Czechoslovak Mathematical Journal

Edoardo Ballico

# Codimension 1 subvarieties $\mathcal{M}_g$ and real gonality of real curves

*Czechoslovak Mathematical Journal*, Vol. 53 (2003), No. 4, 917–924

Persistent URL: http://dml.cz/dmlcz/127849

Czechoslovak Mathematical Journal, 53 (128) (2003), 917–924

# CODIMENSION 1 SUBVARIETIES OF $\mathcal{M}_g$ AND REAL GONALITY OF REAL CURVES

E. Ballico, Trento

(Received January 15, 2001)

*Abstract.* Let $\mathcal{M}_g$ be the moduli space of smooth complex projective curves of genus $g$. Here we prove that the subset of $\mathcal{M}_g$ formed by all curves for which some Brill-Noether locus has dimension larger than the expected one has codimension at least two in $\mathcal{M}_g$. As an application we show that if $X \in \mathcal{M}_g$ is defined over $\mathbb{R}$, then there exists a low degree pencil $u\colon X \to \mathbb{P}^1$ defined over $\mathbb{R}$.

*Keywords*: moduli space of curves, gonality, real curves, Brill-Noether theory, real algebraic curves, real Riemann surfaces

*MSC 2000*: 14H10, 14H51, 14P99

## 1. Introduction

Let $X$ be a smooth complex projective curve of genus $g$ and let $G^r_d(X) := \{(L, V)\colon L \in \operatorname{Pic}^d(X),\ V \subseteq H^0(X, L),\ \dim(V) = r + 1\}$ be the scheme of all $r$-dimensional linear systems of degree $d$ on $X$. Set $W^r_d(X) := \{L \in \operatorname{Pic}^d(X), h^0(X, L) \geqslant r + 1\}$. The geometry of the schemes $G^r_d(X)$ and $W^r_d(X)$ is quite well understood when $X$ is a general curve of genus $g$ ([1]). In particular for a general $X$ every $G^r_d(X)$ is smooth, equidimensional, non-empty if and only if $\varrho(g, r, d) := g - (r + 1)(g + r - d) \geqslant 0$ and connected if and only if $\varrho(g, r, d) > 0$. It is natural to try to give upper bounds for the codimension in the moduli space $\mathcal{M}_g$ of the set of all curves for which some of these properties fail. For all integers $g$, $d$, $r$, $i$ with $g \geqslant 3$, $0 < d < 2g - 2$, $0 < r < g$ and $i > 0$ set $B(g, r, d, i) := \{X \in \mathcal{M}_g\colon$ there is an irreducible component $T$ of $G^r_d(X)$ such that the Zariski tangent space of $G^r_d(X)$ has dimension at least

The author was partially supported by MURST and GNSAGA of INdAM (Italy).

$\varrho(g,r,d)+i$ at a general point of $T\}$ and $D(g,r,d,i) := \{X \in \mathcal{M}_g\colon W^r_d(X)$ has dimension at least $\varrho(g,r,d)+i\}$. It is possible to define several other failure loci. We will need only the following ones. For all integers $x, y$ with $x > 0$ and $y > 0$ set $E(g,d,x,y) := \{X \in \mathcal{M}_g\colon$ there is a $y$-dimensional family of $L \in \mathrm{Pic}^d(X)$ with $h^0(X,L) \geqslant 2$ and $h^1(X,L^{\otimes 2}) \geqslant x\}$. In Section 2 we will prove the following result.

**Theorem 1.** *For all integers $g$, $d$ with $g \geqslant 4$ and $0 < d < 2g-2$ every irreducible component of $B(g,1,d,3)$, $E(g,d,2,1)$ and $E(g,d,1,2)$ has codimension at least two in $\mathcal{M}_g$.*

