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Tadeusz Jankowski

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BOUNDARY VALUE PROBLEMS FOR ODES

Tadeusz Jankowski, Gdańsk

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Abstract. We use the method of quasilinearization to boundary value problems of ordinary differential equations showing that the corresponding monotone iterations converge to the unique solution of our problem and this convergence is quadratic.

Keywords: quasilinearization, monotone iterations, quadratic convergence

MSC 2000: 34A45, 34B99

1. Introduction

In this paper we consider the differential problem

$$(1)\qquad \begin{cases} x'(t) = h(t, x(t)), & t \in J = [0, T],\ T > 0, \\ x(0) = \lambda x(T) + k, \end{cases}$$

where $h \in C(J \times \mathbb{R}, \mathbb{R})$ and $\lambda, k \in \mathbb{R}$.

It is well known [1], [2] that the method of quasilinearization offers an approach for obtaining approximate solutions to nonlinear differential problems. Recently it has been generalized and extended under less restrictive assumptions. In this paper we apply this metod to boundary value problems of type (1). Note that if $\lambda = 0$, then (1) reduces to the initial value problem for differential equations and this case is considered, for example, in [4], [5], [7], [8]. If $\lambda = 1$ and $k = 0$, then we have a periodic boundary problem considered, for example, in [6], [8], while if $\lambda = -1$ and $k = 0$, then we have an anti-periodic boundary problem discussed in [8], [9].

In this paper we extend some results of [4]–[9] to boundary value problems of type (1) when $\lambda \in \mathbb{R}$ and $k \in \mathbb{R}$ under the assumption that $f + \Delta$ is convex and $g + \Psi$ is concave for some convex function Δ and a concave function Ψ with $h = f + g$.

We show that it is possible to construct monotone sequences that converge to the unique solution of (1) and the convergence is quadratic.

2. Case $\lambda \geqslant 0$

A function $u \in C^1(J, \mathbb{R})$ is said to be a lower solution of problem (1) if

$$\begin{cases} u'(t) \leqslant h(t, u(t)), & t \in J, \\ u(0) \leqslant \lambda u(T) + k, \end{cases}$$

and an upper solution of (1) if the inequalities are reversed.

Let $\Omega = \{u\colon\ y_0(t) \leqslant u \leqslant z_0(t),\ \ t \in J\}$. The notation $h \in C^{0,2}(J \times \mathbb{R}, \mathbb{R})$ means that $h, h_x, h_{xx} \in C(J \times \mathbb{R}, \mathbb{R})$.

We introduce the following assumptions for later use.

(H_1) $h \in C(J \times \mathbb{R}, \mathbb{R}),\ \ \lambda \geqslant 0$,

(H_2) $y_0, z_0 \in C^1(J, \mathbb{R})$ are lower and upper solutions of (1), respectively, and such that $y_0(t) \leqslant z_0(t),\ \ t \in J$,

(H_3) $f, g, \Delta, \Psi \in C^{0,2}(J \times \mathbb{R}, \mathbb{R})$ with $h = f + g$, and moreover

- (a) $F_{xx}(t, u) \geqslant 0,\ \ \Delta_{xx}(t, u) \geqslant 0,\ \ G_{xx}(t, u) \leqslant 0,\ \ \Psi_{xx}(t, u) \leqslant 0$ on $J \times \Omega$ for $F = f + \Delta,\ \ G = g + \Psi$,
- (b) $\int_0^T L(s)\,\mathrm{d}s < -\ln(\lambda)$ for $\lambda > 0$ with $L(s) = F_x(s, z_0(s)) + G_x(s, y_0(s)) - \Delta_x(t, y_0(s)) - \Psi_x(s, z_0(s))$,

(H_4) $h_x \in C(J \times \mathbb{R}, \mathbb{R}),\ \ \lambda H(T) \neq 1$ with $H(t) = \mathrm{e}^{\int_0^t h_x(s, u(s))\,\mathrm{d}s}$ for $u \in \Omega$.

We will formulate some lemmas which are important in our considerations.

Lemma 1. *Let assumptions* H_1, H_4 *hold. Then problem* (1) *has at most one solution on the segment* $[y_0, z_0]$.

P r o o f. Assume that problem (1) has two distinct solutions x and y. Put $p = x - y$, so $p(0) = \lambda p(T)$. Using the mean value theorem we have

$$p'(t) = h(t, x) - h(t, y) = h_x(t, \xi)p(t), \quad t \in J,$$

where ξ is between x and y. Hence $p(t) = H(t)p(0)$, $t \in J$. Moreover, the boundary condition yields $p(0)[1 - \lambda H(T)] = 0$. Since $\lambda H(T) \neq 1$, we have $p(0) = 0$ showing that $p(t) = 0$ on J. It means that $x(t) = y(t)$ on J.

