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ON THE LONG-TIME BEHAVIOUR OF COMPRESSIBLE FLUID FLOWS SUBJECTED TO HIGHLY OSCILLATING EXTERNAL FORCES

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Abstract. We show that the global-in-time solutions to the compressible Navier-Stokes equations driven by highly oscillating external forces stabilize to globally defined (on the whole real line) solutions of the same system with the driving force given by the integral mean of oscillations. Several stability results will be obtained.

Keywords: compressible Navier-Stokes equations, global-in-time solutions, large time bahaviour

MSC 2000: 35Q30, 35B35

1. Introduction and statement of result

There seems to be a common belief that highly oscillating driving forces of zero time average do not influence the long-time dynamics of dissipative systems. Thus for instance the solutions of the semilinear parabolic equation

$$u_t - \Delta u = f(u) + \sin(t^2)g(x)$$

will behave as solutions of the corresponding autonomous problem when the time t tends to infinity. Averaging a function over a short time interval should be considered analogous to making a macroscopic measurement in a physical experiment. The result of such an experiment being close to zero, the effect on the solutions of robust dynamical systems, if any, should be negligible at least in the long run. From the

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mathematical point of view, these ideas have been made precise in the paper by Chepyzhov and Vishik [1] dealing with trajectory attractors of evolution equations. They showed that a trajectory attractor of a dissipative dynamical system perturbed by a highly oscillating forcing term is the same as for the unperturbed system. These results apply to a vast class of equations including the wave equation with weak dissipation and the Navier-Stokes equations of incompressible fluids in three space dimensions. Note, however, that the theory of trajectory attractors itself is based on considering the time averages rather than the instantaneous values of solutions.

The time evolution for $t \in \mathbb{R}^+ = (0, \infty)$ of the density $\varrho = \varrho(t, x)$ and the velocity $\boldsymbol{u} = [u^1(t,x), u^2(t,x), u^3(t,x)]$ of a driven compressible fluid flow contained in a bounded domain $\Omega \subset \mathbb{R}^3$ can be described by the Navier-Stokes equations:

$$\left\{ \begin{array}{c} \varrho_t + \operatorname{div}(\varrho\, \boldsymbol{u}) = 0, \\ (\varrho\, \boldsymbol{u})_t + \operatorname{div}(\varrho\, \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \operatorname{div} T(\boldsymbol{u}) + \varrho[\nabla F(x) + \boldsymbol{g}(t,x)]. \end{array} \right\} \tag{1.1}$$

Here T is the Cauchy stress tensor

$$T = T_{i,j}(\boldsymbol{u}) = \mu(u^i_{x_j} + u^j_{x_i}) + \lambda \operatorname{div} \boldsymbol{u} \delta_{i,j}, \quad \mu > 0, \quad \lambda + \mu \geqslant 0$$

and p is the isentropic pressure

$$p = p(\varrho) = a\varrho^\gamma, \quad a > 0, \quad \gamma > 1.$$

We shall assume that Ω has a Lipschitz boundary and impose the no-slip boundary conditions for the velocity

$$\boldsymbol{u}|_{\partial\Omega} = 0. \tag{1.2}$$

The flow is driven by an external force $\boldsymbol{f}(t,x) = \nabla F(x) + \boldsymbol{g}(t,x)$ where F is a globally Lipschitz potential independent of t and $\boldsymbol{g}$ is a measurable bounded perturbation.

In accordance with the available existence theory (see Lions [13] and [4]) we shall deal with the *finite energy weak solutions* of the problem, that is

- the functions $\varrho \geqslant 0$ and $\boldsymbol{u}$ belong to the spaces

$$\varrho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^\gamma(\Omega)), \quad \boldsymbol{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^+; [W^{1,2}_0(\Omega)]^3);$$

- the total energy $E[\varrho, (\varrho\boldsymbol{u})] = \int_\Omega \frac{1}{2}\varrho|\boldsymbol{u}|^2 + a/(\gamma-1)\varrho^\gamma \,\mathrm{d}x$ is locally integrable on $\mathbb{R}^+$ and the energy inequality

