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ON OSCILLATION OF SOLUTIONS OF FORCED NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER

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Abstract. In this paper, necessary and sufficient conditions are obtained for every bounded solution of

$$[y(t)-p(t)y(t-\tau)]^{(n)}+Q(t)G\big(y(t-\sigma)\big)=f(t),\quad t\geqslant 0, \tag{$*$}$$

to oscillate or tend to zero as $t\to\infty$ for different ranges of $p(t)$. It is shown, under some stronger conditions, that every solution of $(*)$ oscillates or tends to zero as $t\to\infty$. Our results hold for linear, a class of superlinear and other nonlinear equations and answer a conjecture by Ladas and Sficas, Austral. Math. Soc. Ser. B *27* (1986), 502–511, and generalize some known results.

Keywords: oscillation, nonoscillation, neutral equations, asymptotic behaviour

MSC 2000: 34C10, 34C15, 34K40

1. Introduction

In recent years, a good deal of work has been done on the oscillation theory of higher order neutral delay-differential equations. In [1]–[4], [11], [17], [18], [23], [25], [26] the authors have considered oscillation of solutions of linear homogeneous equations of the form

$$[y(t)-p(t)y(t-\tau)]^{(n)}+Q(t)y(t-\sigma)=0 \tag{1}$$

or some more general linear homogeneous equations with several delays or variable delays. Sufficient conditions have been obtained under which every solution of (1) oscillates (see [1]–[4], [11], [23], [25]). Some authors (see [17], [18]) have obtained conditions so that every solution of

$$[y(t) - p(t)y(t-\tau)]^{(n)} + Q(t)y\big(\sigma(t)\big) = 0$$

or

$$[y(t) - py(t-\tau)]^{(n)} + \sum_{i=1}^{m} Q_i(t)y(t-\sigma_i(t)) = 0$$

oscillates or tends to zero as $t \to \infty$. In [2]–[4], [11], the results are obtained under the assumption $\int_0^\infty Q(t)\,\mathrm{d}t = \infty$. However, in [1], [25], a weaker condition $\int_0^\infty t^{n-1}Q(t)\,\mathrm{d}t = \infty$ is assumed. In [23], oscillation results are obtained under the assumption $\int_0^\infty Q(t)\,\mathrm{d}t < \infty$. The oscillatory and asymptotic behaviour of solutions of linear nonhomogeneous equations

$$\Big[y(t) + \sum_{i=1}^{l} p_i(t)y(t-\tau_i)\Big]^{(n)} \pm \sum_{j=1}^{m} Q_j(t)y(t-\sigma_j) = f(t)$$

are investigated in [16] under the assumption that f is a very rapidly oscillating function in the sense that

$$\int_0^\infty Q_k(t)F_\pm(t-\sigma_k)\,\mathrm{d}t = \infty$$

for some $k \in \{1,2,\dots,m\}$, where F is a real-valued n-times continuously differentiable function such that $F^{(n)}(t) = f(t)$. Nonlinear homogeneous equations of the form

$$[y(t) - p(t)y(t-\tau)]^{(n)} + Q(t)G\big(y(t-\sigma)\big) = 0 \tag{2}$$

or more general equations of the type (2) are studied in [5], [14], [15], [24]. In [24], sublinear cases satisfying $\lim\limits_{u\to 0}(G(u)/u) > \lambda > 0$ are dealt with under strong assumptions on Q. Sublinear cases satisfying

$$\int_0^{\pm c} \frac{\mathrm{d}u}{G(u)} < \infty \quad \text{for every } c > 0 \tag{3}$$

are considered in [15] under the assumption

$$\int_0^\infty Q(t)G\big((t-\sigma)^{n-1}\big)\,\mathrm{d}t = \infty. \tag{4}$$

On the other hand, superlinear cases satisfying

$$\int_{\pm c}^{\pm\infty} \frac{\mathrm{d}u}{G(u)} < \infty \text{ for every } c > 0 \tag{5}$$

are dealt with under the assumption

$$\int_0^\infty (t-\sigma)^{n-1} Q(t)\,\mathrm{d}t = \infty \tag{6}$$

in [14]. It seems that not much work has been done on nonlinear nonhomogeneous neutral equations of the form

$$[y(t) - p(t)y(t-\tau)]^{(n)} + Q(t)G\bigl(y(t-\sigma)\bigr) = f(t). \tag{7}$$

Equation (7) is studied under the assumptions (5) and (6) in [14] and under the assumptions (3) and (4) in [15]. In both papers, f is small in some sense. In most of these papers $p(t)$ lies in the range $-1 < p(t) \leqslant 0$ or $0 \leqslant p(t) < 1$.

In the literature, the conditions assumed differ from authors to authors due to the different techniques they use and the different type of equations they consider. Even the conditions assumed by different authors for similar type of equations are often not comparable. While considering Eq. (7) for the study of oscillation of its solutions, one is required to consider various ranges for $p(t)$, whether n is even or odd, $Q(t) > 0$ or < 0 or is oscillating, whether G is linear or sublinear or superlinear, and whether f is small in some sense or f is a rapidly oscillating function.

In this paper, we consider equations of the form (7), with $n \geqslant 2$, where p and $f \in C\bigl([0,\infty),\mathbb{R}\bigr)$, $Q \in C\bigl([0,\infty),[0,\infty)\bigr)$, $G \in C(\mathbb{R},\mathbb{R})$, $\tau > 0$ and $\sigma \geqslant 0$. Following assumptions are needed in the sequel:

(H_1) G is nondecreasing and $xG(x) > 0$ for $x \neq 0$;

(H_2) $\liminf\limits_{|u|\to\infty} G(u)/u > \alpha > 0$;

(H_3) $\int_0^\infty t^{n-1}Q(t)\,\mathrm{d}t = \infty$;

(H_4) $\int_0^\infty t^{n-2}Q(t)\,\mathrm{d}t = \infty$;

(H_5) There exists $F \in C^{(n)}\bigl([0,\infty),\mathbb{R}\bigr)$ such that $F^{(n)}(t) = f(t)$ and $\lim\limits_{t\to\infty} F(t) = 0$.

We may note that (H_4) implies (H_3) and (H_3) holds if and only if

$$\int_0^\infty (t-\gamma)^{n-1}Q(t)\,\mathrm{d}t = \infty,$$

where γ is a real number. Further, $\liminf\limits_{t\to\infty} Q(t) > \lambda > 0$ implies that $\int_0^\infty Q(t)\,\mathrm{d}t = \infty$ which is stronger than (H_4). Some authors ([2]–[4], [11], [24]) have worked with these strong conditions.

We consider the following ranges for $p(t)$:

(A_1) $0 \leqslant p(t) \leqslant p_1 < 1$,
(A_2) $-1 < p_2 \leqslant p(t) \leqslant 0$,
(A_3) $p_4 \leqslant p(t) \leqslant p_3 < -1$,
(A_4) $1 < p_5 \leqslant p(t) \leqslant p_6$,
(A_5) $1 \leqslant p(t) \leqslant p_7$,
(A_6) $0 \leqslant p(t) \leqslant p_8$,
(A_7) $0 \leqslant p(t) \leqslant 1$,
(A_8) $-p \leqslant p(t) \leqslant 0$

where p_i is a constant, $1 \leqslant i \leqslant 8$, and p is a positive constant.