This completes the proof. $\square$

Lemma 2. *Let* $\lambda \geqslant 0$, $a, b \in C(J, \mathbb{R})$. *Assume that*

$$\lambda D(T) \neq 1 \quad \textit{for}\ \ D(t) = \mathrm{e}^{\int_0^t a(s)\,\mathrm{d}s}, \ \ t \in J.$$

Then the problem

$$\begin{cases} y'(t) = a(t)y(t) + b(t), & t \in J, \\ y(0) = \lambda y(T) + k \end{cases}$$

has the unique solution given by

$$y(t) = \frac{D(t)}{1 - \lambda D(T)} \left\{ k + \int_0^T G_0(t,s)\mathrm{e}^{-\int_0^s a(r)\,\mathrm{d}r} b(s)\,\mathrm{d}s \right\}, \quad t \in J$$

with the Green function G_0 *defined by*

$$G_0(t,s) = \begin{cases} 1 & \textit{if}\ 0 \leqslant s \leqslant t \leqslant T, \\ \lambda D(T) & \textit{if}\ t < s \leqslant T. \end{cases}$$

Lemma 3. *Assume that* $a \in C(J, \mathbb{R})$ *and*

(2) $$\int_0^T a(s)\,\mathrm{d}s < -\ln(\lambda) \quad \textit{if}\ \lambda > 0.$$

Let $p \in C^1(J, \mathbb{R})$, *and*

$$\begin{cases} p'(t) \leqslant a(t)p(t), & t \in J, \\ p(0) \leqslant \lambda p(T), & \lambda \geqslant 0. \end{cases}$$

Then $p(t) \leqslant 0$ *on* J.

Proof. Note that

$$p(t) \leqslant B(t)p(0), \quad t \in J \ \text{ with } \ B(t) = \mathrm{e}^{\int_0^t a(s)\,\mathrm{d}s}.$$

If $\lambda = 0$, then $p(0) \leqslant 0$ and hence $p(t) \leqslant 0$ on J. Now, we assume that $\lambda > 0$. Then $p(0) \leqslant \lambda p(T) \leqslant \lambda B(T)p(0)$, so $p(0)[1 - \lambda B(T)] \leqslant 0$. By condition (2), $p(0) \leqslant 0$ showing that $p(t) \leqslant 0$ on J.

This completes the proof. □

Lemma 4. *Let assumptions* H_1, H_2, H_3 *hold. Let* $u, v \in C^1(J, \mathbb{R})$ *be lower and upper solutions of* (1), *respectively, and such that* $u, v \in \Omega$ *and* $u(t) \leqslant v(t)$ *on* J. *Let*

$$(3) \qquad \begin{cases} y'(t) &= h(t,u) + W(t,u,v)[y(t) - u(t)], \quad t \in J, \ \ y(0) = \lambda y(T) + k, \\ z'(t) &= h(t,v) + W(t,u,v)[z(t) - v(t)], \quad t \in J, \ \ z(0) = \lambda z(T) + k \end{cases}$$

with $W(t,u,v) = F_x(t,u) + G_x(t,v) - \Delta_x(t,v) - \Psi_x(t,u)$.

Then

$$(4) \qquad u(t) \leqslant y(t) \leqslant z(t) \leqslant v(t), \quad t \in J,$$

and y, z *are lower and upper solutions of* (1), *respectively.*

Proof. Assumption H_3 yields

$$\lambda \mathrm{e}^{\int_0^T W(s,u,v)\,\mathrm{d}s} \leqslant \lambda \mathrm{e}^{\int_0^T L(s)\,\mathrm{d}s} < 1.$$

This and Lemma 2 show that problems (3) have their unique solutions $y, z \in C^1(J, \mathbb{R})$. Now we are going to show that (4) holds. Put $p = u - y$, so $p(0) \leqslant \lambda p(T)$. Moreover,

$$p'(t) \leqslant h(t,u) - h(t,u) - W(t,u,v)[y(t) - u(t)] = W(t,u,v)p(t), \quad t \in J.$$

Indeed, $W(t,u,v) \leqslant L(t)$, $t \in J$. This and Lemma 3 give $p(t) \leqslant 0$ on J showing that $u(t) \leqslant y(t)$ on J. Similarly, we can show that $z(t) \leqslant v(t)$, $t \in J$. Now, we put $q = y - z$, so $q(0) = \lambda q(T)$. By the mean value theorem and assumption H_3 (a) we obtain

$$\begin{aligned} q'(t) &= h(t,u) - h(t,v) + W(t,u,v)[y(t) - u(t) - z(t) + v(t)] \\ &= h_x(t,\xi)[u(t) - v(t)] + W(t,u,v)[y(t) - u(t) - z(t) + v(t)] \\ &\leqslant W(t,u,v)q(t), \quad t \in J, \end{aligned}$$

where $u(t) < \xi(t) < v(t)$, $t \in J$. This and Lemma 3 prove that $y(t) \leqslant z(t)$ on J, which means that (4) holds.

