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ON TOTAL INCOMPARABILITY OF MIXED TSIRELSON SPACES

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Abstract. We give criteria of total incomparability for certain classes of mixed Tsirelson spaces. We show that spaces of the form $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ with index $i(\mathcal{M}_k)$ finite are either c_0 or ℓ_p saturated for some p and we characterize when any two spaces of such a form are totally incomparable in terms of the index $i(\mathcal{M}_k)$ and the parameter θ_k . Also, we give sufficient conditions of total incomparability for a particular class of spaces of the form $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$ in terms of the asymptotic behaviour of the sequence $\left\| \sum_{i=1}^n e_i \right\|$ where (e_i) is the canonical basis.

Keywords: mixed Tsirelson spaces, totally incomparable spaces

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0. INTRODUCTION

Denote by c_{00} the vector space of all real valued sequences which are eventually zero and by $(e_i)_{i=1}^\infty$ its usual unit vector basis. For $E \subset \mathbb{N}$ and $x = \sum_{i=1}^\infty a_i e_i \in c_{00}$ we denote $Ex = \sum_{i \in E} a_i e_i$. Also, for finite subsets $E, F \subseteq \mathbb{N}$, we write $E < F$ (or $E \leq F$) if $\max E < \min F$ ($\max E \leq \min F$). For simplicity, we write $n \leq E$ instead of $\{n\} \leq E$.

Mixed Tsirelson spaces were introduced in full generality in [2]. We can define those spaces, denoted by $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$, as the completion of c_{00} under a norm which satisfies an implicit equation of the following kind:

$$\|x\| = \max \left\{ \|x\|_\infty, \sup_{k \in I} \left\{ \theta_k \sup_{n \in \mathbb{N}} \left\{ \sum_{i=1}^n \|E_i x\| \mid (E_i)_{i=1}^n \text{ } \mathcal{M}_k\text{-admissible} \right\} \right\} \right\}, \quad x \in c_{00}$$

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where the \mathcal{M}_k 's are certain (see Definition 4 below) families of finite subsets of \mathbb{N} , $\theta_k \in (0, 1]$ for all $k \in I \subseteq \mathbb{N}$ and $(E_i)_{i=1}^n$ is \mathcal{M}_k -admissible if there exists $\{m_1, \dots, m_n\} \in \mathcal{M}_k$ such that $m_1 \leq E_1 < m_2 \leq E_2 < \dots < m_n \leq E_n$.

The first remarkable space in this class is the so called Tsirelson space, introduced by Figiel and Johnson [7] in 1974. (It is actually the dual of the space originally constructed by Tsirelson in [12].) In our notation this space is $T[\mathcal{S}, 1/2]$, where \mathcal{S} is Schreier's class, that is, the set of subsets of \mathbb{N} of cardinality smaller than their first element. Since its construction it was usually considered a "pathological space", a place to look for counterexamples to statements in the Banach space theory. In fact, the reason why it was constructed was to provide a counterexample to the assertion "every Banach space contains c_0 or ℓ_p for some $1 \leq p < \infty$ ".

The second space of the class is Tzafriri space, introduced in 1979 in [13] ($T[(\mathcal{A}_k, \gamma/\sqrt{k})_{k \in \mathbb{N}}]$, $0 < \gamma < 1$ in our notation where \mathcal{A}_k is the set of subsets of \mathbb{N} of at most k elements), also constructed as a counterexample to a statement in the Banach space theory. In 1991 a third example, namely the Schlumprecht space $T[(\mathcal{A}_k, 1/\log_2(1+k))_{k \in \mathbb{N}}]$, was considered, see [11], and with its help a fruitful period started when many "classical" problems in the infinite dimensional Banach space theory were solved, such as the distortion problem or the unconditional basic sequence problem.

A common feature of the three Banach spaces mentioned above is that they do not contain any ℓ_p , $1 \leq p < \infty$ or c_0 . (Actually, in the case of Tzafriri spaces this has been proved, as far as we know, only for $0 < \gamma < 10^{-6}$, see [6].) Moreover, since ℓ_p , $1 \leq p < \infty$ and c_0 are minimal (recall that a Banach space X is minimal if every subspace of X contains a further subspace isomorphic to X) it easily follows that they are totally incomparable to any of the three examples above (recall that two Banach spaces X and Y are totally incomparable if no subspace of X is isomorphic to any of Y). We use the word "subspace" here and throughout the paper for "closed infinite dimensional subspace".

In 1986 Bellenot [3] showed that ℓ_p , $1 \leq p < \infty$ and c_0 are isomorphic to mixed Tsirelson spaces of the form $T[(\mathcal{A}_n, \theta)]$, $\theta \in (0, 1]$. This was somewhat surprising as it showed that ℓ_p , $1 \leq p < \infty$ and c_0 belong to a class of spaces up to then considered pathological.

It is well known that ℓ_p , $1 \leq p < \infty$ and c_0 are totally incomparable to each other. Moreover, ℓ_p and c_0 and the three examples, with $0 < \gamma < 10^{-6}$ in the case of Tzafriri space, are all totally incomparable to each other (see [6] for the details and also use the minimality of the Schlumprecht space). This shows that, at least in the examples considered, the modification of the θ_k 's or the \mathcal{M}_k 's produce totally incomparable spaces.

In the first section we discuss in full generality the case when $\theta_k = 1$ for some k . In this case, the spaces c_0 and ℓ_1 will play a crucial role.

In the second section we consider mixed Tsirelson spaces of the form $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$, $\theta_k \in (0, 1)$, with index $i(\mathcal{M}_k)$, as defined in [2], finite and we characterize when any two spaces of such a form are totally incomparable. This is done by following the ideas in [4] and showing that every such space is either c_0 or ℓ_p saturated for some p . Recall that given a Banach space Y , a Banach space X is Y saturated if every subspace of X contains a further subspace isomorphic to Y .

In the third section we focus on spaces of the form $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$, $\theta_k \in (0, 1]$, such that ℓ_1 is finitely block represented in every block subspace. We give sufficient conditions of total incomparability in terms of the asymptotic behaviour of the sequence $\left\| \sum_{i=1}^n e_i \right\|$ where (e_i) is the canonical basis. These conditions apply to cases different from those considered in [9].

Notation. If K is a subset of a Banach space X , $\overline{\text{Span}\{K\}}$ denotes the closure of the algebraic linear span of K . If $x = \sum_{i=1}^\infty a_i e_i \in c_{00}$, the support of x is the set $\text{supp}(x) = \{i \in \mathbb{N} \mid a_i \neq 0\}$. For $x, y \in c_{00}$ we write $x < y$ if $\text{supp}(x) < \text{supp}(y)$. We say that $E_1, \dots, E_n \subset \mathbb{N}$ are successive if $E_1 < E_2 < \dots < E_n$. The vectors x_1, \dots, x_n are successive if their supports are. A block sequence (x_i) is a sequence of successive vectors. The cardinality of a set E is denoted by $|E|$. The standard norm of ℓ_p , $1 \leq p \leq \infty$ is denoted by $\|\cdot\|_p$. Other unexplained notation is standard and can be found for instance in [8].

Definition 1. Let \mathcal{M} be a family of finite subsets of \mathbb{N} . We say that \mathcal{M} is compact if the set $\{\mathbb{N}_E \mid E \in \mathcal{M}\}$ is a compact subset of the Cantor set $\{0, 1\}^\mathbb{N}$ with the product topology.

