N(R)$, and thus

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{2.7}$$

Next, Lemma 2.1 and $C_1(m,R)$ yield that $mx^{m-1}[x, y^m] = [x^m, y^m] = 0$ for all $x, y \in R$. Again using Lemma 2.5 and $Q(m)$, we get $[x, y^m] = 0$ for all $x, y \in R$. Similarly, we have $my^{m-1}[x, y] = [x, y^m] = 0$ and also $[x, y] = 0$ for all $x, y \in R$. Hence R is commutative. □

Step 2.4. *Suppose that R is a semisimple ring in which for every $x, y \in R$ there exists an integer $m = m(x, y) \geqslant 1$ such that $[(xy)^m + y^m x^m, x] = 0 = [(yx)^m + x^m y^m, x]$. Then R is commutative.*

Proof. First observe that the hypothesis is inherited by all subrings and all homomorphic images of R. Note also that no complete matrix ring D_t over a division ring D $(t > 1)$ satisfies our hypothesis if we take $x = e_{22}, y = e_{22} + e_{21}$. By these facts and the structure theory of rings we can assume that R is a division ring. The proof of (2.3) is still true in the present situation, so $[x, [x^{m+1}, y^m]] = 0$ for all $x, y \in R$ and for some $m = m(x, y) \geqslant 1$. By Lemma 2.6 we get the required result. □

Step 2.5. *Suppose that R is a semisimple ring in which for every $x, y \in R$ there exists an integer $m = m(x, y) \geqslant 1$ such that $(yx)^m x^m - x^m (xy)^m \in Z(R)$. Then R is commutative.*

Proof. Keeping the proof of Step 2.4 in mind, we assume that R is a division ring. Let x, y be non-zero elements in R. Then there exists an integer $m = m(x, x^{-1}y) \geqslant 1$ such that $(x^{-1}yx)^m x^m - x^m (xx^{-1}y)^m \in Z(R)$. This implies that $[x, [x^{m+1}, y^m]] = 0$. By Lemma 2.6, this gives the required result. □

Theorem 2.1. *Let $m \geqslant 1$ be a fixed positive integer, and let R be a ring with identity 1, satisfying $Q(m)$. Suppose, further, that R satisfies $C_1(m, R \setminus J(R))$ and $C_6(m, R \setminus J(R))$. Then R is commutative.*

Proof. Suppose that u, v are units in R. Since the proof of (2.4) in Step 2.3 holds, we get

(2.8) $$[u, [u, v^m]] = 0 \text{ for all } u, v \in u(R).$$

By the property $C_1(m, R \setminus J(R))$, we find $[u^m, v^m] = 0$. In view of (2.8) and Lemma 2.1, we obtain $mu^{m-1}[u, v^m] = 0$. This implies that

(2.9) $$[u, v^m] = 0 \text{ for all } u, v \in U(R).$$

Let $a \in N(R)$. Then there exists a minimal positive integer l such that

(2.10) $$[u, a^n] = 0 \text{ for all } n \geqslant l \text{ and } u \in U(R).$$

Let $l > 1$. Then $1 + a^{l-1} \in U(R)$, and (2.9) yields that $[u, (1 + a^{l-1})^m] = 0$. Next, by (2.10), one gets $m[u, a^{l-1}] = 0$, and by the property $Q(m)$, we get $[u, a^{l-1}] = 0$, which contradicts the minimality of l in (2.10); thus $l = 1$. Therefore, in view of (2.10), we get

$$[u, a] = 0 \text{ for all } u \in U(R) \text{ and } a \in N(R). \tag{2.11}$$

Let $j_1, j_2 \in J(R)$. Then, by (2.9), we have

$$[1 + j_1, (1 + j_2)^m] = 0 \text{ for all } j_1, j_2 \in J(R). \tag{2.12}$$

By Step 2.4, a semisimple ring satisfying $C_6(m, R)$ is commutative and hence by our assumption $R/J(R)$ is commutative, so $C(R) \subseteq J(R)$. Further, we claim that $C(R) \subseteq N(R)$. Choose arbitrary elements $x_1, y_1, x_2, y_2, x_3, y_3$ of R, and let $c_1 = [x_1, y_1]$, $c_2 = [x_2, y_2]$ and $c_3 = [x_3, y_3]$ be any commutators. In view of (2.12), c_1, c_2, c_3 are all in $J(R)$, so $(1 + c_1 + c_2 + c_1c_2)$ and $(1 + c_3)$ are in $U(R)$ and hence are not in $J(R)$. By hypothesis, we can write

