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STRUCTURE OF PARTIALLY ORDERED CYCLIC SEMIGROUPS

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Abstract. This paper recalls some properties of a cyclic semigroup and examines cyclic subsemigroups in a finite ordered semigroup. We prove that a partially ordered cyclic semigroup has a spiral structure which leads to a separation of three classes of such semigroups. The cardinality of the order relation is also estimated. Some results concern semigroups with a lattice order.

Keywords: cyclic semigroup, ordered semigroup, lattice order, idempotent element, subidempotent, superidempotent elements

MSC 2000: 06F05, 20M10, 20M30

1. Cyclic semigroups

Our investigations are inspired by Š. Schwarz's work on the semigroup of binary relations ([7], [8]). We want to separate pure semigroup properties used there and in many papers on fuzzy relations (cf. [9] or [5]).

We begin with the notion of periodic semigroup (cf. [4], § I, 2).

Definition 1. A semigroup S is called periodic if every element $a \in S$ has a repetition in the sequence of powers: $a, a^2, a^3, \ldots$. The index of $a \in S$ is the number

$$k = k(a) = \min\{n \in \mathbb{N}\colon\ \exists_{m>n}(a^m = a^n)\}. \tag{1}$$

The period of $a \in S$ is the number

$$d = d(a) = \min\{n \in \mathbb{N}\colon\ a^{k+n} = a^k\}. \tag{2}$$

This definition prepares our fundamental assumption

Hypothesis 1. $(S, *)$ is a periodic semigroup and $a \in S$.

Now we recall the known results on the cyclic semigroup generated by a:

$$\langle a \rangle = \{a, a^2, a^3, \ldots\}. \tag{3}$$

Theorem 1 ([4], Theorem 2.6). *Under Hypothesis* 1 *the semigroup* (3) *has exactly* $k + d - 1$ *different elements,*

$$\langle a \rangle = \{a, a^2, \ldots, a^k, a^{k+1}, \ldots, a^{k+d-1}\}, \tag{4}$$

and contains a cyclic subgroup

$$K_a = \{a^k, a^{k+1}, \ldots, a^{k+d-1}\} \tag{5}$$

of order d*, with the identity* $e = a^r$*, where*

$$r = r(a), \quad k \leqslant r \leqslant k + d - 1, \quad d \mid r \tag{6}$$

and with the generator $q = a^{r+1}$*, i.e.*

$$K_a = \{q, q^2, \ldots, q^d\}. \tag{7}$$

Moreover

$$(a^m = a^n) \Leftrightarrow d \mid (m - n) \quad \textit{for all } n, m \geqslant k. \tag{8}$$

Definition 2 ([4]). The group (5) is called the kernel of the semigroup (4).

Definition 3. An element $p \in S$ is idempotent if

$$p^2 = p. \tag{9}$$

Immediately from (9) we get

$$(p^n = p) \quad \text{for all } n \in \mathbb{N}. \tag{10}$$

As an example of an idempotent element we consider the group identity e. From Theorem 1 we see that the semigroup (3) has at least one idempotent element $e = a^r$. Conversely, we prove that

Lemma 1. *Under Hypothesis* 1 *the semigroup* (3) *has at most one idempotent element.*

P r o o f. Let $p = a^m$, $q = a^n$ be idempotent. Then (10) implies

$$p = p^n = (a^m)^n = (a^n)^m = q^m = q.$$

□

Thus we get

Theorem 2 ([7], Lemma 1.7). *Under Hypothesis* 1 *the semigroup* (3) *has exactly one idempotent element* $p = a^r$ *with* $r = r(a)$ *from* (6).