Remark 1. In Definition 1, $\{0, 1\}^\mathbb{N}$ is identified with the space of all mappings $f: \mathbb{N} \rightarrow \{0, 1\}$ and \mathbb{N}_E is the characteristic function of E . In $\{0, 1\}^\mathbb{N}$, the convergence under the product topology is the pointwise convergence. Therefore if $E \subseteq \mathbb{N}$ is a finite set and \mathbb{N}_{E_k} converges to \mathbb{N}_E pointwise, there exists $N \in \mathbb{N}$ such that $E \subseteq E_k$ for all $k \geq N$.

Definition 2. Let \mathcal{M} be a family of finite subsets of \mathbb{N} . We say that \mathcal{M} is hereditary if $E \in \mathcal{M}$ and $F \subseteq E$ implies that $F \in \mathcal{M}$.

Definition 3. Let \mathcal{M} be a compact family of finite subsets of \mathbb{N} . We define a transfinite sequence $(\mathcal{M}^{(\lambda)})$ of subsets of \mathcal{M} as follows:

1. $\mathcal{M}^{(0)} = \mathcal{M}$.
2. $\mathcal{M}^{(\lambda+1)} = \{E \in \mathcal{M} \mid \mathbb{N}_E \text{ is a limit point of the set } \{\mathbb{N}_E \mid E \in \mathcal{M}^{(\lambda)}\}\}$.

3. If λ is a limit ordinal then $\mathcal{M}^{(\lambda)} = \bigcap_{\mu < \lambda} \mathcal{M}^{(\mu)}$.

We call the least λ for which $\mathcal{M}^{(\lambda)} \subseteq \{\emptyset\}$ the index of \mathcal{M} and denote it by $i(\mathcal{M})$.

Definition 4. Let $I \subseteq \mathbb{N}$. Let $(\mathcal{M}_k)_{k \in I}$ be a sequence of compact hereditary families of finite subsets of \mathbb{N} and let $(\theta_k)_{k \in I} \subset (0, 1]$. We denote by $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ the completion of c_{00} with respect to the norm defined by

$$\|x\| = \max \left\{ \|x\|_\infty, \sup_{k \in I} \left\{ \theta_k \sup_{n \in \mathbb{N}} \left\{ \sum_{i=1}^n \|E_i x\| \mid (E_i)_{i=1}^n \text{ } \mathcal{M}_k\text{-admissible} \right\} \right\} \right\}$$

and we call it the mixed Tsirelson space defined by the sequence $(\mathcal{M}_k, \theta_k)_{k \in I}$.

Remark 2. The existence of such a norm is shown, for instance, in [10]. It follows from the definition of the norm that the sequence $(e_i)_{i=1}^\infty$ is a normalized 1-unconditional basis for $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$.

Remark 3. There are two useful alternative ways to define the norm. Given $x = \sum_{n=1}^\infty a_n e_n \in c_{00}$,

(i) define a non decreasing sequence of norms on c_{00} :

$$\begin{aligned} |x|_0 &= \max_{n \in \mathbb{N}} |a_n|, \\ |x|_{s+1} &= \max \left\{ |x|_s, \sup_{k \in I} \left\{ \theta_k \sup_{n \in \mathbb{N}} \left\{ \sum_{i=1}^n |E_i x|_s \mid (E_i)_{i=1}^n \text{ } \mathcal{M}_k\text{-admissible} \right\} \right\} \right\}. \end{aligned}$$

Then $\|x\| = \sup_{s \in \mathbb{N} \cup \{0\}} |x|_s$.

(ii) Let $K_0 = \{\pm e_n \mid n \in \mathbb{N}\}$. Given $K_s, s \in \mathbb{N} \cup \{0\}$, let

$$\begin{aligned} K_{s+1} &= K_s \cup \{ \theta_k \cdot (f_1 + \dots + f_d) \mid k \in I, d \in \mathbb{N}, f_i \in K_s, i = 1, \dots, d \\ &\quad \text{are successive and } (\text{supp}(f_1), \dots, \text{supp}(f_d)) \text{ } \mathcal{M}_k\text{-admissible} \}. \end{aligned}$$

Let $K = \bigcup_{s=0}^\infty K_s$. Then $\|x\| = \sup\{f(x) \mid f \in K\}$.

The latter definition of the norm provides information about the dual space. Looking at the set K as a set of functionals it is not difficult to see that B_{X^*} is the closed convex hull of K , where the closure is taken either in the weak-* topology or in the pointwise convergence topology.

1. THE CASE $\theta_k = 1$

Let $J = \{k \in I \mid \theta_k = 1\}$. If J is not empty, we give information about the structure of $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ depending on the index $i(\mathcal{M}_k)$, $k \in J$. It is known that if $i(\mathcal{M}_k) \geq 2$ for some $k \in J$, then $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ contains an isomorphic copy of ℓ_1 . Actually it is possible to say much more as our next proposition shows.

Proposition 1. *If $i(\mathcal{M}_{k_0}) \geq 2$ for some $k_0 \in J$, then $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ is ℓ_1 -saturated.*

Proof. By the Bessaga-Pełczyński principle (see e.g. [6], p. 10), it suffices to show that every block subspace contains a further subspace isomorphic to ℓ_1 . Recall that a block subspace is a space of the form $\overline{\text{Span}}\{u_i, i \in \mathbb{N}\}$, with $(u_i)_{i=1}^\infty$ a block sequence.

Let $(u_i)_{i=1}^\infty$ be a block sequence. We are going to construct a subsequence $(u_{i_k})_{k=1}^\infty$ of $(u_i)_{i=1}^\infty$ equivalent to the ℓ_1 basis.

Let $\{p\} \in \mathcal{M}_{k_0}^{(1)}$. We can choose u_{i_1} such that $p < u_{i_1}$. Now, since $\{p\} \in \mathcal{M}_{k_0}^{(1)}$, there exists $n_1 \in \mathbb{N}$ such that $n_1 > u_{i_1}$ and $\{p, n_1\} \in \mathcal{M}_{k_0}$, so we can take u_{i_2} such that $n_1 < u_{i_2}$. Continuing in this manner, we can construct a subsequence $(u_{i_k})_{k=1}^\infty$ of $(u_i)_{i=1}^\infty$ such that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $u_{i_k} < n_k < u_{i_{k+1}}$ and $\{p, n_k\} \in \mathcal{M}_{k_0}$. It is now easy to see that $(u_{i_k})_{k=1}^\infty$ is equivalent to the ℓ_1 basis. \square

The following example shows a Tsirelson type space ℓ_1 -saturated but not isomorphic to ℓ_1 . It was shown to us by I. Deliyanni.

Example 1. Let $\mathcal{M} = \{F \subseteq \mathbb{N} \mid \exists i \in \mathbb{N} \text{ such that } F \subseteq \{1, 2^i\}\}$ and $\theta = 1$.

It is clear that $i(\mathcal{M}) = 2$. If $T[\mathcal{M}, \theta]$ were isomorphic to ℓ_1 then since ℓ_1 has a unique—up to equivalence—normalized unconditional basis, there would exist a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\frac{1}{C} \sum_{i=1}^n |a_i| \leq \left\| \sum_{i=1}^n a_i e_i \right\| \leq C \sum_{i=1}^n |a_i|.$$

Now taking $x = \sum_{i=2^{k+1}}^{2^{k+1}} e_i$ we would obtain $2^k - 1 \leq C$ for all $k \in \mathbb{N}$.

We now examine $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ with $i(\mathcal{M}_k) = 1$, $k \in J$. We will find different subspaces depending on whether the set $\bigcup_{k \in J} \mathcal{M}_k$ contains only a finite number of non singleton sets or not.