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \int_\Omega \mu|\nabla\boldsymbol{u}|^2 + (\lambda+\mu)|\operatorname{div}\ \boldsymbol{u}|^2 \,\mathrm{d}x \leqslant \int_\Omega \varrho[\nabla F + \boldsymbol{g}].\boldsymbol{u} \,\mathrm{d}x \tag{1.3}$$

holds in $\mathscr{D}'(\mathbb{R}^+)$;

- the equations (1.1) are satisfied in $\mathscr{D}'(\mathbb{R}^+ \times \Omega)$;
- the continuity equation holds in the sense of renormalized solutions, i.e.,

$$\partial_t b(\varrho) + \operatorname{div}(b(\varrho)\boldsymbol{u}) + (b'(\varrho)\varrho - b(\varrho)) \operatorname{div} \boldsymbol{u} = 0$$

  in $\mathscr{D}'(\mathbb{R}^+ \times \Omega)$ for any $b \in C^1(\mathbb{R})$ satisfying

$$b'(z) = 0 \quad \text{for all } z \text{ such that } |z| \geqslant M$$

  for a certain constant $M = M(b)$; moreover, we assume

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0$$

  to hold in $\mathscr{D}'(\mathbb{R}^+ \times \mathbb{R}^3)$ provided ϱ, $\boldsymbol{u}$ were prolonged to be zero outside Ω.

Any finite energy weak solution complies with the total mass conservation principle:

$$m = \int_\Omega \varrho(t,x)\, \mathrm{d}x \quad \text{is independent of } t \in \mathbb{R}^+$$

(see [10, Lemma 3.1]). Rescaling a, μ and λ we shall always assume $m = 1$.

As for the unperturbed system, we report the following result (see [6, Theorem 1.1]):

Theorem 1.1. *Let $\gamma > \frac{3}{2}$, $\boldsymbol{g} \equiv 0$, and let F be such that the upper level sets*

$$[F > k] = \{x \in \Omega \mid F(x) > k\} \quad \textit{are connected in } \Omega \textit{ for all } k \in \mathbb{R}.$$

Then

$$(\varrho \boldsymbol{u})(t) \to 0 \quad \textit{strongly in } L^1(\Omega), \quad \varrho(t) \to \varrho_s \quad \textit{strongly in } L^\gamma(\Omega) \quad \textit{as } t \to \infty$$

for any finite energy weak solution of the problem (1.1), (1.2), *where ϱ_s is the unique solution of the stationary problem*

$$a\nabla \varrho_s^\gamma = \varrho_s \nabla F, \quad \int_\Omega \varrho_s\, \mathrm{d}x = 1. \tag{1.4}$$

Related results may be found in [2], Novotný and Straškraba [14], and also Straškraba [15]. Similar problems for mixtures of two incompressible fluids were considered by Gerbeau and Le Bris [11]. The hypothesis of connectedness of the upper level sets $[F > k]$ guarantees uniqueness of solutions to the stationary problem (1.4) (see [5]). An interesting open question is to determine whether this condition is really necessary for the conclusion of Theorem 1.1 to hold. A partial answer may be found in [7].

The goal of the present paper is to show that the conclusion of Theorem 1.1 remains valid provided $\boldsymbol{g}$ is a small or/and rapidly oscillating perturbation.

Highly oscillating sequences converge in the weak topology, i.e., the topology of convergence of integral means. Consider a ball B_G of radius G centered at zero in the space $L^\infty((0,1)\times\Omega)$. The weak-star topology on B_G is metrizable and we denote the corresponding metric by d_G.

The main result of this paper reads as follows:

Theorem 1.2. *Let $\gamma > \frac{5}{3}$ and let F be a globally Lipschitz function such that all the upper level sets $[F > k]$, $k \in \mathbb{R}$ are connected in Ω.*

Then given $G > 0$, $\varepsilon > 0$ there exists $\delta = \delta(G, \varepsilon) > 0$ such that

$$\limsup_{t\to\infty} [\|\varrho(t) - \varrho_s\|_{L^\gamma(\Omega)} + \|\varrho\boldsymbol{u}(t)\|_{L^1(\Omega)}] < \varepsilon \tag{1.5}$$

for any finite energy weak solution ϱ, $\boldsymbol{u}$ of the problem (1.1), (1.2) *provided*

$$\left\{ \begin{array}{c} \limsup\limits_{t\to\infty} \|\boldsymbol{g}\|_{L^\infty((t,\infty)\times\Omega))} < G, \\ \limsup\limits_{t\to\infty} d_G[\boldsymbol{g}(t+s)|_{s\in[0,1]}, 0] < \delta. \end{array} \right\} \tag{1.6}$$

Here ϱ_s is the unique solution of the stationary problem (1.4).