Using the Lagrange theorem (cf. e.g. [6], p. 122), as a corollary from Theorems 1, 2 we obtain

Theorem 3. *Under Hypothesis* 1 *for every* $b \in \langle a \rangle$ *the semigroup* $\langle b \rangle$ *has the same idempotent element as* $\langle a \rangle$*. Moreover (cf.* (1)–(6)*)*

$$K_b \subset K_a, \quad d(b) \mid d(a). \tag{11}$$

Observe that in the case of the index $k(b)$ for $b \in \langle a \rangle$ we only have the inequality $k(b) \leqslant k(a)$. More precisely, if $b = a^m$, then

$$m(k(b) - 1) < k(a) \leqslant mk(b).$$

2. Partially ordered semigroups

Now we consider a semigroup with an order relation (cf. [1], Chapter XIV).

Definition 4 ([3]). A semigroup (group) $(S, *, \leqslant)$ with a partial order relation "$\leqslant$" is partially ordered if

$$a \leqslant b \Rightarrow (a * c \leqslant b * c, \;\; c * a \leqslant c * b) \quad \text{for all } \; a, b, c \in S. \tag{12}$$

Now the additional assumption has the form

Hypothesis 2. $(S, *, \leqslant)$ is a partially ordered semigroup.

Definition 5 (cf. [2]). An element b of a partially ordered semigroup is subidempotent if

$$b^2 \leqslant b \tag{13}$$

and superidempotent if

$$b^2 \geqslant b. \tag{14}$$

Using (12) we see that

$$(15)\qquad b^2 \leqslant b \Rightarrow b^{n+1} \leqslant b^n \leqslant b, \ (b^n)^2 \leqslant b^n \quad \text{for all } n,$$

$$(16)\qquad b^2 \geqslant b \Rightarrow b^{n+1} \geqslant b^n \geqslant b, \ (b^n)^2 \geqslant b^n \quad \text{for all } n.$$

As a consequence of the inequalities on the right-hand sides we obtain

Lemma 2. *Assume Hypothesis* 2. *If $b \in S$ is subidempotent (superidempotent) then all its powers are subidempotent (superidempotent).*

We compose conditions from Definitions 1 and 4.

Theorem 4. *Assume Hypotheses* 1, 2. *If a is subidempotent (superidempotent), then all elements of the semigroup* (3) *are subidempotent (superidempotent) and form a descending chain $a \geqslant a^2 \geqslant \ldots \geqslant a^r$ (an ascending chain $a \leqslant a^2 \leqslant \ldots \leqslant a^r$), where $r(a) = k(a)$, $d(a) = 1$. The kernel* (5) *reduces to the singleton $\{a^r\}$, and a^r is the zero element of the semigroup* (3).

P r o o f. By Lemma 2 all elements of the semigroup (3) are of the same kind and the sequence of powers is monotonic by (15) or (16). But a is of finite order and a suitable inequality $a^k \leqslant a^{k+1} \leqslant \ldots \leqslant a^{k+d} = a^k$ changes into the equality $a^k = a^{k+1} = a^{k+2} = \ldots$. Therefore, $d(a) = 1$ and $r(a) = k(a)$ by (6). Moreover $a^i a^r = a^{r+i} = a^r$ for $i \in \mathbb{N}$, i.e. a^r is the zero element of the semigroup (3). □

Under the assumptions of the above theorem all powers in (3) are comparable. Conversely, if all powers of a are comparable then $a^2 \leqslant a$ or $a \leqslant a^2$. Thus we have

Corollary 1. *Assume Hypotheses* 1, 2. *The semigroup* (3) *is linearly ordered iff a is subidempotent or superidempotent.*

In general the comparability of elements of a cyclic semigroup is not necessary. There exist cyclic subsemigroups of a semigroup with a partial order without pairs of comparable elements.

Example 1. Let us consider

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 4 & 4 & 5 & 6 & 9 & 8 \end{pmatrix}.$$

We can find that all elements of $\langle f \rangle = \{f, f^2, f^3, f^4\}$ are non-comparable and $f^5 = f^3$, $f^6 = f^4$, $k(f) = 3$, $d(f) = 2$, $r(f) = 4$. Similarly for the restrictions

$$g = f|_{\{1,\ldots,7\}}, \quad h = f|_{\{8,9\}}$$

we get

$$\langle g\rangle = \{g, g^2, g^3\},\quad k(g)=3,\quad d(g)=1,\quad r(g)=3,$$
$$\langle h\rangle = \{h, h^2\},\quad k(h)=1,\quad d(h)=2,\quad r(h)=2,$$

with all elements non-comparable.