Proposition 2. Let $I' \subseteq I$ be such that $\bigcup_{k \in I'} \mathcal{M}_k$ contains only a finite number of non singleton sets.

- (1) If $I' \neq I$, then $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ is isomorphic to $T[(\mathcal{M}_k, \theta_k)_{k \in I \setminus I'}]$.
- (2) If $I' = I$, then $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ is isomorphic to c_0 .

Proof. (1) Let $\|\cdot\|$ and $\|\cdot\|'$ be the norms of the spaces $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ and $T[(\mathcal{M}_k, \theta_k)_{k \in I \setminus I'}]$, respectively. We will see that they are equivalent. Clearly, $\|\cdot\|' \leq \|\cdot\|$.

For the other inequality let $M = \max\left\{\max E \mid E \in \bigcup_{k \in I'} \mathcal{M}_k, \text{ non singleton}\right\}$ and write $x = \sum_{i=1}^{\infty} a_i e_i = \sum_{i=1}^M a_i e_i + \sum_{i=M+1}^{\infty} a_i e_i := x_1 + x_2$.

We have $\|x_1\| \leq M\|x\|'$ since $\|x_1\| = \left\|\sum_{i=1}^M a_i e_i\right\| \leq \sum_{i=1}^M |a_i| \leq \sum_{i=1}^M \|x\|_{\infty} \leq M\|x\|'$.

On the other hand, we show first by induction over s that $|x_2|_s \leq |x_2|'_s$. For $s = 0$ it is clear. Suppose now that it is true for s and let E_1, \dots, E_n be a sequence of finite subsets of \mathbb{N} , \mathcal{M}_k -admissible for some k . There are two possibilities, either $k \in I \setminus I'$ and then $\theta_k \sum_{i=1}^n |E_i x_2|_s \leq \theta_k \sum_{i=1}^n |E_i x_2|'_s \leq |x_2|'_{s+1}$, or $k \in I'$ and then, by hypothesis, $n = 1$, E_1 is \mathcal{M}_k -admissible and $\theta_k |E_1 x_2|_s \leq \theta_k |x_2|_s \leq |x_2|'_s \leq |x_2|'_{s+1}$.

Therefore, $\|x_2\| \leq |x_2|'$ and by 1-unconditionality, $\|x_2\|' \leq \|x\|'$. Thus, $\|x\| \leq (M+1)\|x\|'$. \square

For (2), it is easy to see that $T(\mathcal{M}_0, \theta_0)$ is isomorphic to c_0 , where $\mathcal{M}_0 = \{\{i\} \mid i \in \mathbb{N}\}$, and $\theta_0 = 1$. Now use (1) to get that $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ is isomorphic to $T[(\mathcal{M}_k, \theta_k)_{k \in I \cup \{0\}}]$ and once again to see that the latter is isomorphic to $T(\mathcal{M}_0, \theta_0)$.

Proposition 2 for $I' = J$ yields

Proposition 3. Let $J = \{k \in I \mid \theta_k = 1\}$.

- (1) Let $\bigcup_{k \in J} \mathcal{M}_k$ contain only a finite number of non singleton sets.
 - 1.1. If $J = I$, then $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ is isomorphic to c_0 .
 - 1.2. If $J \neq I$, then $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ is isomorphic to $T[(\mathcal{M}_k, \theta_k)_{k \in I \setminus J}]$.
- (2) Let $\bigcup_{k \in J} \mathcal{M}_k$ contain an infinite number of non singleton sets. Then $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ contains a subspace isomorphic to ℓ_1 .

Proof. (1) follows from Proposition 2. For (2), we will construct a subsequence $(e_{n_i})_{i=1}^{\infty}$ of $(e_i)_{i=1}^{\infty}$ equivalent to the ℓ_1 basis.

Let $M_1 \in \bigcup_{k \in J} \mathcal{M}_k$ be a non singleton. Let $n_1 = \min M_1$. Having chosen n_i , we can take $M_{i+1} \in \bigcup_{k \in J} \mathcal{M}_k$ a non singleton such that $\min M_{i+1} > \max M_i$, and take $n_{i+1} = \min M_{i+1}$.

Consider the sequence $(e_{n_i})_{i=1}^\infty$ and let's show that it is equivalent to the ℓ_1 basis.

Let $x = \sum_{i=1}^\infty a_i e_{n_i}$. By the definition of the norm and the fact that for every $N \in \mathbb{N}$ and $i < N$, $(\{n_i\}, [n_{i+1}, n_N] \cap \mathbb{N})$ is \mathcal{M}_k -admissible for some $k \in J$ we have

$$\|x\| \geq |a_1| + \left\| \sum_{i=2}^N a_i e_{n_i} \right\| \geq \dots \geq |a_1| + |a_2| + \dots + |a_N|.$$

The proof is complete since always $\|x\| \leq \|x\|_1$. □

Observe that in statement (2) of Proposition 3 we do not ensure ℓ_1 saturation. Actually, in some cases we can also find c_0 as a subspace. This is a consequence of the following general result.

Proposition 4. *Let \mathcal{M}_k be compact and hereditary for all $k \in I \subseteq \mathbb{N}$, $\theta_k \in (0, 1]$ for all $k \in I$. If for all $N \in \mathbb{N}$ there exists $n \geq N$ such that for all $M \in \bigcup_{k \in I} \mathcal{M}_k$ either $n < \min M$ or $n \geq \max M$, then $T[(\mathcal{M}_k, \theta_k)_{k \in I}]$ contains a subspace isomorphic to c_0 . Moreover, if $\theta_k = 1$ for all $k \in I$, the converse is true.*

Proof. We will construct a subsequence $(e_{n_i})_{i=1}^\infty$ of the basis $(e_i)_{i=1}^\infty$ equivalent to the basis of c_0 .

Let $N_1 = 1$. By hypothesis there exists $n_1 \geq N_1$ such that for all $M \in \bigcup_{k \in I} \mathcal{M}_k$, $n_1 < \min M$ or $n_1 \geq \max M$.

Suppose that n_i is chosen and write $N_{i+1} = n_i + 1$. Then there exists $n_{i+1} \geq N_{i+1}$ verifying the hypothesis. Now, consider the sequence $(e_{n_i})_{i=1}^\infty$.

Let $x = \sum_{i=1}^\infty a_i e_{n_i} \in c_{00}$ and write $|x|_0 = \|x\|_\infty$ as in Remark 3.

Let $(E_i)_{i=1}^n$ be a sequence of finite subsets of \mathbb{N} , \mathcal{M}_k -admissible for some $k \in I$. Then we have $\theta_k \sum_{i=1}^n |E_i x|_0 = \theta_k |E_{i_0} x|_0 \leq |x|_0$ and so $|x|_1 \leq |x|_0$. Indeed, the first equality is true since by the construction of (n_i) , there exists at most one E_i such that $\text{supp}(x) \cap E_i \neq \emptyset$ and the inequality is straightforward by 1-unconditionality. So we have proved that $|x|_1 = |x|_0$ and therefore $|x|_n = |x|_{n+1}$ and $\|x\| = \|x\|_\infty$. □

The converse is a consequence of the following

Claim. *If there is an N_0 such that for all $n \geq N_0$, there exists $M \in \bigcup_{k \in I} \mathcal{M}_k$ such that $\min M \leq n < \max M$, then every normalized block sequence in $T[(\mathcal{M}_k, 1)_{k \in I}]$ has a subsequence equivalent to the canonical basis of ℓ_1 and in particular, $T[(\mathcal{M}_k, 1)_{k \in I}]$ is ℓ_1 -saturated.*

Proof of Claim. Let $(x_i)_{i=1}^\infty$ be a normalized block sequence. Let i_1 be such that $N_0 \leq \min x_{i_1}$. We split $x_{i_1} = \sum_{k=p_1+1}^{p_2} a_k e_k$ in the following manner:

Let $A^{(1)}(x_{i_1}) = \left\{ j > \min x_{i_1} \mid \{t, j\} \in \bigcup_{k \in I} \mathcal{M}_k, t \leq \min x_{i_1} \right\}$. By hypothesis $A^{(1)}(x_{i_1})$ is not empty and $j^{(1)}(x_{i_1}) := \min A^{(1)}(x_{i_1}) > \min x_{i_1}$.