Note that (1.6) allows for rapidly oscillating perturbations both in space and time. Of course, Theorem 1.2 remains valid if d_G is replaced by the (strong) norm distance in, say, $L^1((0,1)\times\Omega))$.

Theorem 1.2 has a corollary concerning the stability of forced time-periodic solutions. Consider a perturbation $\boldsymbol{g}$ which is bounded and periodic with respect to t with a period $\omega > 0$. Then the problem (1.1)–(1.2) possesses at least one finite energy weak solution periodic in time with the same period and the same mass (see [3, Theorem 1.1]). In fact, the proof in [3] is done for rectangular domains with no-stick boundary conditions for the velocity. However, the proof for a general Ω and the boundary conditions (1.2) requires only one modification, namely, one has to have a priori estimates ensuring the square integrability of ϱ up to the boundary $\partial\Omega$. This type of result being now available (see [10]), the existence of time-periodic solutions can be carried over with no additional effort. Accordingly, Theorem 1.2 gives rise to the following

Corollary 1.1. *Under the hypotheses of Theorem* 1.2, *let $h = h(t)$ be a bounded time-periodic function with zero mean and period ω and $w = w(x)$ a function in $L^\infty(\Omega)$.*

Then given $\varepsilon > 0$ there exists n_0 such that

$$\limsup_{t\to\infty}[\|\varrho(t) - \overline{\varrho}(t)\|_{L^\gamma(\Omega)} + \|(\varrho\boldsymbol{u})(t) - (\overline{\varrho}\,\overline{\boldsymbol{u}})(t)\|_{L^1(\Omega)}] < \varepsilon$$

for any finite energy weak solution ϱ, $\boldsymbol{u}$ of the problem (1.1), (1.2) *provided $\boldsymbol{g} = h(nt)w(x)$ or $\boldsymbol{g} = \frac{1}{n}h(t)w(x)$ and $n \geqslant n_0$. Here $\overline{\varrho}$, $\overline{\boldsymbol{u}}$ is a time-periodic solution of the problem* (1.1), (1.2).

The rest of the paper is devoted to the proof of Theorem 1.2. It is worth-while to note that the analysis goes well beyond the proof of Theorem 1.1. The main difficulty of the perturbed problem lies in the fact that, unlike in the (unperturbed) potential case, there is no Lyapunov function and, consequently, there are no a priori estimates on the velocity field which would allow one to conclude that $\boldsymbol{u}$ is close to zero for large times. Consequently, one must take care of possible oscillations of the density resulting from the action of the external force. Moreover, it is by no means clear that the solution stays bounded *uniformly* in time, i.e., that there are no resonance phenomena due to the presence of $\boldsymbol{g}$.

Our approach is based on two properties of the system (1.1), (1.2) established in [8], [9]. According to (1.6), the function $\boldsymbol{g}$ is uniformly bounded on $\mathbb{R}^+ \times \Omega$. Consequently, making use of [9, Theorem 1.1] we are allowed to conclude that the energy of any finite energy weak solution is bounded uniformly in time. Moreover, a careful analysis of propagation of oscillations carried over in [8] enables us to prove the existence of a trajectory attractor in the spirit of Chepyzhov and Vishik [1] with respect to the strong L^1-topology in the density and the weak L^p-topology in the velocity (momenta) component. In the present case, the trajectory attractor happens to be a small neighbourhood of the singleton $[\varrho_s, 0]$ where ϱ_s is the solution of (1.4) (see Section 3).

Finally, analyzing the behaviour of the energy E in the neighbourhood of the trajectory attractor, we conclude that the convergence of $\varrho(t)$ is in fact strong in L^γ and that $(\varrho\boldsymbol{u})(t)$ converges strongly in L^1 as claimed in Theorem 1.2 (see Section 4).