This example leads us to the question of existence of comparable elements and their properties. By analogy to Definition 1 we put

Definition 6. Assume Hypotheses 1, 2. The comparability index of a is the number

$$c = c(a) = \min\{n \in \mathbb{N}\colon\ \exists\, m > n\colon\ (a^m \leqslant a^n \ \text{ or }\ a^m \geqslant a^n)\}. \tag{17}$$

In virtue of (1) we see that $c(a) \leqslant k(a)$. Sometimes this inequality changes into the equality (e.g. in Example 1). The problem arises if the equality $c = k$ characterizes semigroups (4) without comparable elements. First we prove that (8) can be generalized to the case of comparability.

Theorem 5 (cf. [7], [5]). *Under Hypotheses* 1, 2

$$(a^m \leqslant a^n) \Rightarrow d \mid (n-m) \quad \textit{for all } m, n. \tag{18}$$

P r o o f (cf. [5], Theorem 3.3). If $m = n$, then $d \mid (n-m)$. Let $m < n$, $p = n-m$. By assumption we get

$$a^m \leqslant a^{m+p} \leqslant a^{m+2p} \leqslant \ldots$$

but this increasing sequence has a finite number of different elements, and there exists h such that $a^{m+hp} = a^{m+(h+1)p}$. Therefore $m + hp \geqslant k$, by (1) and $d \mid p$ by (8). For $m > n$ the argument is similar. □

Using the inequalities (13) and (14) for $b = a^n$ we see that $m - n = 2n - n = n$ and we obtain

Corollary 2 (cf. [7], Lemma 1.8). *Assume Hypotheses* 1, 2. *If a^n is a subidempotent or superidempotent element then $d \mid n$. Therefore all subidempotent or superidempotent powers of a are contained in the subsemigroup generated by $b = a^d$.*

The same situation is in the kernel (5) and we obtain

Corollary 3. *Under Hypotheses* 1, 2 *the unique subidempotent (superidempotent) element of K_a is a^r.*

Since exponents of elements of K_a differ by less than d, then by (18) we get

Corollary 4. *Under Hypotheses* 1, 2 *if* $d > 1$, *then all elements of* K_a *are non-comparable (antichain).*

In order to distinguish the three possible cases in (17) we introduce (cf. [3], p. 154)

Definition 7. Assume Hypotheses 1, 2. The semigroup (4) is indifferent, if $c(a) = k(a)$. It is semi-positive (semi-negative) if

$$c(a) < k(a), \quad \text{and} \quad a^c \leqslant a^m \ \ (a^c \geqslant a^m) \tag{19}$$

for a certain $m > c$.

We will explain the meaning of this definition. First, directly from equality $c(a) = k(a)$, none of elements $a, \dots, a^{k-1}$ is comparable with other powers of a. Next, the elements $a^k, \dots, a^{k+d-1}$ are non-comparable because of Corollary 4. Therefore we have

Theorem 6. *Assume Hypotheses* 1, 2. *The semigroup* (4) *is indifferent iff all its elements are non-comparable.*

Lemma 3 (cf. [7], Lemma 1.4). *Assume Hypotheses* 1, 2. *If* a^n *is comparable with* a^m *for some* $m > n \geqslant c$, *then there exists* $s \geqslant k$ *such that* a^n *is comparable with* $a^s \in K_a$ *and both inequalities have the same direction (increasing or decreasing with respect to exponents).*

Proof. If $a^n \leqslant a^m$, then by (12)

$$a^n \leqslant a^{n+(m-n)} \leqslant a^{n+2(m-n)} \leqslant \ldots \leqslant a^{n+l(m-n)}.$$

Since $n+l(m-n) \geqslant k$ for sufficiently large l, then $a^{n+l(m-n)} \in K_a$, i.e. $s = n+l(m-n)$ and $a^n \leqslant a^s$. For $a^n \geqslant a^m$ the proof is similar. □

As an immediate consequence we have

Corollary 5. *Assume Hypotheses* 1, 2. *If* $c(a) < k(a)$, *then all comparable elements are bounded by some elements of* K_a.