Therefore,

$$x_{i_1} = \sum_{k=p_1+1}^{p_2} a_k e_k = \sum_{k=p_1+1}^{j^{(1)}(x_{i_1})-1} a_k e_k + \sum_{k=j^{(1)}(x_{i_1})}^{p_2} a_k e_k := x_{i_1}^{(1)} + u^{(1)}.$$

Let $y_{i_1}^{(1)} = x_{i_1}^{(1)} / \|x_{i_1}^{(1)}\|$. Suppose $y_{i_1}^{(l)}$ is defined and we have $x_{i_1} = x_{i_1}^{(1)} + \dots + x_{i_1}^{(l)} + u^{(l)}$. If $u^{(l)} \neq 0$, define $x_{i_1}^{(l+1)} = (u^{(l)})^{(1)}$ and $y_{i_1}^{(l+1)} = x_{i_1}^{(l+1)} / \|x_{i_1}^{(l+1)}\|$ and keep going until we have $u^{(d_1)} = 0$ for some $d_1 \in \mathbb{N}$. Then we have $x_{i_1} = \sum_{l=1}^{d_1} \|x_{i_1}^{(l)}\| y_{i_1}^{(l)}$.

Now, take i_2 such that $\text{supp}(x_{i_2}) > j^{(d_1)}(x_{i_1})$ and split it as before. Continuing in this manner, we obtain a sequence

$$(y_{i_1}^{(1)}, y_{i_1}^{(2)}, \dots, y_{i_1}^{(d_1)}, y_{i_2}^{(1)}, \dots, y_{i_2}^{(d_2)}, \dots, y_{i_n}^{(1)}, \dots, y_{i_n}^{(d_n)}, \dots) := (u_k)_{k=1}^\infty.$$

For this sequence we have

$$\left\| \sum_{k=1}^n a_k u_k \right\| = |a_1| + \left\| \sum_{i=2}^n a_k u_k \right\| = \dots = \sum_{k=1}^n |a_k|,$$

that is, $(u_k)_{k=1}^\infty$ is equivalent to the canonical basis of ℓ_1 . But $(x_{i_k})_{k=1}^\infty$ is a block sequence of $(u_k)_{k=1}^\infty$ and therefore it is also equivalent to the canonical basis of ℓ_1 . \square

Remark 4.

1. Observe that, in particular, the hypothesis of Proposition 4 implies that $i(\mathcal{M}_k) = 1$ for all $k \in I$.
2. The proof of the converse of Proposition 4 states that either $T[(\mathcal{M}_k, 1)_{k \in I}]$ contains a subspace isomorphic to c_0 or $T[(\mathcal{M}_k, 1)_{k \in I}]$ is ℓ_1 -saturated.

We now give an example of a Tsirelson type space which contains ℓ_1 and c_0 .

Example 2. Let $\mathcal{M} = \{F \subseteq \mathbb{N} \mid \exists i \in \mathbb{N} \text{ such that } F \subseteq \{2i - 1, 2i\}\}$. $T(\mathcal{M}, 1)$ contains ℓ_1 by Proposition 3 and c_0 by Proposition 4. Moreover, it is easy to see that the space is isomorphic to $\ell_1 \oplus c_0$.

2. THE CASE $(\mathcal{M}_k, \theta_k)_{k=1}^l$

In view of the previous results, in this section we will consider Tsirelson type spaces defined by finite sequences $(\mathcal{M}_k, \theta_k)_{k=1}^l$, with $\theta_k \in (0, 1)$ for all $k = 1, \dots, l$. The main result of the section is

Theorem 1. *Let $i(\mathcal{M}_k) = n_k \in \mathbb{N}$ and $\theta_k \in (0, 1)$ for all $k = 1, \dots, l$.*

1. *If $\theta_k \leq 1/n_k$ for all k then $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ is c_0 -saturated.*
2. *If $\theta_k > 1/n_k$ for some k then $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ is ℓ_p -saturated for some $p \in (1, +\infty)$.*

Our proof of this theorem is based on Theorem 2 below, proved in [4]. In order to state it we first need some definitions.

Definition 5. Let $m \in \mathbb{N}$ and $\varphi \in K_m \setminus K_{m-1}$. An analysis of φ is any sequence $\{K^s(\varphi)\}_{s=0}^m$ of subsets of K such that for every s ,

1. $K^s(\varphi)$ consists of successive elements of K_s and $\bigcup_{f \in K^s(\varphi)} \text{supp}(f) = \text{supp}(\varphi)$.
2. If $f \in K^{s+1}(\varphi)$ then either $f \in K^s(\varphi)$ or there exists k and successive $f_1, \dots, f_d \in K^s(\varphi)$ with $(\text{supp}(f_1), \dots, \text{supp}(f_d))$ \mathcal{M}_k -admissible and $f = \theta_k(f_1 + \dots + f_d)$.
3. $K^m(\varphi) = \{\varphi\}$.

Definition 6.

1. Let $\varphi \in K_m \setminus K_{m-1}$ and let $\{K^s(\varphi)\}_{s=0}^m$ be a fixed analysis of φ . Then for a given finite block sequence $(x_k)_{k=1}^l$ we set for every $k \in \{1, \dots, l\}$

$$s_k = \begin{cases} \max\{s \mid 0 \leq s < m, \text{ and there are at least two } f_1, f_2 \in K^s(\varphi) \\ \text{such that } |\text{supp}(f_i) \cap \text{supp}(x_k)| > 0, i = 1, 2\}, \\ \text{when this set is non-empty} \\ 0 \text{ if } |\text{supp}(x_k) \cap \text{supp}(\varphi)| \leq 1. \end{cases}$$

2. For $k = 1, \dots, l$ we define the initial part and the final part of x_k with respect to $\{K^s(\varphi)\}_{s=0}^m$, and denote them respectively by x'_k and x''_k , as follows: If $\{f \in K^{s_k}(\varphi) \mid \text{supp}(f) \cap \text{supp}(x_k) \neq \emptyset\} := \{f_1, \dots, f_d\}$ with $f_1 < \dots < f_d$, we set $x'_k = (\text{supp}(f_1))x_k$ and $x''_k = \left(\bigcup_{i=2}^d \text{supp}(f_i)\right)x_k$.

Notation. Let $m \in \mathbb{N}$, $\varphi \in K^m \setminus K^{m-1}$, let $\{K^s(\varphi)\}_{s=0}^m$ be an analysis of φ , $(v_i)_{i=1}^\infty$ a block sequence and $(x_j)_{j=1}^\infty$ a block sequence with $x_j \in \text{Span}\{v_i \mid i \in \mathbb{N}\}$.