In earlier papers [19], [20], [21], the authors studied oscillatory and asymptotic behaviour of solutions of (7) with $n = 1$, (H_1), (H_3) and (H_5) and for the different ranges of $p(t)$. Both necessary and sufficient conditions were obtained. The present study deals with Eq. (7) with $n \geqslant 2$ and superlinear assumption (H_2). However, some of the results in this paper also hold for sublinear cases. We may note that (H_2) includes the linear case. The prototype of G are

$$G(u) = |u|^{\gamma} \operatorname{sgn} u, \quad \gamma \geqslant 1 \text{ and } G(u) = u^{\delta}(\beta + |u|^{\gamma}),$$

where $\beta > 0$, $\gamma > 0$, and $\delta \geqslant 1$ is a ratio of odd integers. Our work also holds for homogeneous neutral delay equations of order n.

By a solution of (7) we mean a real-valued continuous function y on $[T_y - \varrho, \infty)$ for some $T_y \geqslant 0$, where $\varrho = \max\{\tau, \sigma\}$, such that $y(t) - p(t)y(t-\tau)$ is n-times continuously differentiable and (7) is satisfied for $t \in [T_y, \infty)$. A solution of (7) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

In Section 2, some lemmas are given. Sufficient conditions are obtained in Section 3 for oscillation and asymptotic behaviour of solutions of (7). Section 4 deals with necessary conditions.

2. Some lemmas

In this section we obtain some lemmas which are needed in Section 3.

Lemma 2.1. *Let $Q \in C\bigl([0,\infty),[0,\infty)\bigr)$ and $Q(t) \not\equiv 0$ on any interval of the form $[T,\infty)$, $T \geqslant 0$, and $G \in C(\mathbb{R},\mathbb{R})$ with $uG(u) > 0$ for $u \neq 0$. Let $y \in C\bigl([0,\infty),\mathbb{R}\bigr)$ with $y(t) > 0$ or $y(t) < 0$ for $t \geqslant t_0 \geqslant 0$. If $w \in C^{(n)}\bigl([0,\infty),\mathbb{R}\bigr)$ with*

$$w^{(n)}(t) = -Q(t)G\bigl(y(t-\sigma)\bigr), \quad t \geqslant t_0 + \sigma, \quad \sigma \geqslant 0, \tag{8}$$

and there exists an integer $n^* \in \{0, 1, 2 \ldots, n-1\}$ *such that* $\lim_{t\to\infty} w^{(n^*)}(t)$ *exists and* $\lim_{t\to\infty} w^{(i)}(t) = 0$ *for* $i \in \{n^*+1, \ldots, n-l\}$, *then*

$$w^{(n^*)}(t) = w^{(n^*)}(\infty) - \frac{(-1)^{n-n^*}}{(n-n^*-1)!} \int_t^{\infty} (s-t)^{n-n^*-1} Q(s) G\big(y(s-\sigma)\big)\,\mathrm{d}s$$

for large t.

Integrating (8) repeatedly $(n-n^*)$-times, the lemma is obtained.

Remark 1. Suppose that the conditions of Lemma 2.1 hold. If $y(t) > 0$ for $t \geqslant t_0$ and

$$w^{(n)}(t) \leqslant -Q(t)G\big(y(t-\sigma)\big), \quad t \geqslant t_0 + \sigma,$$

with the remaining conditions same as in Lemma 2.1, then

$$w^{(n^*)}(t) \geqslant w^{(n^*)}(\infty) - \frac{(-1)^{n-n^*}}{(n-n^*-1)!} \int_t^{\infty} (s-t)^{n-n^*-1} Q(s) G\big(y(s-\sigma)\big)\,\mathrm{d}s,$$

provided that $n - n^*$ is odd and

$$w^{(n^*)}(t) \leqslant w^{(n^*)}(\infty) - \frac{(-1)^{n-n^*}}{(n-n^*-1)!} \int_t^{\infty} (s-t)^{n-n^*-1} Q(s) G\big(y(s-\sigma)\big)\,\mathrm{d}s$$

provided that $n - n^*$ is even.

If $y(t) < 0$ for $t \geqslant t_0$ and

$$w^{(n)}(t) \geqslant -Q(t)G\big(y(t-\sigma)\big), \quad t \geqslant t_0 + \sigma,$$

with other conditions same as Lemma 2.1, then

$$w^{(n^*)}(t) \leqslant w^{(n^*)}(\infty) - \frac{(-1)^{n-n^*}}{(n-n^*-1)!} \int_t^{\infty} (s-t)^{n-n^*-1} Q(s) G\big(y(s-\sigma)\big)\,\mathrm{d}s,$$

provided that $n - n^*$ is odd. If $n - n^*$ is even then

$$w^{(n^*)}(t) \geqslant w^{(n^*)}(\infty) - \frac{(-1)^{n-n^*}}{(n-n^*-1)!} \int_t^{\infty} (s-t)^{n-n^*-1} Q(s) G\big(y(s-\sigma)\big)\,\mathrm{d}s.$$

Lemma 2.2 ([9], [12, p. 193]). *Let* $y \in C^n\big([0,\infty), \mathbb{R}\big)$ *be of constant* sign. *Let* $y^{(n)}(t)$ *be of constant* sign *and* $\not\equiv 0$ *in any interval* $[T, \infty)$, $T \geqslant 0$, *and* $y^{(n)}(t)y(t) \leqslant 0$. *Then there exists a number* $t_0 \geqslant 0$ *such that the functions* $y^{(j)}(t)$, $j = 1, 2, \ldots, n-1$, *are of constant* sign *on* $[t_0,\infty)$ *and there exists a number* $k\in\{1,3,\dots,n-1\}$ *when* n *is even or* $k\in\{0,2,4,\dots,n-1\}$ *when* n *is odd such that*

$$\begin{aligned} y(t)y^{(j)}(t) &>0 \quad \text{for } j=0,1,2,\dots,k,\ \ t\geqslant t_0,\\ (-1)^{n+j-1}y(t)y^{(j)}(t) &>0 \quad \text{for } j=k+1,k+2,\dots n-1,\ \ t\geqslant t_0. \end{aligned}$$

Lemma 2.3 ([7, p. 19]). *Let* $F,G,p\in C\bigl([t_0,\infty),\mathbb{R}\bigr)$, $t_0\geqslant 0$, *be such that*

$$F(t)=G(t)-p(t)G(t-\tau),\quad t\geqslant t_0+\tau,\ \ \tau\geqslant 0,$$

$G(t)>0$ *for* $t\geqslant t_0$, $\liminf\limits_{t\to\infty}G(t)=0$ *and* $\lim\limits_{t\to\infty}F(t)=L$ *exists. Let* $p(t)$ *satisfy* (A_2) *or* (A_3) *or* (A_6). *Then* $L=0$. *If* $G(t)<0$ *for* $t>t_0$, *then* $\liminf\limits_{t\to\infty}G(t)=0$ *is replaced by* $\limsup\limits_{t\to\infty}G(t)=0$ *in the above statement.*

Lemma 2.4. *Suppose that* (H_1)–(H_3) *and* (H_5) *hold. Let* $p(t)$ *be in the range* (A_5). *Let* $y(t)$ *be a solution of* (7) *such that* $y(t)>0$ *for* $t\geqslant t_0>0$ *and let*

$$w(t)=y(t)-p(t)y(t-\tau)-F(t) \tag{9}$$

for $t\geqslant t_0+\varrho$, *where* $\varrho=\max\{\tau,\sigma\}$. *Then either* $\lim\limits_{t\to\infty}w(t)=-\infty$ *or* $\lim\limits_{t\to\infty}w^{(i)}(t)=0$, $i=0,1,2,\dots,n-1$ *and* $(-1)^{n+k}w^{(k)}(t)<0$ *for* $k=0,1,2,\dots,n-1$ *and* $w^{(n)}(t)\leqslant 0$ *for large* t. *If* $y(t)<0$ *for* $t\geqslant t_0>0$, *then either* $\lim\limits_{t\to\infty}w(t)=\infty$ *or* $\lim\limits_{t\to\infty}w^{(i)}(t)=0$, $i=0,1,2,\dots,n-1$, $(-1)^{n+k}w^{(k)}(t)>0$ *for* $k=0,1,2,\dots,n-1$ *and* $w^{(n)}(t)\geqslant 0$ *for* $t\geqslant t_0+\varrho$.