Now we explain our motivation for showing that some bad sublocus of $\mathcal{M}_g$ has codimension at least two in $\mathcal{M}_g$. Let $X$ be a smooth connected projective curve of genus $g$ defined over $\mathbb{R}$. Hence the set $X(\mathbb{C})$ of its complex points is an oriented compact topological surface (a sphere with $g$ handles) equipped with a complex structure. The set $X(\mathbb{R})$ is the union of $n(X)$ disjoint circles. The real structure is uniquely determined by an anti-holomorphic involution $\sigma\colon X(\mathbb{C}) \to X(\mathbb{C})$. We have $X(\mathbb{R}) = \{P \in X(\mathbb{C})\colon \sigma(P) = P\}$. It is known that either $X(\mathbb{C}) \setminus X(\mathbb{R})$ is connected or $X(\mathbb{C}) \setminus X(\mathbb{R})$ has exactly two connected components (see e.g. [7, Prop. 3.1], or [11, p. 262]); following [7] we will write $a(X) = 0$ if $X(\mathbb{C}) \setminus X(\mathbb{R})$ is not connected and $a(X) = 1$ if $X(\mathbb{C}) \setminus X(\mathbb{R})$ is connected. It is known that we have $1 \leqslant n(X) \leqslant g+1$ and $n(X) \equiv g+1 \bmod(2)$ if $a(X) = 0$ and $0 \leqslant n(X) \leqslant g$ if $a(X) = 1$ and that every such pair $(n(X), a(X))$ arises for some real smooth curve of genus $g$ ([7, Prop. 3.1], or [11, p. 262]). The pair $(n(X), a(X))$ will be called the topological type of the real curve $X$. There exists a connected smooth Teichmüller space, $T(g,c,e)$, parametrizing all genus $g$ real smooth curve with fixed topological type $(c,e)$ ([6] or [10, Th. 5.1]). We learned the notion of gonality over an arbitrary base field from [9]. For every smooth real curve $X$ set $\mathrm{gon}(X,\mathbb{R}) := \inf\{d \in \mathbb{Z}\colon$ there exists a real $L \in \mathrm{Pic}^d(X)$ with $h^0(X,L) \geqslant 2\}$. The integer $\mathrm{gon}(X,\mathbb{R})$ is called the real gonality of the real curve $X$. For any such curve the case $r=1$ of the existence theorem for special divisors ([1]) gives $\mathrm{gon}(X,\mathbb{C}) \geqslant [\frac{1}{2}(g+3)]$. By specialization and semicontinuity, to obtain this upper bound it is sufficient to prove this bound for a general curve $Y$ of genus $g$. For a general curve $Y$ we have $\mathrm{gon}(Y,\mathbb{C}) = [\frac{1}{2}(g+3)]$ ([1]). By [2, Th. 3.1], for every integer $g \geqslant 2$ and every topological type $(c,e)$, $e = 0,1$, with $c > 0$ there is an open non-empty euclidean subset $U(g,c,e)$ of $T(g,c,e)$ such that $\mathrm{gon}(X,\mathbb{R}) = [\frac{1}{2}(g+3)]$ for every $X \in U(g,c,e)$. By specialization for every real curve $X$ in the euclidean closure, $V(g,c,e)$, of $U(g,c,e)$ we have $\mathrm{gon}(X,\mathbb{R}) \geqslant [\frac{1}{2}(g+3)]$. We are interested in the corresponding problem for all real curves. In genus 8 a surprise arose: by [2, Th. 3.2], for every topological type $(c,e)$ admissible for genus 8 and with $c > 0$ there is a non-empty open subset $B(8,c,e)$ of $T(8,c,e)$ such that every $X \in B(8,c,e)$ has $\mathrm{gon}(X,\mathbb{R}) > [\frac{1}{2}(g+3)] = 5$. The connection with Theorem 1 is simple; every solution specializes, i.e. for every integer $d$ the set $\mathrm{gon}(d; g, c, e) := \{X \in T(g, c, e)\colon \mathrm{gon}(X, \mathbb{R}) \leqslant d\}$ is closed in $T(g, c, e)$; every smooth solution generalizes, i.e. it can be extended to nearby curves, but in general non-smooth solutions do not generalize (consider a double root of a real polynomial $f \in \mathbb{R}[t]$. Hence we see that if $g = 8$ for all admissible types $B(8, 5, 1, 1) \cap T(8, c, e)$ disconnects $T(8, c, e)$. Since $T(8, c, e)$ is a smooth differential manifold, this implies that $B(8, 5, 1, 1) \cap T(8, c, e)$ has codimension one in $T(8, c, e)$ and hence $B(8, 5, 1, 1)$ has a component of codimension one in $\mathcal{M}_8$. Using Theorem 1 in Section 3 we will prove the following result.