The proof will be completed if we show that y and z are lower and upper solutions of (1), respectively. Indeed, by the mean value theorem and assumption H_3 (a) we have

$$\begin{aligned} y'(t) &= h(t,u) + W(t,u,v)[y(t) - u(t)] - h(t,y) + h(t,y) \\ &= h(t,y) + W(t,u,v)[y(t) - u(t)] + h_x(t,\xi_1)[u(t) - y(t)] \\ &\leqslant h(t,y) \end{aligned}$$

for $u(t) < \xi_1(t) < y(t)$, $t \in J$, and

$$\begin{aligned} z'(t) &= h(t, v) + W(t, u, v)[z(t) - v(t)] - h(t, z) + h(t, z) \\ &= h(t, z) + W(t, u, v)[z(t) - v(t)] + h_x(t, \xi_2)[v(t) - z(t)] \\ &\geqslant h(t, z), \end{aligned}$$

where $z(t) < \xi_2(t) < v(t)$, $t \in J$.

This completes the proof. □

The main result of this part is

Theorem 1. *Let assumptions* H_1, H_2, H_3 *hold.*

Then there exist monotone sequences $\{y_n\}$, $\{z_n\}$ *which converge monotonically and uniformly to the unique solution of problem* (1) *and the convergence is quadratic.*

Proof. Let

$$\begin{aligned} y'_{n+1}(t) &= h(t, y_n) + W(t, y_n, z_n)[y_{n+1}(t) - y_n(t)], \\ & t \in J, \quad y_{n+1}(0) = \lambda y_{n+1}(T) + k, \\ z'_{n+1}(t) &= h(t, z_n) + W(t, y_n, z_n)[z_{n+1}(t) - z_n(t)], \\ & t \in J, \quad z_{n+1}(0) = \lambda z_{n+1}(T) + k, \end{aligned}$$

where W is defined as in Lemma 4. By Lemma 2, y_1 and z_1 are well defined. Moreover, Lemma 4 yields the relation

$$y_0(t) \leqslant y_1(t) \leqslant z_1(t) \leqslant z_0(t), \quad t \in J.$$

Also, by Lemma 4, y_1 and z_1 are lower and upper solutions of (1), respectively. Now using induction argument we can prove that for all n and $t \in J$,

$$y_0(t) \leqslant y_1(t) \leqslant \ldots \leqslant y_{n-1}(t) \leqslant y_n(t) \leqslant z_n(t) \leqslant z_{n-1}(t) \leqslant \ldots \leqslant z_1(t) \leqslant z_0(t).$$

Employing a standard argument (see [3]), we conclude that the sequences $\{y_n\}$, $\{z_n\}$ converge uniformly and monotonically to solutions y and z of (1). This and Lemma 1 yield that problem (1) has a unique solution x, so $y = z = x$.

We shall now prove that the convergence of $\{y_n\}$, $\{z_n\}$ to x is quadratic. To do this, put $p_{n+1} = x - y_{n+1} \geqslant 0$, $q_{n+1} = z_{n+1} - x \geqslant 0$, and note that $p_{n+1}(0) = \lambda p_{n+1}(T)$, $q_{n+1}(0) = \lambda q_{n+1}(T)$.

Using assumption H_3 (a) and the mean value theorem, we get

$$\begin{aligned} p'_{n+1}(t) &= h(t,x) - h(t,y_n) - W(t,y_n,z_n)[y_{n+1}(t) - y_n(t)] \\ &= h_x(t,\xi_1)p_n(t) - W(t,y_n,z_n)[p_n(t) - p_{n+1}(t)] \\ &\leqslant [F_x(t,x) - F_x(t,y_n) + G_x(t,y_n) - G_x(t,z_n) + \Delta_x(t,z_n) - \Delta_x(t,y_n) \\ &\quad + \Psi_x(t,y_n) - \Psi_x(t,x)]p_n(t) + W(t,y_n,z_n)p_{n+1}(t) \\ &= \{F_{xx}(t,\xi_2)p_n(t) - G_{xx}(t,\xi_3)[p_n(t) + q_n(t)] + \Delta_{xx}(t,\xi_4)[q_n(t) + p_n(t)] \\ &\quad - \Psi_{xx}(t,\xi_5)p_n(t)\}p_n(t) + W(t,y_n,z_n)p_{n+1}(t) \\ &\leqslant \{(A_1 + A_4)p_n(t) + (A_2 + A_3)[p_n(t) + q_n(t)]\}p_n(t) + L(t)p_{n+1}(t), \end{aligned}$$