Proof. Let $y(t)>0$ for $t\geqslant t_0$. From Eq. (7) we obtain

$$w^{(n)}(t)=-Q(t)G\bigl(y(t-\sigma)\bigr)\leqslant 0$$

for $t\geqslant t_0+\varrho$ and $w^{(n)}(t)\not\equiv 0$ in any interval of the form $[T,\infty)$, $T\geqslant 0$. Hence each of $w(t)$, $w'(t),\dots,w^{(n-1)}(t)$ is monotonic in $[t_1,\infty)$, $t_1>t_0+\varrho$. If $\lim\limits_{t\to\infty}w(t)=l$, then $-\infty\leqslant l\leqslant\infty$. Assume that $l=\infty$. Then $w(t)>0$ and $w'(t)>0$ for $t\geqslant t_1$. Since $w^{(n)}(t)\leqslant 0$ for $t\geqslant t_1$, then from Lemma 2.2 it follows that there exist $t_2>t_1$ and an integer n^* such that $0\leqslant n^*\leqslant n-1$, $n-n^*$ is odd,

$$w^{(i)}(t)>0\quad\text{for } i=0,1,2,\dots,n^*,\ \ t\geqslant t_2,$$

and

$$(-1)^{n+i-1}w^{(i)}(t)>0\quad\text{for } i=n^*+1,\dots,n-1,\ \ t\geqslant t_2.$$

Hence $\lim_{t\to\infty} w^{(n^*)}(t)$ exists and $\lim_{t\to\infty} w^{(i)}(t) = 0$ for $i = n^*+1, n^*+2, \dots, n-1$. If $n^* = 0$, then $0 \leqslant l < \infty$, a contradiction. Hence $1 \leqslant n^* \leqslant n-1$. From Lemma 2.1 it follows that

$$w^{(n^*)}(t) = L - \frac{(-1)^{n-n^*}}{(n-n^*-1)!}\int_t^\infty (s-t)^{n-n^*-1} Q(s) G\bigl(y(s-\sigma)\bigr)\,\mathrm{d}s$$

for $t \geqslant t_3 > t_2$, where L is a constant. Hence

$$\int_{t_3}^\infty (s-t_3)^{n-n^*-1} Q(s) G\bigl(y(s-\sigma)\bigr)\,\mathrm{d}s < \infty. \tag{11}$$

From this it follows, due to (H_3), that

$$\liminf_{t\to\infty}\bigl(G\bigl(y(t)\bigr)/t^{n^*}\bigr) = 0.$$

Hence $\liminf_{t\to\infty}(y(t)/t^{n^*}) = 0$ by (H_1) and (H_2). We can choose $M_0 > 0$ such that $w(t) > M_0 t^{n^*-1}$ for $t \geqslant t_4 \geqslant t_3$. Hence, for $0 < M_1 < M_0$, $y(t) - p(t)y(t-\tau) > M_1 t^{n^*-1}$, $t \geqslant t_5 > t_4$, by (H_5), that is,

$$y(t) > y(t-\tau) + M_1 t^{n^*-1}, \quad t \geqslant t_5, \tag{12}$$

due to (A_5). Let

$$T_0 > \max\Bigl\{\frac{(n^*-2)\tau}{3}, t_5\Bigr\}, \qquad M = \min\{y(t)\colon\ T_0 \leqslant t \leqslant T_0+\tau\}$$

and

$$0 < \beta < \min\Bigl\{\frac{M}{(T_0+\tau)^{n^*}}, \frac{M_1}{2n^*\tau}\Bigr\}.$$

Define, for $t \geqslant T_0$,

$$H(t) = \begin{cases} (M_1 - n^*\beta\tau)t^{n^*-1} + \beta\displaystyle\sum_{i=2}^{n^*}(-1)^i c(n^*,i)\tau^i t^{n^*-i}, & n^* \geqslant 2\\ M_1 - \beta\tau, & n^* = 1\end{cases}$$

where

$$c(n,i) = \frac{n!}{i!(n-i)!}.$$

If n^* is odd, we may write

$$\begin{aligned}\sum_{i=2}^{n^*}&(-1)^i c(n^*,i)\tau^i t^{n^*-i}\\ &= \bigl(c(n^*,2)\tau^2 t^{n^*-2} - c(n^*,3)\tau^3 t^{n^*-3}\bigr)\\ &\quad + \bigl(c(n^*,4)\tau^4 t^{n^*-4} - c(n^*,5)\tau^5 t^{n^*-5}\bigr)\\ &\quad + \ldots + (-1)^{n^*-1}(c(n^*,n^*-1)\tau^{n^*-1}t - c(n^*,n^*)\tau^{n^*}\bigr)\end{aligned}$$

to obtain

$$\sum_{i=2}^{n^*}(-1)^i c(n^*,i)\tau^i t^{n^*-i} > 0$$

because

$$c(n^*,i)\tau^i t^{n^*-i} > c(n^*,i+1)\tau^{i+1}t^{n^*-i-1}$$

if and only if

$$t > \frac{c(n^*,i+1)}{c(n^*,i)}\tau = \frac{(n^*-i)\tau}{i+1}$$

for $i = 2, 4, \ldots, n^*-1$. Further, $t \geqslant T_0$ implies that

$$t \geqslant T_0 > \frac{(n^*-2)\tau}{3} > \frac{(n^*-4)\tau}{5} > \ldots > \frac{\tau}{n^*}.$$

If n^* is even, then we put the terms in pair as above with the last positive term $(-1)^{n^*}c(n^*,n^*)\tau^{n^*}$. Thus $H(t) > 0$ for $t \geqslant T_0$. Since $y(t) \geqslant M$ for $T_0 \leqslant t \leqslant T_0+\tau$ and $\beta(T_0+\tau)^{n^*} < M$, then $y(t) > \beta t^{n^*}$ for $T_0 \leqslant t \leqslant T_0+\tau$. Using (12) we obtain, for $t \in [T_0+\tau, T_0+2\tau]$,

$$y(t) > y(t-\tau) + M_1 t^{n^*-1} > \beta(t-\tau)^{n^*} + M_1 t^{n^*-1} > \beta t^{n^*}$$

because, for $n^* \geqslant 2$,

$$\begin{aligned}\beta t^{n^*} < H(t) + \beta t^{n^*} &= (M_1 - n^*\beta\tau)t^{n^*-1} + \beta\big[(t-\tau)^{n^*} - t^{n^*} + n^*\tau t^{n^*-1}\big] + \beta t^{n^*}\\ &= M_1 t^{n^*-1} + \beta(t-\tau)^{n^*}\end{aligned}$$

and, for $n^* = 1$,

$$\beta t < H(t) + \beta t = M_1 + \beta(t-\tau).$$

Proceeding as above we have $y(t) > \beta t^{n^*}$ for $t \geqslant T_0$. Hence $\liminf_{t\to\infty}(y(t)/t^{n^*}) \geqslant \beta > 0$, a contradiction. Consequently, $-\infty \leqslant l < \infty$. Suppose that $-\infty < l < \infty$. Then $(-1)^{n+k}w^{(k)}(t) < 0$ for $k = 1, 2, \ldots, n-1$ and hence $\lim_{t\to\infty} w^{(i)}(t) = 0$, $i = 1, 2, \ldots, n-1$. Whether n is odd or even, we take $n^* = 0$ to obtain

$$w(t) = L_1 + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)G\big(y(s-\sigma)\big)\,\mathrm{d}s$$

for $t \geqslant t_1$ by Lemma 2.1, where L_1 is a constant. Hence

$$\int_{t_1}^\infty (s-t_1)^{n-1}Q(s)G\big(y(s-\sigma)\big)\,\mathrm{d}s < \infty.$$

From this it follows, due to (H_3), that $\liminf_{t\to\infty} G\big(y(t)\big) = 0$ and hence $\liminf_{t\to\infty} y(t) = 0$. If $z(t) = y(t) - p(t)y(t-\tau)$, then $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} w(t) = l$. Hence $\lim_{t\to\infty} z(t) = 0$ by Lemma 2.3. Thus $\lim_{t\to\infty} w(t) = 0$. Consequently, $\lim_{t\to\infty} w^{(i)}(t) = 0$, $i = 0,1,2,\ldots,n-1$ and $(-1)^{n+k}w^{(k)}(t) < 0$ for $k = 0,1,2,\ldots,n-1$.