Suppose that there exists n_φ such that $\text{supp}(\varphi) \subseteq \bigcup_{j=1}^{n_\varphi} \text{supp}(x_j)$ and denote by x'_j and

x_j'' the initial and the final part of x_j , $j \leq n_\varphi$. For all $f = \theta_k(f_1 + \dots + f_d) \in K^s(\varphi)$ and $J \subseteq \{1, \dots, n_\varphi\}$ we define the following sets for (x_j') :

$$I' = \{i \mid 1 \leq i \leq d \text{ and } \text{supp}(f_i) \cap \text{supp}(x_j') \neq \emptyset \text{ for at least two different } j \in J\}$$

and for every $i \in I$,

$$D'_{f_i} = \{j \in J \mid \text{supp}(f_i) \cap \text{supp}(x_j') \neq \emptyset \\ \text{and } (\text{supp}(f) \cap \text{supp}(x_j')) \setminus \text{supp}(f_i) \subseteq \text{supp}(v_t) \text{ for some } t\}$$

and

$$T' = \left\{ j \in J \mid j \notin \bigcup_{i \in I'} D'_{f_i} \text{ and } \exists t_1 \neq t_2 \right. \\ \left. \text{such that } \text{supp}(x_j') \cap \left(\bigcup_{i \notin I'} \text{supp}(f_i) \right) \cap \text{supp}(v_{t_i}) \neq \emptyset, \ i = 1, 2 \right\}.$$

In the same manner we define sets I'' , D''_{f_i} , T'' exchanging x_j' for x_j'' .

Theorem 2 ([4]). *Given $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ with $l \in \mathbb{N}$, $\theta_k \in (0, 1)$ and $i(\mathcal{M}_k) = n_k \in \mathbb{N}$, for all $k = 1, \dots, l$, let $(v_i)_{i=1}^\infty$ be a normalized block sequence. If there exists a sequence $x_j = \sum_{i \in I_j} a_i v_i$ with $(a_i)_{i=1}^\infty \subset \mathbb{R}$ and $(I_j)_{j=1}^\infty \subset \mathbb{N}$ successive such*

that

- (a) $1/2^{j+1} \leq |a_j| < 1/2^j$ and
- (b) for all $m \in \mathbb{N}$, $\varphi \in K^m \setminus K^{m-1}$, each analysis $\{K^s(\varphi)\}_{s=1}^m$ of φ , all $f = \theta_k(f_1 + \dots + f_d) \in K^s(\varphi)$, and all $J \subseteq \{1, \dots, n_\varphi\}$, the inequalities $|I'| + |T'| \leq n_k$ and $|I''| + |T''| \leq n_k$ hold,

then $(x_j)_{j=1}^\infty$ is equivalent to the canonical basis of $T[(\mathcal{A}_{n_k}, \theta_k)_{k=1}^l]$.

Recall, see [4], that the space $T[(\mathcal{A}_{n_k}, \theta_k)_{k=1}^l]$ is either isometrically isomorphic to c_0 , when $n_k \cdot \theta_k \leq 1$ for all k , or isomorphic to ℓ_p , where $p = \min \left\{ \frac{1}{1 - \log_{n_k} \frac{1}{\theta_k}} \mid n_k \cdot \theta_k > 1 \right\}$. So, to prove Theorem 1 we need to find the sequence $(x_j)_{j=1}^\infty$ and the next lemma will be useful for constructing it.

Lemma 1. *Let $l \in \mathbb{N}$, $\theta_k \in (0, 1)$ and \mathcal{M}_k be such that $i(\mathcal{M}_k) = n_k \in \mathbb{N}$ for all $k = 1, \dots, l$. Then for every block sequence $(u_i)_{i=1}^\infty$ in $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ there exists an infinite subset $\mathcal{P} = \{p_i\}_{i=1}^\infty$ of \mathbb{N} and a subsequence $(v_i)_{i=1}^\infty$ of $(u_i)_{i=1}^\infty$ having the following properties:*

- (a) $p_1 \leq \text{supp}(v_1) < p_2 \leq \text{supp}(v_2) < \dots < p_i \leq \text{supp}(v_i) < p_{i+1} \leq \dots$

- (b) For every sequence $E_1 < E_2 \dots < E_{n_k}$ of finite subsets of \mathcal{P} , where $E_i = \{p_{l_1^i}, \dots, p_{l_{t_i}^i}\}$, $i = 1, \dots, n_k$, the family

$$\left(\bigcup_{j=l_1^1}^{l_{t_1}^1} \text{supp}(v_j), \dots, \bigcup_{j=l_1^{n_k}}^{l_{t_{n_k}}^{n_k}} \text{supp}(v_j) \right)$$

is \mathcal{M}_k -admissible.

- (c) If $r \geq n_k + 1$, $S = \{s_1, \dots, s_r\} \subseteq \mathbb{N}$ is such that

$$|\{j \in \mathbb{N} \mid [s_i, s_{i+1}] \cap \text{supp}(v_j) \neq \emptyset\}| \geq 2$$

for all $i = 1, \dots, r - 1$, then $S \notin \mathcal{M}_k$.

Proof. The proof is based on the following result from [4]:

Lemma 2. Let $l, n_1, \dots, n_l \in \mathbb{N}$. Let $\mathcal{M}_k, k = 1, \dots, l$ be such that $i(\mathcal{M}_k) = n_k$. Then there exists an infinite subset Q of \mathbb{N} having the following properties:

1. Let $k \in \{1, \dots, l\}$. Every sequence $E_1 < E_2 \dots < E_{n_k}$ of length n_k of finite subsets of Q is \mathcal{M}_k -admissible.
2. Let $k \in \{1, \dots, l\}$. If $r \geq n_k + 1$, then no sequence $E_1 < E_2 \dots < E_r$ of finite subsets of Q with $|E_i| \geq 2$ for all $i = 1, \dots, r$, is \mathcal{M}_k -admissible.

Now, let $Q = \{k_i\}_{i=1}^\infty$ be the sequence in Lemma 2. Take $p_1 = k_1$, and $v_1 = u_l$ such that $p_1 \leq \text{supp}(u_l)$. Having chosen p_i and v_i with $p_i \leq \text{supp}(v_i)$, since $\{k_i\}_{i=1}^\infty$ is increasing, let k_{j_i} be such that $p_i \leq \text{supp}(v_i) < k_{j_i}$, and take $p_{i+1} = k_{j_i+1}$ and $v_{i+1} = u_l$ such that $p_{i+1} \leq \text{supp}(u_l)$.

The sequences $\{p_i\}_{i=1}^\infty$ and $(v_i)_{i=1}^\infty$ satisfy the assertions of Lemma 1:

- (a) By construction.

- (b) It is sufficient to see that $\bigcup_{j=l_1^i}^{l_{t_i}^i} \text{supp}(v_j) \subseteq [p_{l_1^i}, p_{l_{t_i}^i}]$ and, since the family

$\{\{p_{l_1^i}, p_{l_{t_i}^i}\}\}_{i=1}^{n_k}$ is \mathcal{M}_k -admissible by Lemma 2, $\left(\bigcup_{j=l_1^i}^{l_{t_i}^i} \text{supp}(v_j) \right)_{i=1}^{n_k}$ is also \mathcal{M}_k -admissible.

- (c) Let $S = \{s_1, \dots, s_r\}$ be such that $|\{j \in \mathbb{N} \mid [s_i, s_{i+1}] \cap \text{supp}(v_j) \neq \emptyset\}| \geq 2$ for all $i = 1, \dots, r - 1$, let $d_i = \min\{j \in \mathbb{N} \mid [s_i, s_{i+1}] \cap \text{supp}(v_j) \neq \emptyset\}$. Then $k_{j_{d_i}}$ and $p_{d_i+1} \in [s_i, s_{i+1}] \cap Q$ for all $i = 1, \dots, r - 1$, and by the property (2) of Lemma 2, $S \notin \mathcal{M}_k$. \square

Proof of Theorem 1. It suffices to show that c_0 or l_p is included in every block subspace of $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$.