where $y_n < \xi_1, \xi_2, \xi_5 < x$, $y_n < \xi_3, \xi_4 < z_n$ with L defined as in H_3 (b), and

$$|F_{xx}(t,u)| \leqslant A_1,\ \ |G_{xx}(t,u)| \leqslant A_2,\ \ |\Delta_{xx}(t,u)| \leqslant A_3,\ \ |\Psi_{xx}(t,u)| \leqslant A_4 \ \text{ on } J \times \Omega.$$

Hence, we obtain

$$p'_{n+1}(t) \leqslant L(t)p_{n+1}(t) + B, \quad t \in J \tag{5}$$

for $B = \max_{t\in J}[K_1|p_n(t)|^2 + K_2|q_n(t)|^2]$, and

$$K_1 = A_1 + A_4 + \frac{3}{2}(A_2 + A_3), \quad K_2 = \frac{1}{2}(A_2 + A_3).$$

Put $D(t) = \mathrm{e}^{\int_0^t L(s)\,\mathrm{d}s}$, $M(t) = \int_0^t \mathrm{e}^{-\int_0^s L(r)\,\mathrm{d}r}\,\mathrm{d}s$. Note that (5) yields

$$p_{n+1}(t) \leqslant D(t)[p_{n+1}(0) + BM(t)], \quad t \in J, \tag{6}$$

so $p_{n+1}(0) = \lambda p_{n+1}(T) \leqslant \lambda D(T)[p_{n+1}(0) + BM(T)]$. Assume that $\lambda > 0$. By assumption H_3 (b), $\lambda D(T) < 1$, and hence

$$p_{n+1}(0) \leqslant \frac{\lambda}{1 - \lambda D(T)} D(T)BM(T).$$

This yields

$$p_{n+1}(t) \leqslant D(t)\Big[\frac{\lambda}{1 - \lambda D(T)} D(T)BM(T) + BM(t)\Big] \leqslant \frac{D(T)M(T)}{1 - \lambda D(T)} B,$$

and finally

$$\max_{t\in J}|p_{n+1}(t)| \leqslant \frac{D(T)M(T)}{1 - \lambda D(T)}\Big[K_1 \max_{t\in J}|p_n(t)|^2 + K_2 \max_{t\in J}|q_n(t)|^2\Big]. \tag{7}$$

Note that if $\lambda = 0$, then $p_{n+1}(0) = 0$, and (7) holds in this case as well.

By a similar argument, we can get

$$\max_{t\in J} |q_{n+1}(t)| \leqslant \frac{D(T)M(T)}{1-\lambda D(T)} \Big[K_3 \max_{t\in J} |p_n(t)|^2 + K_4 \max_{t\in J} |q_n(t)|^2\Big]$$

with

$$K_3 = \tfrac{1}{2}(A_1 + A_4), \quad K_4 = A_2 + A_3 + \tfrac{3}{2}(A_1 + A_4).$$

This completes the proof. □

Remark 1. Theorem 1 contains some results of [4], [8] if $\lambda = 0$, and also some results of [6], [8] when $\lambda = 1$ and $k = 0$.

3. Case $\lambda < 0$

Functions $u, v \in C^1(J, \mathbb{R})$ are called weakly coupled lower and upper solutions of problem (1) if

$$\begin{cases} u'(t) \leqslant h(t, u(t)), & t \in J, \ \ u(0) \leqslant \lambda v(T) + k, \\ v'(t) \geqslant h(t, v(t)), & t \in J, \ \ v(0) \geqslant \lambda u(T) + k. \end{cases}$$

Lemma 5. *Let $h, h_x \in C(J \times \mathbb{R}, \mathbb{R})$ and $\lambda < 0$. Then problem* (1) *has at most one solution on the segment* $[y_0, z_0]$.