If $y(t) < 0$ for $t \geqslant t_0$, then proceeding as above we obtain the necessary conclusion. Thus the lemma is proved. $\square$

Remark 2. The part of the proof of Lemma 2.4 following $-\infty < l < \infty$ is independent of the range of $p(t)$.

Lemma 2.5. *Let* (H_1), (H_2), (H_4) *and* (H_5) *hold and let* $p(t)$ *lie in the range* (A_2) *or* (A_7). *If* $y(t)$ *is a solution of* (7) *with* $y(t) > 0$ *for* $t \geqslant t_0 > 0$, *then* $\lim_{t\to\infty} w^{(i)}(t) = 0$, $i = 0,1,2,\ldots,n-1$, *and* $(-1)^{n+k}w^{(k)}(t) < 0$, $k = 0,1,2,\ldots,n-1$, *and* $w^{(n)}(t) \leqslant 0$ *for large* t, *where* $w(t)$ *is given by* (10). *If* $y(t) < 0$ *for* $t \geqslant t_0$, *then* $\lim_{t\to\infty} w^{(i)}(t) = 0$, $i = 0,1,2,\ldots,n-1$, *and* $(-1)^{n+k}w^{(k)}(t) > 0$ *for* $k = 0,1,2,\ldots,n-1$, *and* $w^{(n)}(t) \geqslant 0$ *for large* t.

Proof. Let $y(t) > 0$ for $t \geqslant t_0$. Then $w^{(n)}(t) \leqslant 0$ for $t \geqslant t_0 + \varrho$ and $w^{(n)}(t) \not\equiv 0$ in any interval of the form $[T,\infty)$, $T \geqslant 0$. Hence each of $w(t), w'(t), \ldots w^{(n-1)}(t)$ is monotonic in $[t_1,\infty)$, $t_1 > t_0 + \varrho$. Let $\lim_{t\to\infty} w(t) = l$, $-\infty \leqslant l \leqslant \infty$. If $l = \infty$, then $w(t) > 0$ and $w'(t) > 0$ for $t \geqslant t_1$. Proceeding as in Lemma 2.4, we obtain $n^* \geqslant 1$ and (11). Then (H_4) implies that $\liminf_{t\to\infty}\big(G\big(y(t)\big)/t^{n^*-1}\big) = 0$. Hence $\liminf_{t\to\infty}(y(t)/t^{n^*-1}) = 0$ by (H_1) and (H_2). Since $n^* \geqslant 1$, we can choose $M_0 > 0$ such that $w(t) > M_0 t^{n^*-1}$ for $t \geqslant t_4 \geqslant t_5$. Thus

$$\liminf_{t\to\infty} \frac{y(t)}{w(t)} = 0. \tag{13}$$

Set, for $t \geqslant t_4$,

$$p^*(t) = p(t)w(t-\tau)/w(t).$$

Since $w(t)$ is increasing, then $0 \leqslant p^*(t) < p(t) \leqslant 1$ or $-1 < p_2 \leqslant p(t) < p^*(t) \leqslant 0$, respectively, when $p(t)$ is in the range (A_7) or (A_2). As $\lim_{t\to\infty}\big(F(t)/w(t)\big) = 0$, we have

$$\begin{aligned}
1 = \lim_{t\to\infty} \frac{w(t)}{w(t)} &= \lim_{t\to\infty}\left[\frac{y(t) - p(t)y(t-\tau) - F(t)}{w(t)}\right] \\
&= \lim_{t\to\infty}\left[\frac{y(t)}{w(t)} - \frac{p^*(t)y(t-\tau)}{w(t-\tau)} - \frac{F(t)}{w(t)}\right] \\
&= \lim_{t\to\infty}\left[\frac{y(t)}{w(t)} - \frac{p^*(t)y(t-\tau)}{w(t-\tau)}\right].
\end{aligned}$$

Use of Lemma 2.3 yields, due to (13), that

$$\lim_{t\to\infty}\left[\frac{y(t)}{w(t)}-\frac{p^*(t)y(t-\tau)}{w(t-\tau)}\right]=0,$$

a contradiction. Hence $l\neq\infty$. If possible, let $l=-\infty$. For every $\beta>0$, there exists $t_5>t_1$ such that $w(t)<-\beta$ for $t\geqslant t_5$. If $p(t)$ is in the range (A_7), then for $t\geqslant t_5$,

$$\begin{aligned} y(t) <& -\beta+p(t)y(t-\tau)+F(t)\\ \leqslant& -\beta+y(t-\tau)+F(t). \end{aligned}$$

Hence, for $t\geqslant t_5+k\tau$,

$$\begin{aligned} y(t) <& -2\beta+y(t-2\tau)+F(t)+F(t-\tau)\\ &\vdots\\ <& -k\beta+y(t-k\tau)+F(t)+F(t-\tau)+\ldots+F(t-(k-1)\tau), \end{aligned}$$

where $k>0$ is an integer. For $0<\varepsilon<\beta$, there exists a $t_6>t_5+k\tau$ such that $|F(t)|<\varepsilon$ for $t\geqslant t_6$. Hence, for $t\geqslant t_6+k\tau$,

$$y(t)<-k\beta+y(t-k\tau)+k\varepsilon$$

implies that

$$y(t_6+k\tau)<-k(\beta-\varepsilon)+y(t_6).$$

Thus $y(t_6+k\tau)<0$ for large k, a contradiction. If $p(t)$ is in the range (A_2), then

$$y(t)<-\beta+F(t)<-(\beta-\varepsilon)<0$$

for $t\geqslant t_6$, a contradiction. Hence $l\neq-\infty$. Thus $-\infty<l<\infty$. Then $(-1)^{n+k}w^{(k)}(t)<0$ for $k=1,2,\ldots,n-1$ and hence $\lim\limits_{t\to\infty}w^{(i)}(t)=0$, $i=1,2,\ldots,n-1$. Proceeding as in Lemma 2.4, we may show that $\lim\limits_{t\to\infty}w(t)=0$. Thus $(-1)^{n+k}w^{(k)}(t)<0$ for $k=0,1,2,\ldots,n-1$.