**Theorem 2.** *For every integer $g \geqslant 3$ and every smooth real curve $X$ of genus $g$ with $X(\mathbb{R}) \neq \emptyset$ we have* $\mathrm{gon}(X, \mathbb{R}) \leqslant [\frac{1}{2}(g + 3)] + 3$.

The case of curves without real points is more delicate, because such curves have only real linear systems of even degree. In Section 3 we will prove the following result.

**Theorem 3.** *For every integer $g \geqslant 3$, let $u(g)$ be the first even integer bounded above by $[\frac{1}{2}(g + 3)] + 6$, i.e. set $u(g) := 2([\frac{1}{4}(g + 3)] + 6$. Then for every smooth real curve $X$ of genus $g$ with $X(\mathbb{R}) = \emptyset$ we have* $\mathrm{gon}(X, \mathbb{R}) \leqslant u(g)$.

## 2. Proof of Theorem 1

In the case of spanned pencils, the Petri map is essentially equivalent to the multiplication map $H^0(X, L) \otimes H^0(X, L) \to H^0(X, L^{\otimes 2})$. More precisely, for any smooth curve $X$ and $L \in \mathrm{Pic}^d(X)$ with $h^0(X, L) = 2$ and $|L| = |F| + B$ with $|F|$ base point free pencil and $B \geqslant 0$ the base locus of $|L|$, as in the base point free pencil trick the multiplication map induces an exact sequence

$$0 \to L \otimes F^* \to H^0(X, F) \otimes L \to L \otimes F \to 0. \tag{1}$$

The exact sequence (1) shows that the excess dimension of the Zariski tangent space of $W^1_d(X)$ at $L$ is just $H^1(X, F^{\otimes 2} \otimes B)$. For a spanned pencil $F$ the condition $H^1(X, F^{\otimes 2}) = 0$ has a very nice geometric interpretation: we have $h^1(X, F^{\otimes 2}) \neq 0$ if and only if $F$ is the limit of two different $g^1_d$'s in a family of curves ([3]). We will not need this nice interpretation.

**Remark 1.** Since we may ignore finitely many irreducible subvarieties of $\mathcal{M}_g$ with codimension at least two, we will always consider curves without non-trivial automorphisms, i.e. we will always work on the smooth locus of $\mathcal{M}_g$. Fix any such

$Y \in \mathcal{M}_g$ and take $L \in \mathrm{Pic}(Y)$ with $L$ spanned, $h^0(Y, L) = 2$ and $x := h^1(Y, L^{\otimes 2}) > 0$. For $x$ general points $P_1, \ldots, P_x$ of $Y$ we have $h^1(Y, L^{\otimes 2}(P_1 + \ldots + P_x)) = 0$ (e.g. use Serre duality if $x \geqslant 2$).