$$[1 + c_3, (1 + c_1 + c_2 + c_1c_2)^m] = 0. \tag{2.13}$$

Observe that (2.13) is a polynomial identity which is satisfied by all elements of R. But (2.13) is not satisfied by any 2×2 matrix ring over $GF(p)$ with a prime p, if we take $c_1 = [e_{11}, e_{11} + e_{12}]$, $c_2 = [e_{11} + e_{12}, e_{21}]$ and $c_3 = c_1$. Hence by Lemma 2.2, $C(R) \subseteq N(R)$ and by (2.11) we obtain

$$[1 + j_2, [1 + j_1, 1 + j_2]] = 0 \text{ for all } j_1, j_2 \in J(R). \tag{2.14}$$

By virtue of (2.12) and (2.14), Lemma 2.1 gives that $m(1+j_2)^{m-1}[1+j_1, 1+j_2] = 0$. This implies that $m[j_1, j_2] = 0$. By the property $Q(m)$ one gets $[j_1, j_2] = 0$ for all $j_1, j_2 \in J(R)$. This implies that $J(R)$ is commutative and $(J(R))^2 \subseteq Z(R)$.

Let $m = 1$. We have $[x, y] = [1 + x, y] = [x, 1 + y] = [1 + x, 1 + y]$. Here, our hypothesis $[x^m, y^m] = 0$ implies that $[x, y] = 0$ for all $x, y \in R$, since $x \in J(R)$ implies that $1 + x \notin J(R)$. This gives the required result.

Let $m > 1$. In this case it suffices to show that $[x^n, y^n] = 0$ and $[(xy)^n + y^nx^n, x] = 0 = [(yx)^n + x^ny^n, x]$ for all $n \geqslant 2$, where $x \in J(R)$ or $y \in J(R)$. Combining these facts together with the properties $C_1(m, R \setminus J(R))$ and $C_6(m, R \setminus J(R))$, we observe that R satisfies $C_1(m, R)$ and $C_6(m, R)$. By Step 2.3, R is commutative. □

Theorem 2.2. *Let $m \geqslant 1$ be a fixed positive integer, and let R be a ring with identity* 1 *satisfying $Q(m)$. Suppose, further, that R satisfies $C_1(m, R \setminus N(R))$ and $C_6(m, R \setminus N(R))$. Then R is commutative.*

P r o o f. Keeping the proof of Theorem 2.1 in mind, it is enough to show that $N(R)$ is an ideal of R and hence it is contained in $J(R)$. Note that the arguments used in the proof of (2.11) are still valid in the present situation, and hence the set $N(R)$ is commutative. Now let $a^2 = 0$, and for $r \in R$ let us assume that $ra \notin N(R)$. Replacing x by ra and y by $1 + a$ in $C_6(m, R \setminus N(R))$ we get

$$[(ra(1 + a))^m + (1 + a)^m (ra)^m, ra] = 0.$$

This implies that

$$[(ra)^m + (1 + a)^m (ra)^m, ra] = a(ra)^{m+1} = 0.$$

That is,

$$(ra)^{m+2} = 0.$$

Since $a^2 = 0$ and $r \in R$ imply $ra \in N(R)$ and in view of Step 2.2, one gets the required result. □

Theorem 2.3. *Let $m \geqslant 1$ be a fixed positive integer and let R be a ring with identity* 1 *satisfying $Q(m)$. Suppose, further, that R satisfies $C_1(m, R \setminus J(R))$ and $C_7(m, R \setminus J(R))$. Then R is commutative.*