Now we prove that the power function $h(n) = a^n$, $n \in \mathbb{N}$, restricted to $\{c, \dots, k + d - 1\}$ has a partial monotonicity.

Theorem 7. *Assume Hypotheses* 1, 2. *If the semigroup* (4) *is semi-positive (semi-negative) and* a^m, a^n *are comparable for some* $m > n$, *then* $a^m \geqslant a^n$ ($a^m \leqslant a^n$).

P r o o f. Let the semigroup (4) be semi-positive. Suppose that $a^n > a^m$ for some $m > n \geqslant c$. By Lemma 3 there exists $s \geqslant k$ such that $a^n \geqslant a^s \in K_a$. Similarly, $a^c \leqslant a^{c+p}$ implies $a^n = a^{c+(n-c)} \leqslant a^{c+p+(n-c)}$ and, by Lemma 3, $a^n \leqslant a^t \in K_a$. According to (18) $s = n + \alpha d$, $t = n + \beta d$, i.e. $s - t = (\alpha - \beta)d$. Since $s, t \in \{k, \dots, k+d-1\}$, then $s - t = 0$ and $a^n = a^s = a^t \in K_a$, which proves that $a^n = a^m$ (because both are in K_a), contradictory to the assumption. Therefore $a^n \leqslant a^m$ (concordant with $a^c \leqslant a^{c+p}$), and for the semi-negative semigroup (4) the proof is dual. □

All considered consequences of Theorem 5 can be summarized in the following (cf. [7], Lemma 1.9)

Theorem 8. *Assume Hypotheses* 1, 2. *If* $d > 1$ *and* $k > c$, *then the semigroup* (4) *has a spiral structure depicted on Fig.* 1. *The comparable elements are situated on the same radius and two arbitrary elements from different radii are non-comparable. Moreover, if the semigroup* (4) *is semi-positive (semi-negative), then the kernel* K_a *contains maximal (minimal) elements of* $\langle a \rangle$.