If $y(t)<0$ for $t\geqslant t_0$, then one may proceed as above to arrive at the conclusions. Thus the proof of the lemma is complete. □

Lemma 2.6. *Suppose that* (H_1), (H_3) *and* (H_5) *hold and* $p(t)$ *is in one of the ranges* (A_2), (A_3), *and* (A_6). *If* $y(t)$ *is a bounded solution of* (7) *such that* $y(t)>0$ *for* $t\geqslant t_0\geqslant 0$, *then* $\lim\limits_{t\to\infty}w^{(i)}(t)=0$, $i=0,1,2,\ldots,n-1$ *and* $(-1)^{n+k}w^{(k)}(t)<0$ *for* $k=0,1,2,\ldots,n-1$ *and* $w^{(n)}(t)\leqslant 0$ *for large* t, *where* $w(t)$ *is given by* (10). *If*

$y(t) < 0$ for $t \geqslant t_0$, then $\lim\limits_{t\to\infty} w^{(i)}(t) = 0$, $i = 0,1,2,\dots,n-1$ and $(-1)^{n+k}w^{(k)}(t) > 0$ for $k = 0,1,2,\dots,n-1$ and $w^{(n)}(t) \geqslant 0$ for large t.

Proof. Since $y(t)$ is bounded, then $w(t)$ is bounded. If $y(t) > 0$ for $t \geqslant t_0$, then $w^{(n)}(t) \leqslant 0$ for $t \geqslant t_0 + \varrho$ but $\not\equiv 0$. Hence $-\infty < l < \infty$, where $l = \lim\limits_{t\to\infty} w(t)$. The rest of the proof is similar to that of Lemma 2.4. The proof for the case $y(t) < 0$ for $t \geqslant t_0$ is similar. Hence the lemma is proved. □

3. SUFFICIENT CONDITIONS

In this section we study oscillatory and asymptotic behaviour of solutions of Eq. (7) and the associated homogeneous equation

$$[y(t) - p(t)y(t-\tau)]^{(n)} + Q(t)G\big(y(t-\sigma)\big) = 0, \quad t \geqslant 0. \tag{14}$$

Theorem 3.1. *Let n be odd. Suppose that (H_1)–(H_3) hold. If $p(t)$ is in the range (A_5), then every nonoscillatory solution of (14) tends to $+\infty$ or $-\infty$ as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (14). Hence $y(t) > 0$ or < 0 for $t \geqslant t_0 > 0$. Let $y(t) > 0$ for $\geqslant t_0$. The case $y(t) < 0$ for $t \geqslant t_0$ may be treated similarly. Setting

$$z(t) = y(t) - p(t)y(t-\tau) \tag{15}$$

for $t \geqslant t_0 + \varrho$, we obtain from Lemma 2.4 that either $\lim\limits_{t\to\infty} z(t) = -\infty$ or $\lim\limits_{t\to\infty} z^{(i)}(t) = 0$, $i = 0,1,2,\dots,n-1$ and $(-1)^{n+k}z^{(k)}(t) < 0$, $k = 0,1,2,..,n-1$ for large t. If the latter holds, then $z(t) > 0$ for large t, because n is odd. We may take $n^* = 0$ to obtain, by Lemma 2.1,

$$z(t) = z(\infty) + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)G\big(y(s-\sigma)\big)\,\mathrm{d}s$$

for $t \geqslant t_1 > t_0 + \varrho$. Hence

$$\int_{t_1}^\infty (s-t_1)^{n-1}Q(s)G\big(y(s-\sigma)\big)\,\mathrm{d}s < \infty.$$

Thus (H_3) implies that $\liminf\limits_{t\to\infty} G\big(y(t)\big) = 0$. Consequently, $\lim\inf\limits_{t\to\infty} y(t) = 0$ by (H_1). On the other hand, $z(t) > 0$ for $t \geqslant t_2 > t_1$ implies that $y(t) > p(t)y(t-\tau) \geqslant y(t-\tau)$ by (A_5). Hence $\liminf\limits_{t\to\infty} y(t) > 0$, a contradiction. Thus $\lim\limits_{t\to\infty} z(t) = -\infty$. Since $z(t) > -p(t)y(t-\tau) > -p_7y(t-\tau)$, then $\lim\limits_{t\to\infty} y(t) = \infty$. Thus the theorem is proved. □

Remark 3. Theorem 3.1 extends Theorem 1(a) in [10] and Theorem 1(a) in [1].

Corollary 3.2. *Let the conditions of Theorem* 3.1 *hold. Then every bounded solution of* (14) *oscillates.*

Example. The equation

$$\big(y(t) - 2y(t-\pi)\big)''' + 3y\left(t - \frac{3\pi}{2}\right) = 0, \quad t \geqslant 0,$$

admits a bounded oscillatory solution $y(t) = \sin t$. This illustrates Corollary 3.2.

Theorem 3.3. *Let* (H_1)–(H_3) *and* (H_5) *hold. Let* $p(t)$ *be in the range* (A_4). *If* $y(t)$ *is a bounded nonoscillatory solution of* (7), *then* $y(t) \to 0$ *as* $t \to \infty$. *If* $y(t)$ *is an unbounded nonoscillatory solution of* (7), *then* $\lim_{t\to\infty} |y(t)| = \infty$ *or* $\liminf_{t\to\infty} |y(t)| = 0$.

Proof. If $y(t)$ is a nonoscillatory solution of (7), then $y(t) > 0$ or < 0 for $t \geqslant t_0 > 0$. Let $y(t) > 0$ for $t \geqslant t_0$. The proof for the case $y(t) < 0$ for $t \geqslant t_0$ is similar. Set $w(t)$ as in (10) and $z(t)$ as in (15), for $t \geqslant t_0 + \varrho$. Hence either $\lim_{t\to\infty} w(t) = -\infty$ or $\lim_{t\to\infty} w^{(i)}(t) = 0$, $i = 0, 1, 2, \ldots, n-1$, and $(-1)^{n+k} w^{(k)}(t) < 0$, $k = 0, 1, 2, \ldots, n-1$, for large t, by Lemma 2.4. If $y(t)$ is bounded, then $w(t)$ is bounded and hence $\lim_{t\to\infty} w(t) \neq -\infty$. Thus, the latter holds. Then $\lim_{t\to\infty} w(t) = 0$ and hence $\lim_{t\to\infty} z(t) = 0$ by (H_5). Further, $z(t) \leqslant y(t) - p_5 y(t-\tau)$ and $\lim_{t\to\infty} z(t) = 0$ imply that

$$\begin{aligned} 0 &\leqslant \liminf_{t\to\infty} [y(t) - p_5 y(t-\tau)] \\ &\leqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} [-p_5 y(t-\tau)] \\ &= (1-p_5) \limsup_{t\to\infty} y(t). \end{aligned}$$

Hence $\limsup_{t\to\infty} y(t) = 0$. Thus $\lim_{t\to\infty} y(t) = 0$. Next suppose that $y(t)$ is unbounded. If $\lim_{t\to\infty} w(t) = -\infty$, then $\lim_{t\to\infty} y(t) = \infty$. If $\lim_{t\to\infty} w^{(i)}(t) = 0$, $i = 0, 1, 2, \ldots, n-1$ and $(-1)^{n+k} w^{(k)}(t) < 0$, $k = 0, 1, 2, \ldots, n-1$, then we take $n^* = 0$ whether n is odd or even and apply Lemma 2.1 to obtain, for $t \geqslant t_1 > t_0 + \varrho$,

$$\int_{t_1}^{\infty} (s - t_1)^{n-1} Q(s) G\big(y(s-\sigma)\big)\, \mathrm{d}s < \infty$$

as in the proof of Lemma 2.4. Hence $\liminf_{t\to\infty} G\big(y(t)\big) = 0$. Then $\liminf_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. □

The following example illustrates Theorem 3.3.

Example. The equation

$$\big(y(t)-4y(t-\log 2)\big)''+(2+\mathrm{e}^{-2t})y(t-\log 2)=\frac{1}{2}\mathrm{e}^{-t},\quad t\geqslant 0,$$

admits an unbounded nonoscillatory solution $y(t)=\mathrm{e}^{t}$.