P r o o f. Let u, v be units in R. Then by hypothesis $C_7(m, R \setminus J(R))$, we have

$$(u^{-1}vu)^m u^m - u^m (uu^{-1}v)^m \in Z(R)$$

or

$$[u, [u^{m+1}, v^m]] = 0.$$

This implies that

$$[u, [u, v^m]] = 0 \text{ for all } u, v \in U(R).$$

Here the arguments used in the proof of (2.9) and (2.11) are still valid, and hence

(2.15) $$[u, v^m] = 0 \text{ for all } u, v \in U(R).$$

Also

(2.16) $$[u, a] = 0 \text{ for all } u \in U(R) \text{ and } a \in N(R).$$

Let $j_1, j_2 \in J(R)$. Then in view of (2.15), we get

$$[1 + j_1, (1 + j_2)^m] = 0 \text{ for all } j_1, j_2 \in J(R).$$

Arguments similar to those used to obtain (2.14) from (2.12) yield that $C(R) \subseteq N(R)$, and by (2.16) we have

$$[1 + j_1, 1 + j_2], 1 + j_2]] = 0 \text{ for all } j_1, j_2 \in J(R).$$

Now by Lemma 2.1 we get $[j_1, j_2] = 0$ for all $j_1, j_2 \in J(R)$. Hence $J(R)$ is commutative and

$$(J(R))^2 \subseteq Z(R)$$

Let $m = 1$. Then we use arguments similar to those used in the case of Theorem 2.1.

Let $m > 1$. Clearly, by the induction hypothesis, we have $[x^n, y^n] = 0$ and $(yx)^n(x)^n - x^n(xy)^n \in Z(R)$ for all $n \geqslant 2$, provided $x \in J(R)$ or $y \in J(R)$. Hence by the properties $C_1(m, R \setminus J(R))$ and $C_7(m, R \setminus J(R))$ we observe that R satisfies $C_1(m, R)$ and $C_7(m, R)$ for $m > 1$. Now, by Step 2.1, R is commutative. □

Theorem 2.4. *Let $m \geqslant 1$ be a fixed positive integer and let R be a ring with identity 1 satisfying $Q(m)$. Suppose, further, that R satisfies $C_1(m, R \setminus N(R))$ and $C_7(m, R \setminus N(R))$. Then R is commutative.*

Proof. Let R be a ring with 1 satisfying $Q(m)$, $C_1(m, R \setminus N(R))$ and $C_7(m, R \setminus N(R))$. Then we observe that the proof of (2.16) is still valid in the present situation, and hence $N(R)$ is commutative. Let $a^2 = 0$ and for $r \in R$ assume that $ra \notin N(R)$. Then by $C_7(m, R \setminus N(R))$ we have

$$((1 + a)ra)^m(ra)^m - (ra)^m(ra(1 + a))^m \in Z(R).$$

This implies that

$$[((1 + a)ra)^m(ra)^m - (ra)^m(ra(1 + a))^m, ra] = 0.$$

That is,

$$(ar)^{2m+2} = 0.$$

Hence $a^2 = 0$ and $r \in R$ imply $ra \in N(R)$, and by Step 2.2, $N(R)$ is an ideal and hence it is contained in $J(R)$. Thus R is commutative by Theorem 2.3. □

Now, we provide some counterexamples which show that all the hypotheses in our theorems are individually essential.

Remark 2.1. The ring of 3×3 strictly upper (or lower) triangular matrices over $\mathbb{Z}$, the ring of integers, shows that the existence of unity 1 in the hypotheses of Theorems 2.1–2.4 is necessary.

Next we provide an example to show that the property $Q(m)$ in the hypotheses of Theorems 2.1 and 2.2 is not superfluous even if the properties $[x^m, y^m] = 0$ and $[(xy)^m + y^m x^m, x] = 0 = [(yx)^m + x^m y^m, x]$ hold for all $x, y \in R$.

Example 2.1. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \middle| \, \alpha, \beta, \gamma, \delta \in GF(3) \right\}$.

Clearly R satisfies $[x^3, y^3] = 0$ and $(xy)^3 = y^3 x^3$ for all $x, y \in R$. Hence R satisfies all the hypotheses except $Q(m)$ when $m = 3$.