Proof of Theorem 1. The idea is to use Remark 1. To carry over this idea we prefer to use induction on $g$, the cases $g = 2$ and $g = 3$ being trivial. Hence we assume $g \geqslant 4$ and that the result is true for curves of genus at most $g - 1$. In order to obtain a contradiction we assume the existence of an irreducible subvariety $G$ of $\mathcal{M}_g$ with $\dim(G) = 3g - 4$ and such that the corresponding result is false for all curves $X \in G$. Let $T$ be the closure of $G$ in $\overline{\mathcal{M}}_g$. Since $\mathcal{M}_g$ has no complete subvariety of dimension $3g - 4$ (see [5] for much more), $T$ intersects the boundary $\overline{\mathcal{M}}_g \backslash \mathcal{M}_g$. Since $\overline{\mathcal{M}}_g$ has only quotient singularities, we may easily check using a local smooth finite covering that $T$ intersects at least one of the irreducible components of $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ in an algebraic set of codimension $\leqslant 1$. $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ has $[\frac{1}{2}g] + 1$ irreducible components $Y_i$, $0 \leqslant i \leqslant [\frac{1}{2}g]$: a general member of $Y_0$ is given by an irreducible curve with a unique node, while a general member of $Y_i$, $1 \leqslant i \leqslant [\frac{1}{2}g]$, is the union of two smooth curves, one of genus $g - i$ and the other one of genus $i$, linked at a unique point. First assume that $T$ contains an irreducible hypersurface $F$ of $Y_0$. Fix a general $X \in F$ and let $\pi\colon Y \to X$ be the normalization. Either $F$ is the closure of the irreducible curves of arithmetic genus $g$ with two ordinary nodes or its general member has a unique node. First assume that $F$ is the closure of the irreducible curves of arithmetic genus $g$ with two ordinary nodes. Since $\dim(F) = 3g - 5$, $Y$ is a general curve of genus $g - 2$ and the 4 points $\pi^{-1}(\mathrm{Sing}(X))$ are general points of $Y$. Fix $L \in \mathrm{Pic}(X)$ with $L$ spanned, $h^0(X, L) = 2$ and $h^1(X, L^{\otimes 2}) \geqslant 2$. Fix an effective divisor $B \geqslant 0$ such that $h^1(X, L^{\otimes 2}(B)) = 2$ (Remark 1). Set $R = \pi^*(L)$. We have $h^1(Y, R^{\otimes 2}) \geqslant h^1(X, L^{\otimes 2}) - 2$ with equality if and only if $h^0(Y, R) = h^0(X, L)$. Since $Y$ is general, we have $h^1(X, R^{\otimes 2}) = 0$ by a corollary of the Gieseker-Petri Theorem. Hence we obtain a contradiction unless $h^1(X, L^{\otimes 2}) = 2$ and $h^0(Y, R) = h^0(X, L)$. This case is excluded in the statement of Theorem 1 but even avoiding this observation in this case by Remark 1 we obtain that the set of all offending line bundles on $Y$ has dimension at least two more than the ones on $X$, obtaining a contradiction by the inductive assumption for genus $g - 2$. Now assume that $X$ has a unique node. Since $\dim(F) = 3g - 5$, either $Y$ is a general curve of genus $g - 1$ and one of the two points of $\pi^{-1}(\mathrm{Sing}(X))$ is a general point of $Y$ or $Y$ is a general member of an irreducible codimension one subvariety $G'$ of $\mathcal{M}_{g-1}$ and the two points of $\pi^{-1}(\mathrm{Sing}(X))$ are general points of $Y$. Fix $L \in \mathrm{Pic}(X)$ with $h^0(X, L) = 2$ and $L$ spanned. Fix an effective divisor $B \geqslant 0$ such that $h^1(X, L^{\otimes 2}(B)) \geqslant 2$; again, the case $h^1(X, L^{\otimes 2}(B)) = 2$ is excluded in the statement of Theorem 1 but we need to handle it when at the end of the proof we will consider the case in which $L$ is not locally free. Thus $h^1(X, L^{\otimes 2}) \geqslant 2$. Set $R = \pi^*(L)$. We have $h^1(Y, R^{\otimes 2}) \geqslant h^1(X, L^{\otimes 2}) - 2$ with equality if and only if $h^0(Y, R) = h^0(X, L)$. Hence if $Y$ has general moduli, then the corollary of the Gieseker-Petri Theorem just used gives a contradiction. Now assume that $Y$ has not general moduli. Thus $Y$ must be the general member of a hypersurface $Z$ of $\mathcal{M}_{g-1}$, while the two points of $\pi^{-1}(\mathrm{Sing}(X))$ are general in $Y$. We apply Remark 1 and obtain that the set of all offending line bundles on $Y$ has dimension at least one more than the dimension of the set of all offending line bundles on $X$. We conclude by induction on $g$ using in the inductive assumption the full statement of Theorem 1 involving the algebraic sets $E(*, *, *, *)$. Now assume that $T$ contains an irreducible hypersurface $F$ of $Y_i$ with $1 \leqslant i \leqslant [\frac{1}{2}g]$. First assume that a general $X \in F$ has a unique singular point, i.e. assume that $X$ is the union of two smooth curves, $X_1$ of genus $g - i$ and $X_2$ of genus $i$, linked at one point $P$. If $i \geqslant 2$ at least one of the two components of $X$, say $X_2$, must have general moduli. Fix $L \in \mathrm{Pic}(X)$ with $L$ spanned, $h^0(X, L) \geqslant 2$ and $h^1(X, L) \geqslant 2$. Hence $L\big|X_1$ and $L\big|X_2$ are both spanned and at least one of them is not trivial. For any spanned $M \in \mathrm{Pic}(X)$ consider the Mayer-Vietoris exact sequence