Corollary 3.4. *Let n be even and let* (H_1)–(H_3) *hold. Suppose that $p(t)$ is in the range* (A_4)*. Then bounded nonoscillatory solutions of* (14) *tend to zero as $t\to\infty$. Further, if $y(t)$ is an unbounded nonoscillatory solution of* (14)*, then $\lim\limits_{t\to\infty}|y(t)|=\infty$ or $\liminf\limits_{t\to\infty}|y(t)|=0$.*

Remark 4. The first part of Corollary 3.4 answers a conjecture by Ladas and Sficas (see [10, p. 506]). In fact, the conjecture should state "every bounded solution of (14) tends to zero as $t\to\infty$ when n is even, $G(u)=u$, $p(t)=p>1$ and $Q(t)=q>0$", because such an equation may admit an unbounded solution. The following example illustrates this statement.

Example. The equation

$$\big(y(t)-4y(t-\log 2)\big)''+\mathrm{e}y(t-1)=0,\quad t\geqslant 0$$

admits a positive unbounded solution $y(t)=\mathrm{e}^{t}$.

Theorem 3.5. *Suppose that* (H_1), (H_3) *and* (H_5) *hold. If $p(t)$ is in one of the ranges* (A_1)–(A_4)*, then every bounded solution of* (7) *oscillates or tends to zero as $t\to\infty$.*

Proof. Let $y(t)$ be a bounded solution of (7). If $y(t)$ oscillates, then there is nothing to prove. Let $y(t)>0$ for $t\geqslant t_0>0$. The case $y(t)<0$ for $t\geqslant t_0$ may be treated similarly. From Lemma 2.6 it follows that $\lim\limits_{t\to\infty}w^{(i)}(t)=0$, $i=0,1,2,\dots,n-1$, where $w(t)$ is given by (10). Hence $\lim\limits_{t\to\infty}z(t)=0$, where $z(t)$ is set as in (15). If (A_1) holds, then from (15) it follows that

$$\begin{aligned}0&=\limsup_{t\to\infty}[y(t)-p(t)y(t-\tau)]\\&\geqslant\limsup_{t\to\infty}[y(t)-p_1y(t-\tau)]\\&\geqslant\limsup_{t\to\infty}y(t)+\liminf_{t\to\infty}[-p_1y(t-\tau)]\\&=(1-p_1)\limsup_{t\to\infty}y(t).\end{aligned}$$

Hence $\limsup_{t\to\infty} y(t) = 0$. Then $\lim_{t\to\infty} y(t) = 0$. If ($A_2$) or ($A_3$) holds, then $z(t) > y(t)$ implies that $\limsup_{t\to\infty} y(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$. If ($A_4$) holds, then (15) yields

$$\begin{aligned} 0 &= \liminf_{t\to\infty}[y(t) - p(t)y(t-\tau)] \\ &\leqslant \liminf_{t\to\infty}[y(t) - p_5 y(t-\tau)] \\ &\leqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}[-p_5 y(t-\tau)] \\ &= (1-p_5)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $p_5 > 1$, then $\limsup_{t\to\infty} y(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$. Thus the theorem is proved. ☐

Remark 5. Theorem 3.5 holds for linear, sublinear and superlinear equations.

Remark 6. The assumption (H_3) is not enough to show that every solution of (7) oscillates or tends to zero as $t \to \infty$. The following example illustrates this statement. Hence we consider the stronger assumption (H_4) in our next result.

Example. Consider

$$\begin{aligned} (16)\quad &[y(t) + py(t-1)]''' \\ &+ \left[\frac{1}{t^2(t-1)(\log(t-1)-1)} + \frac{p}{(t-1)^3(\log(t-1)-1)}\right] y(t-1) = 0, \quad t \geqslant 14, \end{aligned}$$

where $0 \leqslant p < 1$. Hence $p(t)$ is in the range (A_2). Further,

$$\int_{14}^{\infty} t^2 Q(t)\,\mathrm{d}t > \int_{14}^{\infty} \frac{\mathrm{d}t}{(t-1)(\log(t-1)-1)} = \int_{(\log 13)-1}^{\infty} \frac{\mathrm{d}z}{z} = \infty$$

and

$$\begin{aligned} \int_{14}^{\infty} tQ(t)\,\mathrm{d}t &= \int_{14}^{\infty} \frac{\mathrm{d}t}{t(t-1)(\log(t-1)-1)} + p\int_{14}^{\infty} \frac{t\mathrm{d}t}{(t-1)^3(\log(t-1)-1)} \\ &< \int_{(\log 13)-1}^{\infty} \frac{\mathrm{d}z}{z^2} + p\int_{(\log 13)-1}^{\infty} \frac{\mathrm{d}z}{z\mathrm{e}^{1+z}} + p\int_{(\log 13)-1}^{\infty} \frac{\mathrm{d}z}{z\mathrm{e}^{2(1+z)}} < \infty. \end{aligned}$$

Thus (H_3) holds but (H_4) fails. We may note that $y(t) = t(logt - 1)$ is an unbounded positive solution of (16) which $\to \infty$ as $t \to \infty$.

Theorem 3.6. *Let $p(t)$ be in one of the ranges* (A_1) *and* (A_2). *If* (H_1), (H_2), (H_4) *and* (H_5) *hold, then every solution of* (7) *oscillates or tends to zero as $t \to \infty$.*

Proof. If $y(t)$ is a nonoscillatory solution of (7), then $y(t) > 0$ or < 0 for $t \geqslant t_0 > 0$. Let $y(t) > 0$ for $t \geqslant t_0$. The proof for the case $y(t) < 0$ for $t \geqslant t_0$ is similar. It is enough to show that $\limsup_{t\to\infty} y(t) = 0$. From Lemma 2.5 it follows that $\lim_{t\to\infty} w(t) = 0$ and hence $\lim_{t\to\infty} z(t) = 0$, where $w(t)$ and $z(t)$ are given, respectively, by (10) and (15). We claim that $y(t)$ is bounded. If not, then there exists a sequence $\{tn\}$ such that $t_0 + \varrho < t_1 < t_2 < \ldots$, $t_n \to \infty$ and $y(t_n) \to \infty$ as $n \to \infty$ and

$$y(t_n) = \max\{y(t)\colon\ t_0 + \varrho \leqslant t \leqslant t_n\}.$$

If (A_1) holds, then, for large n,

$$z(t_n) \geqslant (y(t_n) - p_1 y(t_n - \tau) \geqslant (1 - p_1) y(t_n).$$

Hence $z(t_n) \to \infty$ as $n \to \infty$, a contradiction. If (A_2) holds, then $z(t_n) > y(t_n)$ implies that $\lim_{n\to\infty} z(t_n) = \infty$, a contradiction. Thus our claim holds. Proceeding as in Theorem 3.5 we may obtain $\limsup_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. $\square$

Remark 7. Theorem 3.6 is an extension of Theorem 2 in [2], and Theorem 1(b) and Theorem 2 in [1]. The following example illustrates Theorem 3.6.

Example. The equation

$$\bigl(y(t) - py(t - 2\pi)\bigr)'' + 2e^{-3\pi/2}(e^{2\pi} - p)y\Bigl(t - \frac{\pi}{2}\Bigr) = 0$$

admits an unbounded oscillatory solution $y(t) = e^t \cos t$, where $0 \leqslant p < 1$ or $-1 < p \leqslant 0$.

Theorem 3.7. *Let* $p(t)$ *be in the range* (A_8). *Suppose that* (H_1), (H_2) *and* (H_5) *hold. Let* $Q^*(t) = \min\{Q(t), Q(t - \tau)\}$. *If*

(H_6) $\int_0^\infty t^{n-2} Q^*(t)\,dt = \infty$,

(H_7) *for* $u > 0$ *and* $v > 0$, $G(uv) \leqslant G(u)G(v)$,

(H_7') $G(-u) = -G(u)$ *and*

(H_8) *for* $u > 0$ *and* $v > 0$, *there exists a* $\delta > 0$ *such that* $G(u) + G(v) \geqslant \delta G(u + v)$

hold, then every solution of (7) *oscillates or tends to zero as* $t \to \infty$.