To conclude, let us remark that our result adapts easily to the case of dimension $N = 1, 2$. The long-time behaviour of solutions for $N = 1$ was studied in a recent paper by Hoff and Ziane [12]. Note, however, that their hypotheses require much more regularity of the driving force, in particular, they do not cover the case of rapidly oscillating perturbations. The restriction $\gamma \geqslant \frac{9}{5}$ is irrelevant if $N = 1$ and the pressure $p = p(\varrho)$ can be taken an arbitrary increasing function with at least linear growth for large values of ϱ.

2. Uniform boundedness

The components of a finite energy weak solution, namely, the density ϱ and the momenta $\varrho\boldsymbol{u}$ are a priori defined only for a.a. $t \in \mathbb{R}^+$. However, it can be shown (see e.g. Lions [13]) that the continuity equation is satisfied also in the sense of renormalized solutions, in particular,

$$\varrho \in C(J; L^1(\Omega)) \cap C(J; L^\gamma_{\text{weak}}(\Omega)) \quad \text{for any compact interval } J \subset \mathbb{R}^+.$$

Moreover, the fact that the time derivative of the momenta can be expressed by means of the equations of motion yields

$$(\varrho\boldsymbol{u}) \in C(J; L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\Omega)) \quad \text{for any compact } J \subset \mathbb{R}^+.$$

Thus it makes sense to consider instantaneous values of both the quantities. Moreover, it can be shown (see [2, formula (1.12)]) that they satisfy the seemingly obvious relation

$$(\varrho\boldsymbol{u})(t,x) = 0 \ \text{ for a.a. } \ x \in V(t) = \{x \mid \varrho(t,x) = 0\} \quad \text{for any } t \in \mathbb{R}^+. \tag{2.1}$$

Finally, redefining the energy on a set of measure zero if necessary we set

$$E = E[\varrho, (\varrho\boldsymbol{u})] = \frac{1}{2}\int_{\varrho>0} \frac{|(\varrho\boldsymbol{u})|^2}{\varrho}\,\mathrm{d}x + \frac{a}{\gamma-1}\int_\Omega \varrho^\gamma\,\mathrm{d}x.$$

Now, E is defined for *any* $t \in \mathbb{R}^+$; and is a lower-semicontinuous function of t (see [2, Corollary 1.1]).

By virtue of (1.6) the driving force $\nabla F + \boldsymbol{g}$ in (1.1) is uniformly bounded for t large enough by a constant depending only on G and the norm of ∇F. Thus we can apply [9, Theorem 1.1] on the existence of bounded absorbing sets; specifically, there is a constant E_∞ depending only on the amplitude of the driving force such that

$$E(t) \leqslant E_\infty \quad \text{for all } t \geqslant T_0 \tag{2.2}$$

where T_0 depends only on the value of E at an arbitrary Lebesgue point $t \in [0,1]$. Moreover, by virtue of (1.6), T_0 can be chosen so large that

$$\operatorname*{ess\,sup}_{t>T_0,\, x\in\Omega} |\boldsymbol{g}(t,x)| < G \tag{2.3}$$

and

$$d_G(\boldsymbol{g}(t+s)|_{s\in[0,1]}, 0) < \delta \quad \text{for all } t \geqslant T_0. \tag{2.4}$$

In view of these arguments, the conclusion of Theorem 1.2 will follow from the next relatively simpler assertion:

Lemma 2.1. *Assume $\gamma > \frac{5}{3}$ and F satisfies the hypotheses of Theorem* 1.2.

Then given G, E_∞, $\varepsilon > 0$, there exists $\delta = \delta(G, \varepsilon, E_\infty) > 0$ such that (1.5) *holds for any finite energy weak solution of the problem* (1.1), (1.2) *satisfying*

$$E[\varrho(t), (\varrho \boldsymbol{u})(t)] \leqslant E_\infty \quad \textit{for all } t > 0$$

with

$$\|\boldsymbol{g}\|_{L^\infty((0,\infty)\times\Omega)} \leqslant G, \quad d_G[\boldsymbol{g}(t+s)|_{s\in[0,1]}, 0] < \delta \quad \textit{for all } t > 0.$$