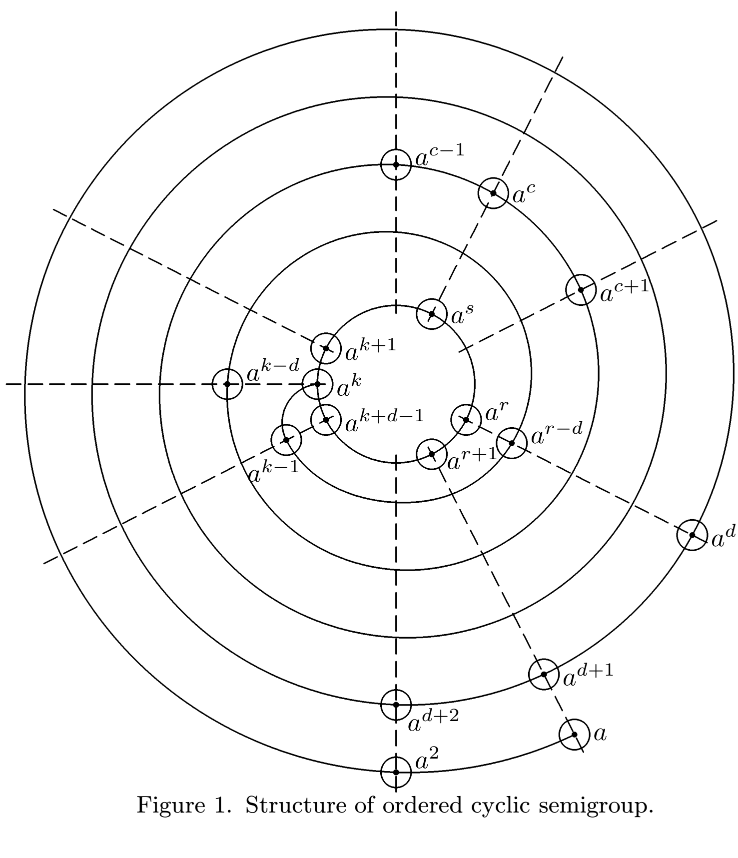

Figure 1. Structure of ordered cyclic semigroup.

We omit here the examination of maximal chains in the semigroup (4). The next example shows that all maximal chains can have length 2.

Example 2. Let $\mathbf{B}_n$ denote the set of all $n\times n$ Boolean matrices. For $R,S\in\mathbf{B}_n$ we use the max-min product $R\circ S$ and the partial order relation $R\leqslant S$

$$(R\circ S)_{ij}=\bigvee_{k=1}^{n} r_{ik}\wedge s_{kj}, \tag{20}$$

$$R\leqslant S\Leftrightarrow r_{ik}\leqslant s_{ik}\quad\text{for all } 1\leqslant i\leqslant n,\ 1\leqslant k\leqslant n. \tag{21}$$

For $R,S,T\in\mathbf{B}_n$ it is known that (cf. [10])

$$R\circ(S\circ T)=(R\circ S)\circ T,$$
$$R\leqslant S\Rightarrow R\circ T\leqslant S\circ T,\quad T\circ R\leqslant T\circ S.$$

So $(\mathbf{B}_n,\circ,\leqslant)$ is a partially ordered semigroup. Let $S\in\mathbf{B}_4$. If we put

$$S=\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&1\\1&1&1&1\end{pmatrix},$$

then we obtain the following list of maximal chains in $\langle S\rangle=\{S,\dots,S^4\}$: $S\leqslant S^4$, $S^2\leqslant S^4$, $S^3\leqslant S^4$. Thus elements S, S^2, S^3 are minimal and element S^4 is maximal. Moreover $k=r=4$, $d=1$, $c=1$.

A similar discussion can be lead in the case of subidempotent and superidempotent elements. By Corollary 2 all such elements lie on radius from a^d to a^r. But their existence depends on a position of c. Directly from Theorem 8 (cf. Fig. 1) we obtain

Theorem 9. *Assume Hypotheses* 1, 2. *If $c>r-d$, then $\langle a\rangle\setminus K_a$ does not contain subidempotent or superidempotent elements. If $c\leqslant r-d$, then a^{r-d} is comparable with $a^{2(r-d)}=a^r$, i.e. a^{r-d} is subidempotent in the semi-negative case (superidempotent in the semi-positive case). Moreover if $c\leqslant\frac{1}{2}r$, then the number of such elements is greater than $\frac{1}{2}r/d$.*

The case $c>r-d$ appeared in Example 1. In Example 2 we have $k=r=4$, $d=1$, $c=1$. Thus $c=1<2=\frac{1}{2}r$ and we find at least $\left[\frac{1}{2}r/d\right]=1$ superidempotent element in $\langle S\rangle\setminus K_S$. Actually S^2, S^4 and S^3, $S^9=S^4$ are comparable, i.e. S^2, S^3 are superidempotent (simultaneously $S^2=S^{r-d}$).

We see that exponents of subidempotent (superidempotent) powers are divisible by d. Since the successive multiples r and $r+d$ have this property, then d is the greatest common divisor of the exponents (cf. [7], Theorem 1.2):

$$\gcd\{s>0\colon\ a^s \text{ is sub-(super-)idempotent}\} = d.$$

Now we return to the general case described in Theorems 6–8. For indifferent semigroups it suffices to consider the two parameter model as in Theorem 1 (cyclic semigroups are represented by pairs $(k,d)\in\mathbb{N}\times\mathbb{N}$). Semi-positive and semi-negative semigroups have dual properties with parameters $k,d,c\in\mathbb{N}$, $c<k$. However, these parameters do not suffice in order to describe the family of partially ordered cyclic semigroups.