P r o o f. Assume that problem (1) has two distinct solutions x and y. Put $p = x - y$, so $p(0) = \lambda p(T)$. Using the mean value theorem we have

$$p'(t) = h(t, x) - h(t, y) = h_x(t, \xi)p(t), \quad t \in J,$$

where ξ is between x and y. Hence $p(t) = H(t)p(0)$, $t \in J$ with $H(t) = \mathrm{e}^{\int_0^t h_x(s,\xi)\,\mathrm{d}s}$. Moreover, the boundary condition yields $p(0)[1 - \lambda H(T)] = 0$. Since $\lambda < 0$, we have $1 - \lambda H(T) > 0$. It proves that $p(0) = 0$ showing that $p(t) = 0$ on J. It means that $x(t) = y(t)$ on J.

This completes the proof. □

Lemma 6. *Let $\lambda < 0$, $a, b_1, b_2 \in C(J, \mathbb{R})$. Assume that*

$$-\lambda A(T) \neq 1 \quad \text{for } A(t) = \mathrm{e}^{\int_0^t a(s)\,\mathrm{d}s}, \quad t \in J.$$

Then there exists a unique solution $(y, z) \in C^1(J, \mathbb{R}) \times C^1(J, \mathbb{R})$ of the problem

$$\begin{cases} y'(t) = a(t)y(t) + b_1(t), & t \in J, \ y(0) = \lambda z(T) + k, \\ z'(t) = a(t)z(t) + b_2(t), & t \in J, \ z(0) = \lambda y(T) + k. \end{cases}$$

Proof. Let $B_i(t) = \int_0^t \mathrm{e}^{-\int_0^s a(r)\,\mathrm{d}r} b_i(s)\,\mathrm{d}s$, $i = 1, 2$. Note that

$$\begin{cases} y(t) = A(t)[y(0) + B_1(t)], & t \in J, \\ z(t) = A(t)[z(0) + B_2(t)], & t \in J. \end{cases} \tag{8}$$

This and the boundary conditions give

$$y(0) = \lambda A(T)[z(0) + B_2(T)] + k = \lambda A(T)\{\lambda A(T)[y(0) + B_1(T)] + k + B_2(T)\} + k.$$

Since $|\lambda A(T)| \neq 1$, it yields

$$y(0) = \frac{1}{1 - [\lambda A(T)]^2}\{\lambda A(T)[\lambda A(T)B_1(T) + B_2(T) + k] + k\}.$$

Similarly, we have

$$z(0) = \frac{1}{1 - [\lambda A(T)]^2}\{\lambda A(T)[\lambda A(T)B_2(T) + B_1(T) + k] + k\}.$$

Inserting the values of $y(0)$ and $z(0)$ into formulas (8), we obtain the unique solution (y, z) of our problem.

This completes the proof. □

Lemma 7. *Let $\lambda < 0$, $a \in C(J, \mathbb{R})$, and*

$$\int_0^T a(s)\,\mathrm{d}s < -\ln(-\lambda). \tag{9}$$

Assume that $p \in C^1(J, \mathbb{R})$ and

$$\begin{cases} p'(t) \leqslant a(t)p(t), & t \in J, \\ p(0) \leqslant -\lambda p(T). \end{cases}$$

Then $p(t) \leqslant 0$ on J.

Proof. Indeed, $p(t) \leqslant A(t)p(0)$ with $A(t) = \mathrm{e}^{\int_0^t a(s)\,\mathrm{d}s}$. Hence $p(0) \leqslant -\lambda A(T)p(0)$, so $p(0)[1 + \lambda A(T)] \leqslant 0$. Because of (9), $1 + \lambda A(T) > 0$, so $p(0) \leqslant 0$ showing that $p(t) \leqslant 0$ on J.

This completes the proof. □

Lemma 8. *Assume that* $\lambda < 0$, $a \in C(J,\mathbb{R})$ *and let condition* (9) *hold. Let* $p, q \in C^1(J,\mathbb{R})$, *and*

$$\begin{cases} p'(t) \leqslant a(t)p(t), & t \in J, \\ p(0) \leqslant -\lambda q(T), \end{cases} \qquad \begin{cases} q'(t) \leqslant a(t)q(t), & t \in J, \\ q(0) \leqslant -\lambda p(T). \end{cases}$$

Then $p(t) \leqslant 0$ *and* $q(t) \leqslant 0$ *on* J.

P r o o f. Indeed,

$$p(t) \leqslant A(t)p(0), \quad q(t) \leqslant A(t)q(0), \quad t \in J \quad \text{with } A(t) = \mathrm{e}^{\int_0^t a(s)\,\mathrm{d}s}.$$

Since $-\lambda > 0$, we have

$$p(0) \leqslant -\lambda q(T) \leqslant -\lambda A(T)q(0) \leqslant \lambda^2 A(T)p(T) \leqslant [\lambda A(T)]^2\, p(0),$$

so $p(0)\{1 - [\lambda A(T)]^2\} \leqslant 0$. By condition (9),

$$[\lambda A(T)]^2 < \lambda^2 \mathrm{e}^{-2\ln(-\lambda)} = 1.$$

This gives $p(0) \leqslant 0$ showing that $p(t) \leqslant 0$ on J. By the same argument we obtain $q(0) \leqslant 0$, which proves that $q(t) \leqslant 0$ on J as well.