Example 2.2. Consider R as in Example 2.1, but with the elements in $GF(2)$. Obviously, R satisfies $[x^2, y^2] = 0$ and $(yx)^2 x^2 - x^2 (xy)^2 \in Z(R)$ for all $x, y \in R$. This shows that for $m = 2$ the property $Q(m)$ cannot be omitted from the hypotheses of Theorems 2.3 and 2.4.

Remark 2.2. The ring R from Example 2.1 satisfies the identity $(xy)^2 = y^2 x^2$. Clearly R satisfies $C_6(2, R)$ and $Q(2)$. This demonstrates that the property $C_1(m, R \setminus J(R))$ ($C_1(m, R \setminus N(R))$) is essential in the hypotheses of Theorem 2.1 (Theorem 2.2).

Remark 2.3. Clearly the ring R from Example 2.1 satisfies $(yx)^4 x^4 - x^4 (xy)^4 \in Z(R)$ and $Q(4)$. Hence R satisfies all the hypotheses of Theorem 2.3 (Theorem 2.4) except $C_1(4, R \setminus J(R))$ ($C_1(4, R \setminus N(R))$).

Remark 2.4. The following example demonstrates that a ring R with identity 1 satisfying $C_1(m, R)$ and $Q(m)$ need not be commutative.

Example 2.3. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \middle| \, \alpha, \beta, \gamma \in GF(4) \right\}$.

Clearly the non-commutative ring R satisfies $C_1(3, R)$ and $Q(3)$. This shows the necessity of the property $C_6(m, R \setminus J(R))$ ($C_7(m, R \setminus J(R))$) in Theorem 2.1 (Theorem 2.3).

3. A commutativity theorem for periodic rings

In what follows, a ring R is called periodic if for each $x \in R$ there exist distinct positive integers p, q such that $x^p = x^q$. Recently Abu-Khuzam and Yaqub [3, Theorem 3] proved that a periodic ring R is commutative if R satisfies $C_5(m, R \setminus N(R))$. Also they established that if $N(R)$ is commutative in a periodic ring R and R is an $m(m+1)$-torsion-free ring satisfying $C_5(m, R \setminus N(R))$, then R is commutative. It is natural to ask a question: Is the above result valid if the property $C_5(m, R \setminus N(R))$ is replaced by $C_7(m, R \setminus N(R))$? Now we provide an affirmative answer to this question:

Theorem 3.1. *Let $m \geqslant 1$ be a fixed positive integer and let R be a periodic ring satisfying $Q(m(m+1))$ and $C_7(m, R \setminus N(R))$. Suppose, further, that $N(R)$ is commutative. Then R is commutative.*

Lemma 3.1 [2]. *Let R be a periodic ring such that $N(R)$ is commutative. If for each $a \in N(R)$ and $x \in R$ there exists an integer $m = m(x, a) \geqslant 1$ such that $[x^m [x^m, a]] = 0$ and $[x^{m+1}, [x^{m+1}, a]] = 0$, then R is commutative. In particular: if R is a periodic ring such that $N(R)$ is commutative and $[x, [x, a]] = 0$ for all $a \in N(R)$, $x \in R$, then R is commutative.*

Lemma 3.2 [5]. *Let R be a periodic ring such that $N(R)$ is commutative. Then the commutator ideal of R is nil, and $N(R)$ forms an ideal.*

Lemma 3.3 [6]. *Let R be a periodic ring and let $f\colon R \to S$ be a homomorphism of R onto S. Then the nilpotents of S coincide with $f(N(R))$, where $N(R)$ is the set of nilpotents of R.*

Proof of Theorem 3.1. Since R is periodic and $N(R)$ is commutative, Lemma 3.2 yields that the commutator ideal $C(R)$ of R is nil; that is $C(R) \subseteq N(R)$ and $N(R)$ forms an ideal of R. But $N(R)$ is commutative, and also $(N(R))^2 \subseteq Z(R)$.

We break the proof into two cases.

Case 1. Let R have identity 1 ($1 \in R$). Suppose that $a \in N(R)$ and $b \in R \setminus N(R)$. Then by hypothesis $C_7(m, R \setminus N(R))$, we can write

(3.1) $b^m(1+a)^m - (1+a)^{m+1} b^m (1+a)^{-1} \in Z(R)$ for all $a \in N(R)$, $b \in R \setminus N(R)$.