Let $(u_i)_{i=1}^\infty$ be a normalized block sequence. Let $\mathcal{P} = \{p_i\}_{i=1}^\infty$ and $(v_i)_{i=1}^\infty$ be the sequences associated to $(u_i)_{i=1}^\infty$ from Lemma 1.

If $\sup_{m \in \mathbb{N}} \left\| \sum_{i=1}^m v_i \right\|$ is finite, then $(v_i)_{i=1}^\infty$ is equivalent to the canonical basis of c_0 , and from Corollary 1 of [4] we have $n_k \cdot \theta_k \leq 1$.

Suppose now that $\lim_{m \rightarrow \infty} \left\| \sum_{i=1}^m v_i \right\| = \infty$. Then we can construct a sequence $(y_i)_{i=1}^\infty$ supported by the subsequence $(v_i)_{i=1}^\infty$ with the following properties: For every j , $y_j = \frac{1}{2^{j+1}} \sum_{i \in I_j} v_i$, where

- (i) I_j are successive intervals of \mathbb{N} , and
- (ii) $1 - \frac{1}{2^{j+1}} \leq \|y_j\| \leq 1$.

If $x_j = y_j / \|y_j\|$, the sequence x_j satisfies condition (a) of Theorem 2.

We prove condition (b) of Theorem 2 for the initial parts of (x_j) since for the final parts the proof is analogous. Suppose that φ , f and J are fixed. Let $m_1 \leq \text{supp}(f_1) < m_2 \leq \text{supp}(f_2) < \dots < m_d \leq \text{supp}(f_d)$. We define $B \subseteq \{m_1, \dots, m_d\}$ as follows:

$$m_{i_s} \in B \leftrightarrow \begin{cases} i_s \in I', \\ i_s = \min\{i \notin I' \mid \text{supp}(x'_j) \cap \text{supp}(f_i) \neq \emptyset\} \text{ for some } j \in T'. \end{cases}$$

Let $m_{i_1} < \dots < m_{i_r}$ be the elements of B . Observing that

$$|\{t \in \mathbb{N} \mid [m_{i_s}, m_{i_{s+1}}] \cap \text{supp}(v_t)\}| \geq 2, \quad \forall 1 \leq s \leq r - 1$$

and using property (c) of Lemma 1 we get that $r = |B| \leq n_k$. So $|I'| + |T'| \leq n_k$. \square

The proof of the next two corollaries easily follows from Theorem 1 from this paper and Corollaries 1 and 2 from [4].

Corollary 1. Let $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$, $1 < p < \infty$, $n_k = i(\mathcal{M}_k)$ and $\theta_k \in (0, 1)$. The following conditions are equivalent:

- i) $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ contains a subspace isomorphic to ℓ_p .
- ii) $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ is ℓ_p -saturated.
- iii) $i(\mathcal{M}_k)$ is finite, $\theta_k > 1/n_k$ for some $k = 1, \dots, l$ and

$$p = \min \left\{ \frac{1}{1 - \log_{n_k} \frac{1}{\theta_k}} \mid n_k \cdot \theta_k > 1 \right\}.$$

Corollary 2. Let $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$, $\theta_k \in (0, 1)$. The following conditions are equivalent:

- i) $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ contains a subspace isomorphic to c_0 .
- ii) $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ is c_0 -saturated.
- iii) $i(\mathcal{M}_k)$ is finite and $\theta_k \leq 1/i(\mathcal{M}_k)$ for all $k = 1, \dots, l$.

In view of Proposition 1 and the previous corollaries we can include the case ℓ_1 in the discussion.

Corollary 3. Let $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$, $2 \leq i(\mathcal{M}_k) \in \mathbb{N}$ and $\theta_k \in (0, 1]$. The following conditions are equivalent:

- i) $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ contains a subspace isomorphic to ℓ_1 .
- ii) $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ is ℓ_1 -saturated.
- iii) $\theta_k = 1$ for some $k = 1, \dots, l$.

So in particular we have proved the following criterion, which is useful to show when two Tsirelson type Banach spaces are totally incomparable.

Theorem 3. Let $l, l' \in \mathbb{N}$, $\theta_k \in (0, 1)$ and $i(\mathcal{M}_k) = n_k \in \mathbb{N}$ for all $k = 1, \dots, l$ and $\theta'_k \in (0, 1)$ and $i(\mathcal{M}'_k) = n'_k \in \mathbb{N}$ for all $k = 1, \dots, l'$. Then $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ and $T[(\mathcal{M}'_k, \theta'_k)_{k=1}^{l'}]$ are totally incomparable if and only if one of the following situations occurs:

1. $\theta_k \leq 1/n_k$ for all $k = 1, \dots, l$ and $\theta'_k > 1/n'_k$ for some $k \in \{1, \dots, l'\}$, or
2. $\theta'_k \leq 1/n'_k$ for all $k = 1, \dots, l'$ and $\theta_k > 1/n_k$ for some $k \in \{1, \dots, l\}$, or
3. $\theta_k > 1/n_k$ for some $k \in \{1, \dots, l\}$ and $\theta'_k > 1/n'_k$ for some $k \in \{1, \dots, l'\}$ and

$$\min \left\{ \frac{1}{1 - \log_{n_k} \frac{1}{\theta_k}} \mid n_k \cdot \theta_k > 1 \right\} \neq \min \left\{ \frac{1}{1 - \log_{n'_k} \frac{1}{\theta'_k}} \mid n'_k \cdot \theta'_k > 1 \right\}.$$

Also we obtain a characterization of the reflexivity of this kind of spaces as in [1].

Proposition 5. Let $l \in \mathbb{N}$. Let $\theta_k \in (0, 1)$ and $i(\mathcal{M}_k) = n_k \in \mathbb{N}$ for all $k = 1, \dots, l$. Then the following conditions are equivalent:

1. $T[(\mathcal{M}_k, \theta_k)_{k=1}^l]$ is reflexive.
2. $\theta_k > 1/i(\mathcal{M}_k)$ for some $k \in \{1, \dots, l\}$.

3. A CRITERION OF TOTAL INCOMPARABILITY FOR SPACES
OF THE FORM $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$

We will suppose throughout the section that $(\theta_k)_{k=1}^\infty \subset (0, 1]$ is a non increasing null sequence since $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$ is easily seen to be isometric to $T[(\mathcal{A}_k, \theta'_k)_{k=1}^\infty]$ where $\theta'_k = \sup\{\theta_j \mid j \geq k\}$ and $\inf\{\theta_k\} > 0$ implies that $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$ is isomorphic to ℓ_1 .

The following properties of such spaces, stated as lemmas, are known.

Lemma 3. *Let $(u_i)_{i=1}^n$ be a normalized block sequence in $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$. Then for all $a_1, \dots, a_n \in \mathbb{R}$,*

$$\left\| \sum_{i=1}^n a_i e_i \right\| \leq \left\| \sum_{i=1}^n a_i u_i \right\|.$$

Proof. It is easy to prove by induction on s that $\left\| \sum_{i=1}^n a_i e_i \right\|_s \leq \left\| \sum_{i=1}^n a_i u_i \right\|$. □

The following lemma was proved in [11] with $\theta_k = (\log_2(1+k))^{-1}$, but the same proof works for any θ_k converging to zero.