This completes the proof. □

Lemma 9. *Let* $\lambda < 0$ *and assumption* H_3 (a) *hold. Let* u, v *be weakly coupled lower and upper solutions of* (1), *and such that* $u, v \in \Omega$, *and* $u \leqslant v$. *Moreover, let condition* (9) *hold with* L *instead of* a, *and let*

$$\begin{cases} y'(t) = h(t,u) + W(t,u,v)[y(t) - u(t)], & t \in J, \ y(0) = \lambda z(T) + k, \\ z'(t) = h(t,v) + W(t,u,v)[z(t) - v(t)], & t \in J, \ z(0) = \lambda y(T) + k \end{cases}$$

with W *defined as in assumption Lemma* 4.

Then

$$u(t) \leqslant y(t) \leqslant z(t) \leqslant v(t), \quad t \in J, \tag{10}$$

and moreover, y, z *are weakly coupled lower and upper solutions of* (1).

P r o o f. Because,

$$-\lambda \mathrm{e}^{\int_0^T W(s,u,v)\,\mathrm{d}s} \leqslant -\lambda \mathrm{e}^{\int_0^T L(s)\,\mathrm{d}s} < -\lambda \mathrm{e}^{-\ln(-\lambda)} = 1$$

by Lemma 6, there exists a unique solution (y, z) of our problem.

Now we are going to show that (10) holds. Put $p = u - y$, $q = z - v$ so that $p(0) \leqslant -\lambda q(T)$, $q(0) \leqslant -\lambda p(T)$. Moreover,

$$\begin{cases} p'(t) \leqslant h(t,u) - h(t,u) - W(t,u,v)[y(t) - u(t)] = W(t,u,v)p(t), & t \in J, \\ q'(t) \leqslant h(t,v) + W(t,u,v)[z(t) - v(t)] - h(t,v) = W(t,u,v)q(t), & t \in J. \end{cases}$$

Since $\int_0^T W(t,u,v)\,\mathrm{d}s \leqslant \int_0^T L(s)\,\mathrm{d}s \leqslant -\ln(-\lambda)$ by Lemma 8, we have $p(t) \leqslant 0$ and $q(t) \leqslant 0$ on J showing that $u(t) \leqslant y(t)$ and $z(t) \leqslant v(t)$ on J. Now, we put $p = y - z$, so $p(0) = -\lambda p(T)$. By the mean value theorem and assumption H_3 (a) we obtain

$$\begin{aligned} p'(t) &= h(t,u) - h(t,v) + W(t,u,v)[y(t) - u(t) - z(t) + v(t)] \\ &= h_x(t,\xi)[u(t) - v(t)] + W(t,u,v)[y(t) - u(t) - z(t) + v(t)] \\ &\leqslant W(t,u,v)p(t), \quad t \in J, \end{aligned}$$

where $u(t) < \xi(t) < v(t)$, $t \in J$. This and Lemma 7 prove that $y(t) \leqslant z(t)$ on J, which means that (10) holds.

The proof will be completed if we show that y and z are weakly coupled lower and upper solutions of (1). Indeed, by the mean value theorem and assumption H_3 (a), we have

$$\begin{aligned} y'(t) &= h(t,u) + W(t,u,v)[y(t) - u(t)] - h(t,y) + h(t,y) \\ &= h(t,y) + W(t,u,v)[y(t) - u(t)] + h_x(t,\xi_1)[u(t) - y(t)] \\ &\leqslant h(t,y) \end{aligned}$$

for $u(t) < \xi_1(t) < y(t)$, $t \in J$, and

$$\begin{aligned} z'(t) &= h(t,v) + W(t,u,v)[z(t) - v(t)] - h(t,z) + h(t,z) \\ &= h(t,z) + W(t,u,v)[z(t) - v(t)] + h_x(t,\xi_2)[v(t) - z(t)] \\ &\geqslant h(t,z), \end{aligned}$$

where $z(t) < \xi_2(t) < v(t)$, $t \in J$.

This completes the proof. □

Now we are able to formulate the main result for (1) when $\lambda < 0$.