Remark 8. We may note that (H_8) and (H_7') imply that, for $u < 0$, $v < 0$, there exists a $\beta > 0$ such that $G(u) + G(v) \leqslant \beta G(u + v)$.

Proof of Theorem 3.7. Let $y(t)$ be a nonoscillatory solution of (7). Then $y(t) > 0$ or < 0 for $t \geqslant t_0 > 0$. Let $y(t) > 0$ for $t \geqslant t_0$. Setting $w(t)$ and $z(t)$ as in (10) and (15), respectively, we obtain $z(t) > 0$, $w(t) = z(t) - F(t)$,

$$w^{(n)}(t) = -Q(t)G\bigl(y(t-\sigma)\bigr) \leqslant 0$$

for $t \geqslant t_0 + \varrho$ and $w^{(n)}(t) \not\equiv 0$ in any neighbourhood of infinity. Hence $w(t)$, $w'(t)$, $w''(t), \ldots, w^{(n-1)}(t)$ are monotonic and $\lim\limits_{t\to\infty} w(t) = l$, where $-\infty \leqslant l \leqslant \infty$. If $-\infty \leqslant l < 0$, then $z(t) < 0$ for large t, a contradiction. Hence $0 \leqslant l \leqslant \infty$. If $l = 0$, then $\lim\limits_{t\to\infty} z(t) = 0$ and hence $z(t) \geqslant y(t)$ implies that $\lim\limits_{t\to\infty} y(t) = 0$. Let $0 < l \leqslant \infty$. Then $w(t) > 0$ for large t. From Lemma 2.2 it follows that there exists an integer n^*, $0 \leqslant n^* \leqslant n-1$ and $t_1 > t_0 + \varrho$ such that $n - n^*$ is odd, $w^{(j)}(t) > 0$ for $j = 0, 1, 2, \ldots, n^*$ and $(-1)^{n+j-1} w^{(j)}(t) > 0$ for $j = n^*+1, n^*+2, \ldots, n-1$, $t \geqslant t_1$. Hence $\lim\limits_{t\to\infty} w^{(n^*)}(t)$ exists and $\lim\limits_{t\to\infty} w^{(i)}(t) = 0$ for $i = n^*+1, n^*+2, \ldots, n-1$. Further, for $n^* \geqslant 1$, it is possible to choose $M_0 > 0$ such that $w(t) > M_0 t^{n^*-1}$ for $t \geqslant t_2 > t_1$. Hence

$$\liminf_{t\to\infty} (z(t)/t^{n^*-1}) \geqslant M_0 > 0. \tag{17}$$

For $t \geqslant t_2 + \varrho$, (H$_7$) and (H$_8$) yield

$$\begin{aligned}
0 &= w^{(n)}(t) + Q(t)G\bigl(y(t-\sigma)\bigr) \\
&= w^{(n)}(t) + Q(t)G\bigl(y(t-\sigma)\bigr) + G\bigl(-p(t-\sigma)\bigr) \\
&\quad \times \bigl[w^{(n)}(t-\tau) + Q(t-\tau)G\bigl(y(t-\tau-\sigma)\bigr)\bigr] \\
&\geqslant w^{(n)}(t) + G(p)w^{(n)}(t-\tau) + Q^*(t)\bigl[G\bigl(y(t-\sigma)\bigr) + G\bigl(-p(t-\sigma)\bigr)G\bigl(y(t-\tau-\sigma)\bigr)\bigr] \\
&\geqslant w^{(n)}(t) + G(p)w^{(n)}(t-\tau) + Q^*(t)[G\bigl(y(t-\sigma)\bigr) + G(-p(t-\sigma)y(t-\tau-\sigma)] \\
&\geqslant w^{(n)}(t) + G(p)w^{(n)}(t-\tau) + \delta Q^*(t)G\bigl(y(t-\sigma) - p(t-\sigma)y(t-\tau-\sigma)\bigr),
\end{aligned}$$

that is,

$$[w(t) + G(p)w(t-\tau)]^{(n)} \leqslant -\delta Q^*(t)G\bigl(z(t-\sigma)\bigr).$$

Hence, for $t \geqslant t_3 > t_2 + \varrho$,

$$\begin{aligned}
w^{(n^*)}(t)G(p)w^{(n^*)}(t-\tau) \geqslant \bigl(1 + G(p)\bigr)w^{(n^*)}(\infty) \\
+ \frac{\delta}{(n-n^*-1)!}\int_t^\infty (s-t)^{n-n^*-1} Q^*(s)G\bigl(z(s-\sigma)\bigr)\,\mathrm{d}s
\end{aligned}$$

due to Remark 1. In particular,

$$\int_{t_3}^\infty (s-t_3)^{n-n^*-1} Q^*(s)G\bigl(z(s-\sigma)\bigr)\,\mathrm{d}s < \infty.$$

Hence $\liminf\limits_{t\to\infty}\bigl(G\bigl(z(t)\bigr)/t^{n^*-1}\bigr) = 0$ by (H_6). If $n^* = 0$, then $\liminf\limits_{t\to\infty} tG\bigl(z(t)\bigr) = 0$ implies that $\liminf\limits_{t\to\infty} z(t) = 0$, a contradiction to the fact that $\lim\limits_{t\to\infty} z(t) = \lim\limits_{t\to\infty} w(t) = l$ and $0 < l \leqslant \infty$. Hence $n^* \geqslant 1$. Consequently, $\liminf\limits_{t\to\infty}(z(t)/t^{n^*-1}) = 0$ due to (H_1) and (H_2). This is a contradiction to (17). Hence $0 < l \leqslant \infty$ is not possible. If $y(t) < 0$ for $t \geqslant t_0$, then one may use (H_7') and proceed as above to obtain $\lim\limits_{t\to\infty} y(t) = 0$. Thus the theorem is proved. □

Remark 9. The prototype of G in Theorem 3.7 is $G(u) = (\beta + |u|^\mu)|u|^\lambda \operatorname{sgn} u$, where $\beta \geqslant 1$, $\lambda > 0$, $\mu > 0$ and $\lambda + \mu \geqslant 1$ (see [8, p. 292]). Further, we may note that (H_6)⇒(H_4).

4. NECESSARY CONDITIONS

In the following we show that the condition (H_3) is necessary for every solution of (7) to oscillate or tend to zero as $t \to \infty$.