Lemma 4 ([11]). *Let $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$, let θ_k converge to 0. Let $(y_n)_{n=1}^\infty$ be a block sequence, let a strictly decreasing null sequence $(\varepsilon_n)_{n=1}^\infty \subset \mathbb{R}^+$ and a strictly increasing sequence $(k_n)_{n=1}^\infty \subset \mathbb{N}$ be such that for each n there is a normalized block sequence $(y(n, i))_{i=1}^{k_n}$, $(1 + \varepsilon_n)$ -equivalent to the $l_1^{k_n}$ basis and $y_n = \frac{1}{k_n} \sum_{i=1}^{k_n} y(n, i)$. Then for all $l \in \mathbb{N}$,*

$$\lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \dots \lim_{n_l \rightarrow \infty} \left\| \sum_{i=1}^l y_{n_i} \right\| = \left\| \sum_{i=1}^l e_i \right\|.$$

We will consider spaces such that ℓ_1 is finitely block represented in every block subspace of the space but not containing ℓ_1 . The role of ℓ_1 in this context, as well as that of c_0 , can be easily described:

Proposition 6. *The following conditions are equivalent:*

- i) *The identity is an isometric isomorphism from $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$ onto c_0 .*
- ii) *$T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$ contains a subspace isomorphic to c_0 .*
- iii) *For all $n \in \mathbb{N}$, $\left\| \sum_{i=1}^n e_i \right\| = 1$.*
- iv) *$\theta_k \leq 1/k$ for all $k \in \mathbb{N}$.*

Proof. ii) \Rightarrow iii): By the Bessaga-Pełczyński Principle and a theorem of R. C. James (see e.g. [8], p. 97), for every $\varepsilon > 0$ there exists a normalized block sequence $(u_i)_{i=1}^\infty$ such that for all $l \in \mathbb{N}$,

$$\max |a_i| \leq \left\| \sum_{i=1}^l a_i u_i \right\| \leq (1 + \varepsilon) \max |a_i|, \quad a_1 \dots a_l \in \mathbb{R}$$

and so by Lemma 3, $\left\| \sum_{i=1}^l e_i \right\| \leq (1 + \varepsilon)$ and iii) follows.

iii) \Rightarrow iv): This is clear since $\theta \cdot l \leq \left\| \sum_{i=1}^l e_i \right\|$.

iv) \Rightarrow i): By induction on $m \in \mathbb{N}$ it easily follows that $|\cdot|_m = |\cdot|_0$ on c_{00} . \square

Proposition 7. Let $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$, let θ_k converge to 0. The following conditions are equivalent:

- i) The identity is an isometric isomorphism from $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$ onto ℓ_1 .
- ii) $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$ contains a subspace isomorphic to ℓ_1 .
- iii) For all $n \in \mathbb{N}$, $\left\| \sum_{i=1}^n e_i \right\| = n$.
- iv) $\theta_2 = 1$.

Proof. ii) \Rightarrow iii). Choose a strictly decreasing sequence $(\varepsilon_n)_{n=1}^\infty \subset \mathbb{R}_+$ converging to 0 and $k_n = n$. We will construct a block sequence $(y_n)_{n=1}^\infty$ as in Lemma 4 above.

By James' Theorem let $(u_i)_{i=1}^\infty$ be a normalized block sequence $(1 + \varepsilon_1)$ -equivalent to the unit vector basis of ℓ_1 . Let $y_1 = u_1$. Again by James' theorem there exist a normalized block sequence $(u'_i)_{i=1}^\infty$ with $u'_i \in \text{Span}\{u_i \mid i \in \mathbb{N}\}$ and $y_1 < u'_1$, $(1 + \varepsilon_2)$ -equivalent to the unit vector basis of ℓ_1 . Let $y_2 = \frac{1}{2}(u'_1 + u'_2)$. We continue in the same way.

Let $l \in \mathbb{N}$. Since any block sequence $(y_{n_i})_{i=1}^l$ is $(1 + \varepsilon_1)$ -equivalent to the unit vector basis of ℓ_1^l , by Lemma 4 we have

$$(1 - \varepsilon_1)l \leq \left\| \sum_{i=1}^l e_i \right\| \leq l$$

and the result follows.

iii) \Rightarrow iv): Just notice that $2 = \|e_1 + e_2\| = 2\theta_2$.

iv) \Rightarrow i): This follows by induction on $|\text{supp}(x)|$. \square

We now give sufficient conditions, in terms of the behaviour of $\lambda_n := \left\| \sum_{i=1}^n e_i \right\|$, guaranteeing that in a space of this kind ℓ_1 is finitely block represented in every block subspace.

Proposition 8 ([5]). Let $n, l \in \mathbb{N}$, $0 < \varepsilon < 1$. Let $(X, \|\cdot\|)$ be a normed space with a normalized 1-unconditional normalized basis $(e_i)_{i=1}^{n^l}$ such that

$$(n - \varepsilon)^l \leq \left\| \sum_{i=1}^{n^l} e_i \right\| \leq n^l.$$

Then there exists a normalized block sequence $(y_i)_{i=1}^n$ of $(e_i)_{i=1}^{n^l}$ such that

$$n - \varepsilon \leq \left\| \sum_{i=1}^n y_i \right\| \leq n.$$

Moreover, $(y_i)_{i=1}^n$ is $\frac{1}{1-\varepsilon}$ -equivalent to the canonical basis of ℓ_1^n .

Proposition 9. Let $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$, let θ_k converge to 0. If there exists $(n_k)_{k=1}^\infty \subseteq \mathbb{N}$ unbounded and $(l_k)_{k=1}^\infty$ such that

$$\lim_{k \rightarrow \infty} \left[n_k - \left(\lambda_{n_k}^{l_k} \right)^{1/l_k} \right] = 0,$$

then ℓ_1 is finitely block represented in every block subspace of $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$.

Proof. Given $n \in \mathbb{N}$ and $0 < \varepsilon < 1$, take $k \in \mathbb{N}$ such that $n_k > n$ and $n_k - \left(\lambda_{n_k}^{l_k} \right)^{1/l_k} < \varepsilon$. Let $(u_i)_{i=1}^\infty$ be a normalized block sequence. Then

$$n_k^{l_k} \geq \left\| \sum_{i=1}^{n_k} u_i \right\| \geq \left\| \sum_{i=1}^{n_k} e_i \right\| = \lambda_{n_k}^{l_k} \geq (n_k - \varepsilon)^{l_k}$$

and, by Proposition 8, $\ell_1^{n_k}$ is finitely block represented in blocks of $(u_i)_{i=1}^\infty$. \square

Remark 5. By similar arguments it is easy to prove that the following condition is also sufficient:

1. There exists $m \geq 2$ such that $\lim_{l \rightarrow \infty} (\lambda_{m^l})^{1/l} = m$.

We can also give sufficient conditions for the sequence $(\theta_k)_{k=1}^\infty$:

2. There exists $(n_k)_{k=1}^\infty \subseteq \mathbb{N}$ unbounded and $(l_k)_{k=1}^\infty$ such that $\lim_{k \rightarrow \infty} n_k \left[1 - \left(\theta_{n_k}^{l_k} \right)^{1/l_k} \right] = 0$ or
3. There exists $m \geq 2$ such that $\lim_{l \rightarrow \infty} (\theta_{m^l})^{1/l} = 1$ or, equivalently, $\lim_{l \rightarrow \infty} (\theta_{m^l})^{\frac{1}{l}} = 1$ for all $m \geq 2$.