Example 3. Let $S\in\mathbf{B}_4$ (cf. Example 2). Putting

$$S=\begin{pmatrix}1&0&1&0\\1&0&0&0\\1&0&1&1\\1&1&1&1\end{pmatrix},$$

we get $\langle S\rangle=\{S,\dots,S^4\}$ with parameters $k=r=4$, $d=1$, $c=1$ as in Example 2. But here we have one maximal chain: $S\leqslant S^2\leqslant S^3\leqslant S^4$ (and S^2, S^3 are superidempotent elements from $\langle S\rangle\setminus K_S$).

We look for the next parameter characterizing ordered cyclic semigroups.

Definition 8. Assume Hypotheses 1, 2. The comparability number of a is the number

$$p=p(a)=\operatorname{card}(\text{“}<\text{”}\cap(\langle a\rangle\times\langle a\rangle)), \tag{22}$$

where “ $<$ ” $\Leftrightarrow$ “ $\leqslant$ ” and “ $\neq$ ”.

We have $p=0$ in Example 1, $p=3$ in Example 2 and $p=6$ in Example 3. The values of comparability numbers are not arbitrary and depend on the parameters k, d and c.

Theorem 10. *Assume Hypotheses* 1, 2. *If*

$$k-c=\alpha d+\beta,\quad 0\leqslant\beta<d, \tag{23}$$

then

$$k-c\leqslant p\leqslant k-c+\frac{\alpha(\alpha-1)}{2}\,d+\alpha\beta. \tag{24}$$

P r o o f. The left inequality in (24) is a direct consequence of Lemma 3. Additional pairs of comparable elements can be found on radii of Fig. 1. In view of (23) we have β radii with at most $\alpha+1$ comparable elements and $d-\beta$ radii with at most α comparable elements in $\langle a\rangle \setminus K_a$. Therefore we must add

$$\beta\frac{(\alpha+1)\alpha}{2}+(d-\beta)\frac{\alpha(\alpha-1)}{2}\,d=\frac{\alpha(\alpha-1)}{2}\,d+\alpha\beta$$

pairs of comparable elements, which proves the right inequality in (24). □

We see that the lower bound $p = 3$ was obtained in Example 2 and the upper bound $p = 6$ was obtained in Example 3. Thus the inequalities (24) give a sharp estimation of the comparability number. However we do not know if this parameter admits gaps in the sequence of values.

Conjecture 1. *For every $c, d, k, p \in \mathbb{N}$, $c \leqslant k$, satisfying* (24) *there exists an ordered cyclic semigroup* $(\langle a\rangle, \leqslant)$*, such that*

$$c = c(a), \quad d = d(a), \quad k = k(a), \quad p = p(a). \tag{25}$$

Example 4. The parameters above considered do not suffice to distinguish order relations on cyclic semigroups. If we consider

$$S=\begin{pmatrix}0&1&0&0\\0&0&1&0\\1&0&0&1\\1&1&1&1\end{pmatrix},$$

then the resulting cyclic semigroup $\langle S\rangle = \{S, \dots S^4\}$ has all the parameters:

$$c=1,\ \ k=4,\ \ d=1,\ \ p=3$$

as in Example 2. We also see (cf. Theorem 9) that outside of the kernel group there exist superidempotent elements: S^2, S^3.

3. Semigroups with a lattice order

Now we consider stronger assumptions on order relations in $(S,*,\leqslant)$ (cf. [1]).

Hypothesis 3. $(S,*,\vee,\wedge)$ is a partially ordered semigroup with a lattice order.