The proof of Lemma 2.1 will be carried out in the next two sections. It seems convenient to argue by contradiction. Specifically, we shall assume there is a sequence ϱ_n, $\boldsymbol{u}_n$ of finite energy weak solutions of the problem (1.1), (1.2) with the forcing term $\nabla F + \boldsymbol{g}_n$ such that

$$E[\varrho_n, (\varrho_n \boldsymbol{u}_n)] \leqslant E_\infty \quad \text{for all } t > 0, \quad n = 1, 2, \dots \tag{2.5}$$

$$\left\{ \begin{array}{c} \|\boldsymbol{g}_n\|_{L^\infty((0,\infty)\times\Omega)} < G \\ d_G[\boldsymbol{g}(t+s)|_{s\in[0,1]}, 0] < \dfrac{1}{n} \quad \text{for all } t > 0 \end{array} \right\} \tag{2.6}$$

but

$$\left\{ \begin{array}{c} \|\varrho_n(T_n) - \varrho_s\|_{L^\gamma(\Omega)} + \|\varrho_n \boldsymbol{u}_n(T_n)\|_{L^1(\Omega)} \geqslant \kappa > 0 \\ \text{for a certain sequence } T_n \to \infty \end{array} \right\}. \tag{2.7}$$

3. Weak convergence

Consider a sequence ϱ_n, $\boldsymbol{u}_n$ of finite energy weak solutions as in (2.5), (2.6). Let $t_n \to \infty$ and take the corresponding time-shifts

$$\varrho_n(t_n + t), \quad (\varrho_n \boldsymbol{u}_n)(t_n + t) \quad \text{on } (-t_n, \infty).$$

Now, it is relatively straightforward to pass to the limit (taking subsequences if necessary) for $t_n \to \infty$ to conclude that

$$\varrho_n(t_n + t) \to \overline{\varrho} \quad \text{in } C(J; L^\gamma_{\text{weak}}(\Omega))$$

and

$$(\varrho_n \boldsymbol{u}_n)(t_n + t) \to (\overline{\varrho\,\boldsymbol{u}}) \quad \text{in } C(J; L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\Omega)) \quad \text{for any compact } J \subset \mathbb{R}$$

where

$$\left\{ \begin{array}{c} \overline{\varrho}_t + \operatorname{div}(\overline{\varrho\,\boldsymbol{u}}) = 0, \\ (\overline{\varrho\,\boldsymbol{u}})_t + \operatorname{div}(\overline{\varrho\,\boldsymbol{u} \otimes \boldsymbol{u}}) + \nabla\overline{p(\varrho)} = \operatorname{div} T(\overline{\boldsymbol{u}}) + \overline{\varrho}\nabla F \end{array} \right\}$$

in $\mathscr{D}'(\mathbb{R} \times \Omega)$ (see [2, Section 3] for details). Here $\overline{p(\varrho)}$ denotes a weak limit of $p(\varrho_n(t_n + t))$. Note that the perturbation term disappears in the limit equation as, in accordance with (2.6),

$$\boldsymbol{g}_n(t_n + t) \to 0 \quad \text{weakly star in } L^\infty((0, \infty) \times \Omega).$$

Moreover, by virtue of (2.5), the energy $E[\overline{\varrho}, (\overline{\varrho\,\boldsymbol{u}})] \leqslant E_\infty$ a.e. on $\mathbb{R}$. Note that, for the time being, we do not know if the energy corresponding to the limit functions $\overline{\varrho}$, $(\overline{\varrho\,\boldsymbol{u}})$ satisfies the energy inequality (1.3). This will follow as soon as we are able to show the *strong* convergence of the density component which is equivalent to saying that $\overline{p(\varrho)} = p(\overline{\varrho})$.

The arguments to show the strong convergence or compactness of the density $\varrho_n(t_n + t)$ in, say, $L^1((0, 1) \times \Omega)$, are more delicate. Note that the only available result in this direction, namely that of Lions [13], requires the "initial values", i.e., the values $\varrho_n(t_n)$ to be precompact in $L^1(\Omega)$. A priori, there is no reason this should be the case, i.e., there could be oscillations of the density component developing as $t \to \infty$ due to the action of the rapidly oscillating $\boldsymbol{g}$. In other words, we have to prove a uniform in time decay of possible oscillations which is independent of the initial state. This is the main result of [8] and [12] we shall now briefly sketch.