Theorem 2. *Let* $\lambda < 0$ *and assumption* H_3 (a) *hold. Assume that*

$$\int_0^T L(s)\,\mathrm{d}s < -\ln(-\lambda)$$

with L *defined as in assumption* H_3 (b). *Assume that* $y_0, z_0 \in C^1(J,\mathbb{R})$ *are weakly coupled lower and upper solutions of* (1) *such that* $y_0(t) \leqslant z_0(t)$ *on* J.

Then there exist monotone sequences $\{y_n\}$, $\{z_n\}$ *which converge monotonically and uniformly to the unique solution of problem* (1) *and the convergence is quadratic.*

Proof. Let

$$\begin{aligned} y'_{n+1}(t) &= h(t,y_n) + W(t,y_n,z_n)[y_{n+1}(t)-y_n(t)],\\ &\qquad t\in J,\;\; y_{n+1}(0)=\lambda z_{n+1}(T)+k,\\ z'_{n+1}(t) &= h(t,z_n) + W(t,y_n,z_n)[z_{n+1}(t)-z_n(t)],\\ &\qquad t\in J,\;\; z_{n+1}(0)=\lambda y_{n+1}(T)+k, \end{aligned}$$

where W is defined as in Lemma 4. By Lemma 6, y_1 and z_1 are well defined. Moreover, Lemma 9 yields the relation

$$y_0(t) \leqslant y_1(t) \leqslant z_1(t) \leqslant z_0(t), \quad t\in J.$$

Also, by Lemma 9, y_1 and z_1 are weakly coupled lower and upper solutions of (1). Now using induction argument we can prove that for all n and $t\in J$,

$$y_0(t)\leqslant y_1(t)\leqslant\ldots\leqslant y_{n-1}(t)\leqslant y_n(t)\leqslant z_n(t)\leqslant z_{n-1}(t)\leqslant\ldots\leqslant z_1(t)\leqslant z_0(t).$$

Employing a standard argument (see [3]), it is easy to conclude that the sequences $\{y_n\}$, $\{z_n\}$ converge uniformly and monotonically to the limit functions y and z, respectively, where y and z satisfy the equations

$$\begin{cases} y'(t)=h(t,y), & t\in J,\;\; y(0)=\lambda z(T)+k,\\ z'(t)=h(t,z), & t\in J,\;\; z(0)=\lambda y(T)+k. \end{cases}$$

Now, we are going to show that y and z are solutions of (1). Put $p=y-z$, so $p(0)=-\lambda p(T)$. By the mean value theorem we see that

$$p'(t)=h(t,y)-h(t,z)=h_x(t,\xi)p(t), \quad t\in J,$$

where ξ is between y and z. Since

$$\int_0^T h_x(t,\xi)\,\mathrm{d}t \leqslant \int_0^T L(t)\,\mathrm{d}t < -\ln(-\lambda),$$

we obtain by Lemma 7 that $p(t) \leqslant 0$ on J, so $y(t) \leqslant z(t)$ on J. By the same argument we can show that $z(t) \leqslant y(t)$ on J if we put $p = z - y$. This proves that $y = z$, so y and z are solutions of problem (1). By Lemma 5, problem (1) has at most one solution, so $y = z = x$, where x is the unique solution of (1).

It remains to show quadratic convergence. To this end, we put $p_{n+1} = x - y_{n+1} \geqslant 0$, $q_{n+1} = z_{n+1} - x \geqslant 0$ and note that $p_{n+1}(0) = -\lambda q_{n+1}(T)$, $q_{n+1}(0) = -\lambda p_{n+1}(T)$. Hence

$$\begin{aligned}
p'_{n+1}(t) &= h(t,x) - h(t,y_n) - W(t,y_n,z_n)[y_{n+1}(t) - y_n(t)] \\
&= h_x(t,\xi_1)p_n(t) - W(t,y_n,z_n)[p_n(t) - p_{n+1}(t)] \\
&\leqslant [F_x(t,x) - F_x(t,y_n) + G_x(t,y_n) - G_x(t,z_n) + \Delta_x(t,z_n) - \Delta_x(t,y_n) \\
&\quad + \Psi_x(t,y_n) - \Psi_x(t,x)]p_n(t) + W(t,y_n,z_n)p_{n+1}(t) \\
&= \{F_{xx}(t,\xi_2)p_n(t) - G_{xx}(t,\xi_3)[p_n(t) + q_n(t)] + \Delta_{xx}(t,\xi_4)[q_n(t) + p_n(t)] \\
&\quad - \Psi_{xx}(t,\xi_5)p_n(t)\}p_n(t) + W(t,y_n,z_n)p_{n+1}(t) \\
&\leqslant \{(A_1 + A_4)p_n(t) + (A_2 + A_3)[p_n(t) + q_n(t)]\}p_n(t) + L(t)p_{n+1}(t) \\
&\leqslant L(t)p_{n+1}(t) + D_1,
\end{aligned}$$