For $a\in S$, $k=k(a)$, $d=d(a)$, we use the following notations (cf. (4)–(7)):

$$u=u(a)=\sup K_a=\bigvee_{l=0}^{d-1}a^{k+l},\quad v=v(a)=\inf K_a=\bigwedge_{l=0}^{d-1}a^{k+l}, \tag{26}$$

$$\bar a=\bigvee_{n\geqslant 1}a^n=\bigvee_{n=1}^{k+d-1}a^n,\quad \underline{a}=\bigwedge_{n\geqslant 1}a^n=\bigwedge_{n=1}^{k+d-1}a^n. \tag{27}$$

More exactly, the first is a notation (cf. [7]), and the last equality is a simple consequence of Theorem 1. All the above elements exist in S as finite meets and joins of powers and we have

$$\underline{a}\leqslant v(a)\leqslant u(a)\leqslant\bar a. \tag{28}$$

In general these elements do not belong to the semigroup (3) (except under the conditions of Theorem 4).

Since the Hypothesis 3 is a generalization of Hypothesis 2, then for arbitrary $n\in\mathbb{N}$ we get

Lemma 4. *Assume Hypothesis* 3. *For every $c,b_l\in S$, $l=1,\dots,n$, we have*

$$c*\left(\bigwedge_{l=1}^{n}b_l\right)\leqslant\bigwedge_{l=1}^{n}(c*b_l),\quad \left(\bigwedge_{l=1}^{n}b_l\right)*c\leqslant\bigwedge_{l=1}^{n}(b_l*c), \tag{29}$$

$$c*\left(\bigvee_{l=1}^{n}b_l\right)\geqslant\bigvee_{l=1}^{n}(c*b_l),\quad \left(\bigvee_{l=1}^{n}b_l\right)*c\geqslant\bigvee_{l=1}^{n}(b_l*c). \tag{30}$$

These inequalities can be applied to elements from (26)–(27) and we get

Lemma 5. *Under Hypotheses* 1, 3 *we have*

$$u*a^l\geqslant u,\ \ a^l*u\geqslant u,\ \ v*a^l\leqslant v,\ \ a^l*v\leqslant v\quad \textit{for all } l\geqslant 1. \tag{31}$$

Directly from Lemmas 4, 5 we obtain (cf. [7], Lemma 1.11)

Theorem 11. *Under Hypotheses* 1, 3 *we have*

$$v^2 \leqslant v, \quad u^2 \geqslant u \tag{32}$$

i.e. $v(a)$ *is subidempotent and* $u(a)$ *is superidempotent.*

In view of Theorem 4 we get

Corollary 6. *Under Hypotheses* 1, 3 *there exists a sequence of elements*

$$v^{k(v)} \leqslant \ldots \leqslant v^2 \leqslant v \leqslant u \leqslant u^2 \leqslant \ldots \leqslant u^{k(u)}. \tag{33}$$

Powers of $\underline{a}$ and $\bar{a}$ can be placed inside or outside of this sequence.

Example 5. Using $a = f$ from Example 1 we get

$$\underline{a} \leqslant \underline{a}^2 \leqslant \underline{a}^3 = v(a) \leqslant u(a) = \bar{a}^3 \leqslant \bar{a}^2 \leqslant \bar{a}.$$

Similar situation occurs in Examples 2–4, but we can also obtain another inequality. If

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

then

$$v(S) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \underline{S} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\underline{S}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{S}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{S}^4 = \underline{S}^3.$$

We see that $k(\underline{S}) = 3$, and $\underline{S}^3 \leqslant \underline{S}^2 \leqslant \underline{S} \leqslant v(S)$. Dual properties of powers of $\bar{a}$ can be seen for min-max product of square matrices.