We define a defect measure

$$D(t) = \int_\Omega \overline{\varrho \log(\varrho)}(t) - \overline{\varrho}(t) \log(\overline{\varrho}(t))\,\mathrm{d}x$$

where $\overline{\varrho \log \varrho}(t)$ denotes a weak limit (in $L^1(\Omega)$) of the sequence $\varrho_n \log(\varrho_n)(t_n + t)$. Now, it is proved in [8] (see also [2, Section 2]) that D is a *uniformly* bounded and continuous function on the whole real line $t \in \mathbb{R}$ and, moreover, it satisfies

$$D(t_2) - D(t_1) + \int_{t_1}^{t_2} \Phi(D(t))\,\mathrm{d}t \leqslant 0 \quad \text{for any } t_1 < t_2$$

where Φ is a strictly increasing, continuous function such that $\Phi(0) = 0$. The function Φ represents the rate of time decay of possible oscillations in the density field. Consequently, D being uniformly bounded, we have $D \equiv 0$ and, since $z \log(z)$ is strictly convex, this implies strong L^1-convergence of $\varrho_n(t_n + t)$.

All details of the above mentioned procedure can be found in [2], [8]. Adapting [2, Proposition 3.1] to the present situation, we deduce the following result:

Lemma 3.1. *Let ϱ_n, $\boldsymbol{u}_n$ satisfying* (2.5) *be a sequence of finite energy weak solutions of* (1.1), (1.2) *where $\boldsymbol{g}_n$ are such that* (2.6) *holds. Then any sequence $t_n \to \infty$ contains a subsequence (not relabeled) such that*

$$\varrho_n(t_n + t) \to \overline{\varrho} \quad \text{in } C([0,1]; L^1(\Omega)), \tag{3.1}$$

$$(\varrho_n \boldsymbol{u}_n)(t_n + t) \to (\overline{\varrho}\,\overline{\boldsymbol{u}}) \quad \text{in } C([0,1]; L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\Omega)) \tag{3.2}$$

and

$$E[\varrho_n(t_n + t), (\varrho_n \boldsymbol{u}_n)(t_n + t)] \to E[\overline{\varrho}, (\overline{\varrho}\,\overline{\boldsymbol{u}})] \quad \text{strongly in } L^1(0,1) \tag{3.3}$$

where $\overline{\varrho}$, $\overline{\boldsymbol{u}}$ is a finite energy weak solution of the problem (1.1), (1.2) *with $\boldsymbol{g} \equiv 0$ (the unperturbed problem) defined on the whole real line $t \in \mathbb{R}$ and such that $E[\overline{\varrho}, (\overline{\varrho}\,\overline{\boldsymbol{u}})] \in L^\infty(\mathbb{R})$.*

Now, by virtue of the hypothesis of connectedness of the upper level sets $[F > k]$, the unperturbed problem admits exactly one globally defined (for $t \in \mathbb{R}$) finite energy weak solution with globally bounded energy, namely,

$$\overline{\varrho} = \varrho_s, \quad \overline{\boldsymbol{u}} = 0$$

where ϱ_s is the unique solution of the stationary problem (1.4) (see [2, Proposition 3.2]).

Consequently, (3.1), (3.2) yield

$$\left.\begin{aligned} \varrho_n(t_n + t) &\to \varrho_s \quad \text{in } C([0,1]; L^1(\Omega)), \\ (\varrho_n \boldsymbol{u}_n)(t_n + t) &\to 0 \quad \text{in } C([0,1]; L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\Omega)) \end{aligned}\right\} \text{for any} \quad t_n \to \infty \tag{3.4}$$

while (3.3) gives rise to

$$\int_0^1 \int_\Omega \varrho_n(t_n + t) |\boldsymbol{u}(t_n + t)|^2 \, \mathrm{d}x \, \mathrm{d}t \to 0 \quad \text{for any } t_n \to \infty, \tag{3.5}$$

$$\int_0^1 \int_\Omega \varrho_n^\gamma(t_n + t) \, \mathrm{d}x \, \mathrm{d}t \to \int_\Omega \varrho_s^\gamma \, \mathrm{d}x \quad \text{for any } t_n \to \infty \tag{3.6}$$

where we have used the weak lower-semicontinuity of the L^γ-norm.