This implies that

$$\{b^m(1+a)^m - (1+a)^{m+1}b^m(1+a)^{-1}\}(1+a)$$
$$= (1+a)\{b^m(1+a)^m - (1+a)^{m+1}b^m(1+a)^{-1}\}$$

or

$$b^m(1+a)^{m+1} - (1+a)^{m+1}b^m = (1+a)\{b^m(1+a)^m - (1+a)^{m+1}b^m(1+a)^{-1}\}.$$

Using the binomial expansion and the condition $(N(R))^2 \subseteq Z(R)$, one gets

(3.2) $$(m+1)(b^m a - ab^m) = (1+a)\{b^m(1+a)^m - (1+a)^{m+1}b^m(1+a)^{-1}\}.$$

But $N(R)$ is a commutative ideal, $(1+a)(b^m a - ab^m) = b^m a - ab^m$, and by (3.2) we have

$$(1+a)(m+1)(b^m a - ab^m) = (1+a)\{(b)^m(1+a)^m - (1+a)^{m+1}b^m(1+a)^{-1}\}.$$

Since $a \in N(R)$, $1+a \in U(R)$ and by (3.1) this gives that

$$(m+1)(b^m a - ab^m) = \{b^m(1+a)^m - (1+a)^{m+1}b^m(1+a)^{-1}\} \in Z(R).$$

This implies that $(m+1)[b^m, a] \in Z(R)$. Using the property $Q(m(m+1))$ we get

(3.3) $$[b^m, a] \in Z(R) \text{ for all } a \in N(R),\ b \in R \setminus N(R).$$

Now since $N(R)$ is commutative, (3.3) implies that

(3.4) $$[b^m, a] \in Z(R) \text{ for all } a \in N(R),\ b \in R.$$

Next, let $x_1, x_2, \dots, x_n \in R$. Then $R \setminus C(R)$ is commutative; so, by Lemma 3.2,

$$(x_1 \dots x_n)^m - x_1^m \dots x_n^m \in C(R) \subseteq N(R).$$

Therefore $N(R)$ is commutative, which yields that

(3.5) $$[(x_1 \dots x_n)^m, a] = [x_1^m \dots x_n^m] \text{ for all } a \in N(R).$$

Combining (3.4) and (3.5), we get

(3.6) $$[x_1^m \dots x_n^m, a] \in Z(R) \text{ for all } a \in N(R),\ x_1 \dots x_n \in R \text{ and } n \geqslant 1.$$

Let S be the subring generated by the m-th powers of the elements of R. Then by (3.6) we have

$$(3.7) \qquad [x,a] \in Z(S) \text{ for all } a \in N(S), \ x \in S,$$

where $Z(S)$ and $N(S)$ represent the centre of S and the set of nilpotent elements of S, respectively. Combining the facts that S is periodic, $N(S)$ is commutative, and (3.7), Lemma 3.1 shows that S is commutative, and hence $[x^m, y^m] = 0$ for all $x, y \in R$. This implies that R satisfies $C_1(m, R)$. But R also satisfies $Q(m)$ and $C_7(m, R \setminus N(R))$, and by Theorem 2.4 one gets the required result.

Case 2. Let R have no identity 1; $1 \notin R$. First we prove two facts.

Fact 1. The idempotents of R are central. Let $e^2 = e \in R$ and $r \in R$. Replacing x by e and y by $e + er - ere$ in the hypothesis $C_7(m, Z(R))$, we get

$$((e + er - ere)e)^m e^m - e^m(e(e + er - ere))^m \in Z(R).$$

This implies that $ere - er \in Z(R)$. Thus

$$ere - er = e(ere - er) = (ere - er)e = 0,$$

or

$$ere = er.$$

Similarly, if $x = e$ and $y = e + re - ere$, we obtain

$$ere = re.$$

Thus $er = re$ for all $r \in R$ and the result follows immediately.