$$0 \to M \to M\big|X_1 \oplus M\big|X_2 \to M\big|\{P\} \to 0. \tag{2}$$

Since $M\big|X_1$ and $M\big|X_2$ are spanned, we obtain $h^0(X, M) = h^0(X_1, M\big|X_1) + h^0(X, M\big|X_2) - 1$ and $h^1(X, M) = h^1(X_1, M\big|X_1) + h^1(X, M\big|X_2)$. Take $M = L^{\otimes 2}$. If $M\big|X_2$ is not trivial, we obtain a contradiction either for degree reasons (case $i \leqslant 2$) or by the generality of $X_2$ and the inductive assumption. Hence we may assume $L^{\otimes 2}\big|X_2 \cong \mathcal{O}_{X_2}$. We may avoid this case for the following reason; the curve $X$ must be a limit of coverings $\{u_t\colon\ X_t \to \mathbb{P}^1\}$ with $\deg(u_t) = \deg(L)$ and $X_t$ smooth; by the theory of admissible coverings due to Harris and Mumford ([8]) this family has as limit when $X_t$ goes to $X$ an admissible $\deg(L)$ covering of a curve $E$ such that $X$ is the stable resuction of $E$ and in the corresponding covering both $X_1$ and $X_2$ are mapped non-trivially to a copy of $\mathbb{P}^1$. Now assume that $X$ has at least two singular points. Counting dimensions, we see that either $X$ has two irreducible components, one of them being smooth and with general moduli, or it has 3 irreducible components, all of them with general moduli and linked as a tree at general points of the components. We conclude as in the previous case. Now we drop the assumption of the existence of a spanned $L \in \mathrm{Pic}(X)$ of the same degree as the one on a general element of $G$ and with $h^1(X, L^{\otimes 2}) \geqslant 2$. In the general case in the limit we only have a rank one torsion free sheaf $A$ with $h^0(X, A) \geqslant 2$ and with $h^1(X, (A^{\otimes 2})^{**}) \geqslant 2$. For the formal classification of finitely generated torsion free modules on the completion of a local of a node, see [12, pp. 162–163] (both for the case in which the node is on one irreducible component of $X$ and the case in which the node is on the intersection of two irreducible components of $X$). Let $R$ be the subsheaf of $A$ spanned by $H^0(X, A)$. Thus $R$ is a spanned rank one torsion free sheaf on $X$ with $\deg(R) \leqslant \deg(A)$ and $h^0(X, R) = h^0(X, A) \geqslant 2$. First assume $X \in Y_i$, $i \neq 0$, $X = X_1 \cup X_2$ and call $u\colon X_1 \to X$ and $v\colon X_2 \to X$ the inclusions. Set $R_1 := u^*(R)/\operatorname{Tors}(u^*(R))$ and $R_2 := v^*(R)/\operatorname{Tors}(v^*(R))$. The sheaves $R_1$ and $R_2$ are spanned line bundles. We have $0 \leqslant \operatorname{lenght}(\operatorname{Tors}(u^*(R))) \leqslant 1$, $0 \leqslant \operatorname{lenght}(\operatorname{Tors}(v^*(R))) \leqslant 1$ and $\deg(R) - 2 \leqslant \deg(R_1) + \deg(R_2 \leqslant \deg(R)$. We do the same for the torsion free sheaf $R^{\otimes 2**}$; $R_1^{\otimes 2}$ (resp. $R_2^{\otimes 2}$) is the corresponding line bundle on $X_1$ (resp. $X_2$). By Riemann-Roch we have $h^1(X_1, R_1^{\otimes 2}) + h^1(X_2, R_2^{\otimes 2}) \geqslant h^1(X, (R^{\otimes 2})^{**}) - 1 \geqslant 2$. Since at least one of the curves $X_1$ and $X_2$ has general moduli, we obtain a contradiction. In the same way, just loosing one condition (but with a smaller degree) with respect to the case in which $L$ is locally free, we handle the case $F \subset Y_0$. □