From Lemmas 4, 5 we only get

Corollary 7. *Under Hypotheses* 1, 3 *we have*

$$\Big(\bar{a}^n \geqslant \bigvee_{l \geqslant n} a^l \geqslant u\Big), \ \Big(\underline{a}^n \leqslant \bigwedge_{l \geqslant n} a^l \leqslant v\Big) \quad \textit{for all } n \geqslant 1. \tag{34}$$

In order to obtain more information about these powers we use the following generalizations of Hypothesis 3 (cf. [1]):

Hypothesis 4. Operation $*$ is meet-distributive in lattice $(S, \vee, \wedge)$, i.e.

$$a * (b \wedge c) = (a * b) \wedge (a * c), \quad (b \wedge c) * a = (b * a) \wedge (c * a) \quad \text{for all } a, b, c \in S. \tag{35}$$

Hypothesis 5. Operation $*$ is join-distributive in lattice $(S, \vee, \wedge)$, i.e.

$$a * (b \vee c) = (a * b) \vee (a * c), \quad (b \vee c) * a = (b * a) \vee (c * a) \quad \text{for all } a, b, c \in S. \tag{36}$$

As a simple modification of Lemma 4 we get

Lemma 6. *Let $n \in \mathbb{N}$, $c, b_l \in S$, $l = 1, \dots, n$. Under Hypothesis 4 we have*

$$c * \left(\bigwedge_{l=1}^{n} b_l \right) = \bigwedge_{l=1}^{n} (c * b_l), \quad \left(\bigwedge_{l=1}^{n} b_l \right) * c = \bigwedge_{l=1}^{n} (b_l * c). \tag{37}$$

Under Hypothesis 5 we have

$$c * \left(\bigvee_{l=1}^{n} b_l \right) = \bigvee_{l=1}^{n} (c * b_l), \quad \left(\bigvee_{l=1}^{n} b_l \right) * c = \bigvee_{l=1}^{n} (b_l * c). \tag{38}$$

Now we obtain

Lemma 7. *Under Hypotheses 1, 4 we have*

$$v * a^l = a^l v = v, \quad \underline{a} * a^l = a^l * \underline{a} \geqslant \underline{a} \quad \text{for all } l \geqslant 1, \tag{39}$$

$$u * v \geqslant u, \quad v * u \geqslant u, \tag{40}$$

$$\underline{a} * v = v * \underline{a} = v, \quad \bar{a} * v \geqslant u, \quad v * \bar{a} \geqslant u. \tag{41}$$

P r o o f. This is a simple consequence of Lemmas 5, 6. As an example we verify the right hand parts of (40) and (41). Using Lemmas 5, 6 we have

$$v * \bar{a} \geqslant v * u = \left(\bigwedge_{l=0}^{d-1} a^{k+l} \right) * u = \bigwedge_{l=0}^{d-1} (a^{k+l} * u) \geqslant u.$$

$\square$

Similarly we get

Lemma 8. *Under Hypotheses* 1, 5 *we have*

$$u * a^l = a^l * u = u, \quad \bar{a} * a^l = a^l * \bar{a} \leqslant \bar{a} \quad \text{for all } l \geqslant 1, \tag{42}$$

$$u * v \leqslant v, \quad v * u \leqslant v, \tag{43}$$

$$\underline{a} * u \leqslant v, \quad u * \underline{a} \leqslant v, \quad \bar{a} * u = u * \bar{a} = u. \tag{44}$$

As a consequence of the above lemmas we obtain

Theorem 12. *Under Hypotheses* 1, 4 *we have*

$$v^2 = v, \quad \underline{a}^2 \geqslant \underline{a}. \tag{45}$$

Under Hypotheses 1, 5 *we have*

$$u^2 = u, \quad \bar{a}^2 \leqslant \bar{a}. \tag{46}$$

Under Hypotheses 1, 4, 5 *we have* $u = v$, *i.e. the kernel group is a singleton* $K_a = \{a^r\}$.

Using Theorem 4 for the powers (34) we get

Corollary 8. *Under Hypotheses* 1, 4 *we have*

$$\underline{a} \leqslant \underline{a}^2 \leqslant \ldots \leqslant \underline{a}^{k(\underline{a})} = v.$$

Under Hypotheses 1, 5 *we have*

$$u = (\bar{a})^{k(\bar{a})} \leqslant \ldots \leqslant \bar{a}^2 \leqslant \bar{a}.$$

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