Lemma 5. Let $(X, \|\cdot\|)$ and $(X', \|\cdot\|')$ be Banach spaces not totally incomparable with Schauder bases $(e_i)_{i=1}^\infty$ and $(e'_i)_{i=1}^\infty$. If $(e_i)_{i=1}^\infty$ is shrinking, there exist block sequences $(u_i)_{i=1}^\infty$ and $(u'_i)_{i=1}^\infty$ of $(e_i)_{i=1}^\infty$ and $(e'_i)_{i=1}^\infty$ respectively such that the application $T: \overline{\text{Span}}\{u_i \mid i \in \mathbb{N}\} \rightarrow \overline{\text{Span}}\{u'_i \mid i \in \mathbb{N}\}$, given by $T(u_i) = u'_i$ for all $i \in \mathbb{N}$ is an isomorphism.

Proof. There exist subspaces $Y \subseteq X$ and $Y' \subseteq X'$ and an isomorphism $S: Y \rightarrow Y'$. We will see that for all $\varepsilon > 0$ we can find block sequences $(u_i)_{i=1}^\infty$ and $(u'_i)_{i=1}^\infty$ such that $(1 - \varepsilon)\|S\| \|S^{-1}\| \leq \|T\| \|T^{-1}\| \leq (1 + \varepsilon)\|S\| \|S^{-1}\|$.

Let $\varepsilon > 0$. There exists a normalized block sequence $(x_i)_{i=1}^\infty$ of $(e_i)_{i=1}^\infty$ and $\overline{\text{Span}}\{y_i \mid i \in \mathbb{N}\} \subseteq Y$ such that the linear isomorphism defined by $U(x_i) = y_i$ verifies $\|U\| \|U^{-1}\| \leq 1 + \varepsilon$. Let $y'_i := S(y_i)$ for all $i \in \mathbb{N}$.

Since $\inf_{i \in \mathbb{N}} \|y'_i\| > 0$ and $(e_i)_{i=1}^\infty$ is a shrinking basis, y'_i tends to 0 weakly. So, by the Bessaga-Pełczyński principle, there is a subsequence $(y'_{i_k})_{k=1}^\infty$ and a block sequence $(u'_k)_{k=1}^\infty$ of $(e'_i)_{i=1}^\infty$ such that the isomorphism defined by $V(y'_{i_k}) = u'_k$ verifies $\|V\| \|V^{-1}\| \leq 1 + \varepsilon$. Take $u_k = x_{i_k}$ and $T = V \circ S \circ U$. \square

Remark 6. Let $X = T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$, $\theta_k \in (0, 1)$. Since its canonical basis $(e_i)_{i=1}^\infty$ is unconditional, hence being shrinking is equivalent to ℓ_1 not being isomorphic to any subspace of X and this is the case by Proposition 7.

Theorem 4. Let $X = T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$ and $X' = T[(\mathcal{A}'_k, \theta'_k)_{k=1}^\infty]$ with $\theta_k, \theta'_k \in (0, 1)$ be such that ℓ_1 is finitely block represented in every block subspace of X and X' . If X and X' are not totally incomparable, then there exists $C \geq 0$ such that for all $n \in \mathbb{N}$,

$$(*) \quad \frac{1}{C} \leq \frac{\lambda_l}{\lambda'_l} \leq C.$$

Proof. Denote by $\|\cdot\|$ and $\|\cdot\|'$ the norms of X and X' , respectively. By Lemma 5, there exist block sequences $(u_i)_{i=1}^\infty \subseteq X$ and $(u'_i)_{i=1}^\infty \subseteq X'$ of their respective bases denoted by $(e_i)_{i=1}^\infty$ and $(e'_i)_{i=1}^\infty$, such that $T: \overline{\text{Span}}\{u_i \mid i \in \mathbb{N}\} \rightarrow \overline{\text{Span}}\{u'_i \mid i \in \mathbb{N}\}$, given by $T(u_i) = u'_i$ for all $i \in \mathbb{N}$ is an isomorphism. Therefore, for all $(a_i)_{i=1}^\infty \subseteq \mathbb{R}$ and $n \in \mathbb{N}$,

$$\frac{1}{\|T\|} \left\| \sum_{i=1}^n a_i u'_i \right\|' \leq \left\| \sum_{i=1}^n a_i u_i \right\| \leq \|T^{-1}\| \left\| \sum_{i=1}^n a_i u'_i \right\|'.$$

By Lemma 4, given $\varepsilon > 0$ and $l \in \mathbb{N}$, there exists a normalized block sequence y_1, \dots, y_l of $(u_i)_{i=1}^\infty$, such that

$$\lambda_l - \varepsilon \leq \left\| \sum_{i=1}^l y_i \right\| \leq \lambda_l + \varepsilon.$$

Let $y'_i := T(y_i)$ for all $i = 1, \dots, l$. Then we have

$$\begin{aligned} \lambda_l + \varepsilon &\geq \left\| \sum_{i=1}^l y_i \right\| \geq \frac{1}{\|T\|} \left\| \sum_{i=1}^l y'_i \right\|' \\ &= \frac{1}{\|T\|} \left\| \sum_{i=1}^l \|y'_i\|' \frac{y'_i}{\|y'_i\|'} \right\|' \geq \frac{1}{\|T\|} \min_{1 \leq i \leq l} \|y'_i\|' \left\| \sum_{i=1}^l \frac{y'_i}{\|y'_i\|'} \right\|' \\ &\geq \frac{1}{\|T\| \|T^{-1}\|} \left\| \sum_{i=1}^l e'_i \right\|' = \frac{1}{\|T\| \|T^{-1}\|} \lambda'_l \end{aligned}$$

(note that in the last inequality we use Lemma 3). Since the inequality is true for all $\varepsilon > 0$, we have proved that $\lambda_l \geq (\|T\| \|T^{-1}\|)^{-1} \lambda'_l$.

Now we reverse the roles of X and X' to obtain $(\|T\| \|T^{-1}\|)^{-1} \lambda'_l \leq \lambda_l \leq \|T\| \|T^{-1}\| \lambda'_l$. \square

Remark 7. If X and X' contain isometric subspaces Y and Y' , then $\lambda_l = \lambda'_l$ for all $l \in \mathbb{N}$. Actually, the same equality holds if for every $\varepsilon > 0$, X and X' contain $(1 + \varepsilon)$ -isomorphic subspaces.

Remark 8. There are special cases when the calculus of λ_l is easy. For instance when $(\theta_k), (\theta'_k)$ belong to the so called class \mathcal{F} defined in [11] we have $\lambda_l = l \cdot \theta_l$ and the condition $(*)$ of Theorem 4 yields $1/C \leq \theta_l/\theta'_l \leq C$ for all l or $\theta_l = \theta'_l$ if we can find isometric subspaces or $(1 + \varepsilon)$ -isomorphic subspaces for all $\varepsilon > 0$.

Example 3. Let $f_r(x) = \log_2^r(1 + x)$ with $0 < r < 3 \log 2 - 1$. Then $(f_r^{-1}(k)) \in \mathcal{F}$ and if $0 < r < s < 3 \log 2 - 1$, the spaces $T[(\mathcal{A}_k, 1/f_r(k))_{k=1}^\infty]$ and $T[(\mathcal{A}_k, 1/f_s(k))_{k=1}^\infty]$ are, by Theorem 4, totally incomparable. Moreover, it is easy to check that these spaces are also totally incomparable to ℓ_p , $1 \leq p < \infty$ or c_0 .

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