4. Proof of Theorem 1.2, strong convergence

Now we take $t_n = T_n$, where T_n is the sequence from (2.7), and make use of the energy inequality (1.3). The energy being a lower-semicontinuous function of time, one can use the Gronwall inequality to obtain

$$\begin{aligned} E[\varrho_n, (\varrho_n \boldsymbol{u}_n)](T_n) &\leqslant \sup_{t \in [T_n - \tau/2, T_n]} E[\varrho_n, (\varrho_n \boldsymbol{u}_n)](t) \\ &\leqslant \operatorname{ess} \sup_{t \in [T_n - \tau/2, T_n]} E[\varrho_n, (\varrho_n \boldsymbol{u}_n)](t) \\ &\leqslant \operatorname{ess} \inf_{t \in [T_n - \tau, T_n - \tau/2]} E[\varrho, (\varrho \boldsymbol{u})](t) + \tau M \sqrt{E_\infty}, \end{aligned} \tag{4.1}$$

where the constant M depends only on G and the norm of ∇F in $L^\infty(\mathbb{R}^+ \times \Omega)$, E_∞ is the quantity from (2.5), $0 < \tau < T_n$ arbitrary.

Now, by virtue of (3.5), (3.6),

$$\int_{T_n - \tau}^{T_n - \tau/2} E[\varrho_n, (\varrho_n \boldsymbol{u}_n)] \, \mathrm{d}t \to \frac{\tau}{2} \frac{a}{\gamma - 1} \int_\Omega \varrho_s^\gamma \, \mathrm{d}x,$$

whence, in view of (4.1),

$$\limsup_{T_n \to \infty} E[\varrho_n, (\varrho_n \boldsymbol{u}_n)](T_n) \leqslant \frac{a}{\gamma - 1} \int_\Omega \varrho_s^\gamma \, \mathrm{d}x + \tau M \sqrt{E_\infty}. \tag{4.2}$$

As $\tau > 0$ can be taken arbitrarily small, (4.2) yields

$$\limsup_{T_n \to \infty} \frac{1}{2} \int_{\varrho_n(T_n) > 0} \frac{|(\varrho_n \boldsymbol{u}_n)|^2}{\varrho_n}(T_n) \, \mathrm{d}x + \frac{a}{\gamma - 1} \int_\Omega \varrho_n^\gamma(T_n) \, \mathrm{d}x \leqslant \frac{a}{\gamma - 1} \int_\Omega \varrho_s^\gamma \, \mathrm{d}x.$$

Consequently,

$$\|\varrho_n(T_n)\|_{L^\gamma(\Omega)} \to \|\varrho_s\|_{L^\gamma(\Omega)}$$

and, making use of the uniform convexity of the L^γ-norm, we have

$$\varrho_n(T_n) \to \varrho_s \quad \text{strongly in } L^\gamma(\Omega) \tag{4.3}$$

and

$$\int_{\varrho_n(T_n) > 0} \frac{|(\varrho_n \boldsymbol{u}_n)|^2}{\varrho_n}(T_n) \, \mathrm{d}x \to 0 \quad \text{as } T_n \to \infty. \tag{4.4}$$

By virtue of (2.1), the relation (4.4) yields

$$\begin{aligned} \int_\Omega |(\varrho_n \boldsymbol{u}_n)|(T_n) \, \mathrm{d}x &= \int_{\varrho_n(T_n) > 0} \sqrt{\varrho_n(T_n)} \sqrt{\varrho_n(T_n)} \, |\boldsymbol{u}_n(T_n)| \, \mathrm{d}x \\ &\leqslant \left[\int_{\varrho_n(T_n) > 0} \frac{|\varrho_n \boldsymbol{u}_n|^2}{\varrho_n}(T_n) \, \mathrm{d}x \right]^{\frac{1}{2}} \end{aligned}$$

which, combined with (4.4), gives

$$(\varrho_n \boldsymbol{u}_n)(T_n) \to 0 \quad \text{strongly in } L^1(\Omega). \tag{4.5}$$

The relation (4.3) together with (4.5) contradicts (2.7). This completes the proof of Lemma 2.1 and, consequently, that of Theorem 1.2.

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