Theorem 4.1. *Let n be odd. Suppose that* (H_1) *and* (H_5) *hold and $p(t)$ is in the range* (A_1). *If every bounded solution of* (7) *oscillates or tends to zero as $t \to \infty$, then* (H_3) *is satisfied.*

Proof. If possible, let

$$\int_0^\infty t^{n-1} Q(t)\,\mathrm{d}t < \infty. \tag{18}$$

It is possible to choose large $t_0 > 0$ such that

$$\frac{G(1)}{(n-1)!}\int_{t_0}^\infty t^{n-1}Q(t)\,\mathrm{d}t < \frac{1-p_1}{5} \quad\text{and}\quad |F(t)| < \frac{1-p_1}{10} \quad\text{for } t \geqslant t_0. \tag{19}$$

Let

$$X = \Bigl\{y \in \mathrm{BC}\bigl([t_0,\infty),\mathbb{R}\bigr) \colon\ \frac{1-p_1}{10} \leqslant y(t) \leqslant 1\Bigr\}, \tag{20}$$

where $\mathrm{BC}\bigl([t_0,\infty),\mathbb{R}\bigr)$ is the Banach space of real valued bounded continuous functions on $[t_0,\infty)$ with supremum norm. Let

$$K = \{y \in \mathrm{BC}\bigl([t_0,\infty),\mathbb{R}\bigr) \mid y(t) \geqslant 0 \ \text{ for } \ t \geqslant t_0\}.$$

For $u, v \in \mathrm{BC}\bigl([t_0, \infty)\mathbb{R}\bigr)$, $u \leqslant v$ if and only if $v - u \in K$. If $u_0(t) = \frac{1}{10}(1 - p_1)$ for $t \geqslant t_0$, then $u_0 = \inf X$ and $u_0 \in X$. Let $\Phi \subset X^* \subset X$. If $v_0(t) = \sup\{v(t) \mid v \in X^*\}$, then $v_0 = \sup X^*$ and $v_0 \in X$. For $y \in X$, we define

$$(Ty)(t) = \begin{cases} p(t)y(t-\tau) - \dfrac{(-1)^n}{(n-1)!}\displaystyle\int_t^\infty (s-t)^{n-1}Q(s)G\bigl(y(s-\sigma)\bigr)\,\mathrm{d}s \\ \qquad\qquad + F(t) + \dfrac{1-p_1}{5}, \quad \text{for } t \geqslant t_0 + \varrho, \\ (Ty)(t_0 + \varrho), \quad \text{for } t_0 \leqslant t \leqslant t_0 + \varrho. \end{cases}$$

Clearly, $Ty\colon [t_0, \infty) \to \mathbb{R}$ is continuous. Further, for $t \geqslant t_0$,

$$Ty(t) \leqslant p_1 + \frac{1-p_1}{5} + \frac{1-p_1}{10} + \frac{1-p_1}{5} < 1$$

and

$$Ty(t) > \frac{1-p_1}{5} - \frac{1-p_1}{10} = \frac{1-p_1}{10}$$

due to (19). Hence $T\colon X \to X$. Further, for $u, v \in X$ with $u \leqslant v$, $Tu \leqslant Tv$ since G is nondecreasing. Then T has a fixed point $y_0 \in X$ by the Knaster-Tarski fixed-point theorem (see [7, p. 30]). Since n is odd, then y_0 is a solution of (7) for $t \geqslant t_0 + \varrho$ with $\frac{1}{10}(1 - p_1) \leqslant y_0(t) \leqslant 1$. Clearly, $y_0(t) \nrightarrow 0$ as $t \to \infty$. This completes the proof of the theorem.

Corollary 4.2. *Let n be odd,* (H_1) *and* (H_5) *hold and $p(t)$ be in the range* (A_1)*. Every bounded solution of* (7) *oscillates or tends to zero as $t \to \infty$ if and only if* (H_3) *holds.*

Proof. This follows from Theorems 3.5 and 4.1.

□

Theorem 4.3. *Let n be even and let the conditions of Theorem* 4.1 *hold. Suppose that G is Lispchitzian in intervals of the form $[a, b]$, $0 < a < b$. If every bounded solution of* (7) *oscillates or tends to zero as $t \to \infty$, then* (H_3) *holds.*

Proof. Suppose that (18) holds. There exists a large $t_0 > 0$ such that

$$\frac{L}{(n-1)!}\int_{t_0}^\infty t^{n-1}Q(t)\,\mathrm{d}t < \frac{1-p_1}{20} \quad \text{and} \quad |F(t)| < \frac{1-p_1}{20} \quad \text{for} \quad t \geqslant t_0, \tag{21}$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $\bigl[\frac{1}{10}(1 - p_1), 1\bigr]$. Set X as in (20). Hence X is a complete metric space, where the metric is induced by the supremum norm. For $y \in X$, we define

$$(Ty)(t)=\begin{cases} p(t)y(t-\tau)-\dfrac{(-1)^n}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)G\bigl(y(s-\sigma)\bigr)\,\mathrm{d}s \\ \qquad\qquad +F(t)+\dfrac{1-p_1}{5}, \quad \text{for } t\geqslant t_0+\varrho, \\ (Ty)(t_0+\varrho), \quad \text{for } t_0\leqslant t\leqslant t_0+\varrho. \end{cases}$$

Hence $Ty\colon [t_0,\infty)\to\mathbb{R}$ is continuous and, for $t\geqslant t_0$, $Ty(t)<p_1+\frac{1}{20}(1-p_1)+\frac{1}{5}(1-p_1)<1$ and $Ty(t)>-\frac{1}{20}(1-p_1)-\frac{1}{20}(1-p_1)+\frac{1}{5}(1-p_1)=\frac{1}{10}(1-p_1)$ by (21). Thus $TX\subseteq X$. For $u,v\in X$,

$$d(Tu,Tv)=\operatorname{Sup}\{|Tu(t)-Tv(t)|\colon\ t\geqslant t_0\}\leqslant\Bigl(p_1+\frac{1-p_1}{20}\Bigr)d(u,v).$$

Hence T is a contraction. Thus T has a unique fixed point $y_0\in X$ by the Banach contraction principle. Since n is even, then y_0 is a solution of (7) for $t\geqslant t_0+\varrho$ and $\frac{1}{10}(1-p_1)\leqslant y_0(t)\leqslant 1$. Hence the theorem is proved. □

Corollary 4.4. *Let n be even,* (H$_1$) *and* (H$_5$) *hold, G be Lipschitzian in every interval of the form $[a,b]$, $0<a<b$, and $p(t)$ be in the range* (A$_1$)*. Every bounded solution of* (7) *oscillates or tends to zero as $t\to\infty$ if and only if* (H$_3$) *holds.*

Proof. This follows from Theorems 3.5 and 4.3. □

Theorem 4.5. *Let* (H$_1$) *and* (H$_5$) *hold, G be Lipschitzian in intervals of the form $[a,b]$, $0<a<b$, and $p(t)$ be in the range* (A$_2$)*. If every bounded solution of* (7) *oscillates or tends to zero as $t\to\infty$, then* (H$_3$) *holds.*

Proof. The proof is similar to that of Theorem 4.3. However, if n is odd, then we define, for $y\in X$,

$$(Ty)(t)=\begin{cases} p(t)y(t-\tau)+\dfrac{1}{(n-1)!}\displaystyle\int_t^\infty (s-t)^{n-1}Q(s)G\bigl(y(s-\sigma)\bigr)\,\mathrm{d}s \\ \qquad\qquad +F(t)+\dfrac{1-4p_2}{5}, \quad \text{for } t\geqslant t_0+\varrho, \\ (Ty)(t_0+\varrho), \quad \text{for } t_0\leqslant t\leqslant t_0+\varrho. \end{cases}$$

If n is even, then T is defined as follows:

$$(Ty)(t)=\begin{cases} p(t)y(t-\tau)-\dfrac{1}{(n-1)!}\displaystyle\int_t^\infty (s-t)^{n-1}Q(s)G\bigl(y(s-\sigma)\bigr)\,\mathrm{d}s \\ \qquad\qquad +F(t)+\dfrac{1-4p_2}{5}, \quad \text{for } t\geqslant t_0+\varrho, \\ (Ty)(t_0+\varrho), \quad \text{for } t_0\leqslant t\leqslant t_0+\varrho. \end{cases}$$

□

Corollary 4.6. *Suppose that the conditions of Theorem* 4.5 *hold. Every bounded solution of* (7) *oscillates or tends to zero if and only if* (H_3) *holds.*

Proof. This follows from Theorems 3.5 and 4.5. □

Remark 10. Similar theorems may be established for the ranges (A_3) and (A_4).

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