Fact 2. Let $f\colon R \to S$ be a homomorphism of R onto S. Then the nilpotents of S coincide with $f(N(R))$, where $N(R)$ is the set of nilpotents of R. This has been stated in Lemma 3.2.

To complete the proof of Theorem 3.1, first note that R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i ($i \in \Gamma$). Let $f_i\colon R \to R_i$ be the natural homomorphism of R onto R_i, and let $x_i \in R_i$ and $f_i(x) = x_i$, $x \in R$. Since R is periodic, $x^p = x^q$ for some integers $p > q > 0$, and hence

$$e = x^{(p-q)q} \text{ is an idempotent.}$$

By Fact 1, e is central in R and hence $f_i(e)$ is central idempotent of R_i. Since R_i is subdirectly irreducible, so $f_i(e) = 0$ or $f_i(e) = 1_i$ provided $1_i \in R_i$.

Next, two claims arise for R_i.

Claim I. *Let R_i have no identity; $1_i \notin R_i$. Then $f_i(e) = 0$ and by (3.7) we have $x_i^{(p-q)q} = 0$. Hence R_i is nil and by Fact 2, $R_i = f_i(N(R))$. Since by hypothesis $N(R)$ is commutative, R_i is commutative as well.*

Claim II. *Let R_i have identity 1_i. Note that R_i need not be $Q(m(m+1))$-torsion-free. Let $f_i(e_1) = e_1$, $e_1 \in R$, where R is periodic, so we choose integers $p > q > 0$ such that $e_1^p = e_1^q$. Suppose that $e = e_1^{(p-q)q}$. Then e is an idempotent and, moreover,*

$$f_i(e) = 1_i^{(p-q)q} = 1_i.$$

Thus e is central by Fact 1, and hence e is a non-zero central idempotent of R. Hence eR is a ring with identity e. Obviously, eR inherits all the hypotheses of the ground ring R including the property $Q(m(m+1))$. It follows by the first part of the proof that eR is commutative, and hence $[ex, ey] = 0$ for all $x, y \in R$. Since $f_i(e) = 1_i$, this implies that $[f_i(x), f_i(y)] = 0$ for all $x, y \in R$, and then $R_i = f_i(R)$ is commutative. Hence the ground ring R is also commutative.

Finally, we provide some counterexamples to show that no hypotheses in Theorem 3.1 are superfluous.

Remark 3.1. The following example demonstrates that one cannot drop the hypothesis that $N(R)$ is commutative in Theorem 3.1.

Example 3.1. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \middle| \, \alpha, \beta, \gamma, \delta \in GF(3) \right\}$.

Clearly R satisfies all the hypotheses of Theorem 3.1 except the condition that $N(R)$ is commutative when $m = 4$.

Remark 3.2. The following example strengthens the necessity of the property $C_7(m, R \setminus N(R))$ in the hypotheses of Theorem 3.1.

Example 3.2. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \middle| \, \alpha, \beta, \gamma \in GF(5) \right\}$.

Obviously, the non-commutative ring R satisfies all the hypotheses of Theorem 3.1 except $C_7(m, R \setminus N(R))$ when $m = 2$.

Remark 3.3. The following example shows that the hypothesis $Q(m(m+1))$ in Theorem 3.1 is not superfluous.

Example 3.3. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \middle| \, \alpha, \beta, \gamma \in GF(3) \right\}$.

Clearly the non-commutative ring R satisfies all the hypotheses of Theorem 3.1 except $Q(m(m+1))$.

Remark 3.4. One can ask: Can the property "$Q(m(m+1))$" be replaced by "$Q(m)$" or "$Q(m+1)$" in Theorem 3.1? Example 3.1 shows the following: For $m = 5$, the non-commutative ring R satisfies all the hypotheses of Theorem 3.1 and the commutators in R are 5-torsion-free; for $m = 6$, the non-commutative ring R satisfies all the hypotheses and the commutators are 6-torsion-free. This shows that the property "$Q(m(m+1))$"cannot be replaced by "$Q(m)$" or "$Q(m+1)$".

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Author's address: Department of Mathematics, King Abdulaziz University, P.O. Box 30356, Jeddah-21477, Saudi Arabia, e-mail: `nassb@hotmail.com`.