where $y_n < \xi_1, \xi_2, \xi_5 < x$, $y_n < \xi_3, \xi_4 < z_n$ with L defined as in H_3 (b),

$$\begin{gathered}
|F_{xx}(t,u)| \leqslant A_1, \ \ |G_{xx}(t,u)| \leqslant A_2, \ \ |\Delta_{xx}(t,u)| \leqslant A_3, \ \ |\Psi_{xx}(t,u)| \leqslant A_4 \ \text{ on } J \times \Omega, \\
D_1 = \max_{t\in J}[K_5 p_n^2(t) + K_6 q_n^2(t)], \ \ K_5 = A_1 + A_4 + \tfrac{3}{2}(A_2 + A_3), \ \ K_6 = \tfrac{1}{2}(A_2 + A_3).
\end{gathered}$$

In a similar way, we obtain

$$q'_{n+1}(t) \leqslant L(t)q_{n+1}(t) + D_2 \quad \text{with} \quad D_2 = \max_{t\in J}[K_7 p_n^2(t) + K_8 q_n^2(t)],$$

where

$$K_7 = \tfrac{1}{2}(A_1 + A_4), \quad K_8 = A_2 + A_3 + \tfrac{3}{2}(A_1 + A_4).$$

Put $w = p_{n+1} + q_{n+1}$, so $w(0) = -\lambda w(T)$ and

$$w'(t) \leqslant L(t)w(t) + D, \quad t \in J$$

with $D = D_1 + D_2$. Consequently,

$$w(t) \leqslant A(t)[w(0) + DB(t)], \quad t \in J$$

for

$$A(t) = \mathrm{e}^{\int_0^t L(s)\,\mathrm{d}s}, \quad B(t) = \int_0^t \mathrm{e}^{-\int_0^s L(r)\,\mathrm{d}r}\,\mathrm{d}s.$$

Hence $w(0) = -\lambda w(T) \leqslant -\lambda A(T)[w(0) + DB(T)]$, and finally

$$w(0) \leqslant \frac{-\lambda A(T)B(T)}{1 + \lambda A(T)} D$$

because

$$1 + \lambda A(T) > 1 + \lambda \mathrm{e}^{-\ln(-\lambda)} = 0.$$

This yields

$$w(t) \leqslant A(t)\Big[\frac{-\lambda A(T)B(T)}{1 + \lambda A(T)} D + DB(t)\Big] \leqslant \frac{A(T)B(T)}{1 + \lambda A(T)} D.$$

Hence

$$\max_{t \in J} |w(t)| \leqslant \frac{A(T)B(T)}{1 + \lambda A(T)} \Big[M_1 \max_{t \in J} |p_n(t)|^2 + M_2 \max_{t \in J} |q_n(t)|^2\Big]$$

with $M_i = K_{4+i} + K_{6+i}$, $i = 1, 2$. Since

$$\max_{t \in J} |p_{n+1}(t)| \leqslant \max_{t \in J} |w(t)| \quad \text{and} \quad \max_{t \in J} |q_{n+1}(t)| \leqslant \max_{t \in J} |w(t)|$$

we get the desired quadratic convergence.

The proof is therefore complete. □

Remark 2. Theorem 2 contains some results of [8] and [9] if $\lambda = -1$ and $k = 0$.

Theorems 1 and 2 can be combined in

Theorem 3. *Let $h \in C(J \times \mathbb{R}, \mathbb{R})$ with $h = f + g$. Assume that*

(i) *y_0, z_0 are lower and upper solutions of* (1), *respectively, if $\lambda \geqslant 0$, or*

(ii) *y_0, z_0 are weakly coupled lower and upper solutions of* (1) *if $\lambda < 0$.*

Let $y_0(t) \leqslant z_0(t)$ on J. Let assumption H_3 (a) *hold and*

$$\int_0^T L(s)\,\mathrm{d}s < -\ln(|\lambda|) \quad \text{for} \quad \lambda \neq 0$$

with L defined as in assumption H_3 (b).

Then there exist monotone sequences $\{y_n\}$, $\{z_n\}$ which converge monotonically and uniformly to the unique solution of problem (1) *and the convergence is quadratic.*

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Author's address: Technical University of Gdańsk, Department of Differential Equations, 11/12 G. Narutowicz Str., 80-952 Gdańsk, Poland, e-mail: `tjank@mifgate.pg.gda.pl`.