## 3. Proofs of Theorems 2 and 3

Proof of Theorem 2. Set $A(g, c, e) := \{X \in T(g, c, e)\colon \operatorname{gon}(X, \mathbb{R}) \leqslant [\frac{1}{2}(g + 3)] + 3\}$. By specialization $A(g, c, e)$ is a closed subset of $T(g, c, e)$ and Theorem 2 is equivalent to the assertion that $A(g, c, e)$ contains a dense open subset of $T(g, c, e)$. By [2, Th. 3.1], for every integer $g \geqslant 2$ and every topological type $(c, e)$ with $c > 0$ and $e = 0, 1$ there is a non-empty open (for the euclidean topology) subset $U(g, c, e)$ of $T(g, c, e)$ corresponding to smooth real curves with real gonality $[\frac{1}{2}(g + 3)]$. By Theorem 1 the set of all curves $X \in \mathcal{M}_g$ with $L \in \operatorname{Pic}(X)$, $h^0(X, L) = 2$, $L$ spanned, $\deg(L) = [\frac{1}{2}(g+3)]$ and with $h^1(X, L^{\otimes 2}) \geqslant 3$ has codimension at least two in $\mathcal{M}_g$. We know that the subset of $\mathcal{M}_g$ parametrizing curves with non-trivial automorphisms has codimension at least two in $\mathcal{M}_g$ and hence that its real part does not disconnect any connected component $U(g, c, e)$ of the semialgebraic subset of $\mathcal{M}_g$ formed by real curves. Take a pair $(X, L)$ with $X \in U(g, c, e)$, $c > 0$, $\deg(L) = [\frac{1}{2}(g + 3)]$, $h^0(X, L^{\otimes 2}) = 2$, $L$ spanned and $x := h^1(X, L^{\otimes 2}) > 0$. Since $X(\mathbb{R})$ is infinite, for $x$ general points $P_1, \ldots, P_x$ of any component of $X(\mathbb{R})$ we have $h^1(X, L^{\otimes 2}(P_1 + \ldots + P_x)) = 0$ (Remark 1). Non-base point free pencils propagate to nearby curves, even to spanned complete pencils ([4, Prop. A.3]). Hence for each degree $d \geqslant [\frac{1}{2}(g + 3)] + 2$ outside a subset which does not disconnect $U(g, c, e)$ we may find a smooth pencil of degree $d$. For every real curve $Y$ with real structure induced by an anti-holomorphic involution $\sigma$ and all integers $r$, $d$ the schemes $W^r_d(Y)$ and $G^r_d(Y)$ are real and $W^r_d(Y)_{\mathrm{reg}}(\mathbb{R})$ is either empty or a real manifold of dimension $\varrho(g, r, d)$ whose members propagate to nearby real curves as $\sigma$-invariant linear systems; here we use that if $Y(\mathbb{R}) \neq \emptyset$, then every $\sigma$-invariant line bundle is real ([7, Prop. 2.2]). Hence by the existence theorem of a degree $[\frac{1}{2}(g + 3)]$ real pencil for an euclidean non-empty open subset $U_1(g,c,e)$ of $U(g,c,e)$ ([2, Th. 3.1]) we obtain the existence of a dense open subset $U_2(g,c,e)$ of $U(g,c,e)$ such that every $X \in U_2(g,c,e)$ has a real pencil of degree $[\frac{1}{2}(g+3)]+3$; indeed, by Theorem 1 outside a non-disconnecting subset of $U(g,c,e)$ every point on the closure of $U_1(g,c,e)$ in $U(g,c,e)$ has a real pencil of degree at most $[\frac{1}{2}(g+3)]+3$ which propagates to nearby curves, i.e. for all nearby curves $X$ we have $W^1_{[\frac{1}{2}(g+3)]+3}(X)_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$. By specialization every $X \in U(g,c,e)$ has a real pencil of degree at most $[\frac{1}{2}(g+3)]+3$, proving Theorem 2. □

P r o o f of Theorem 3. First we will check the existence of a non-empty euclidean open subset $U_1(g,0,1)$ of $U(g,0,1)$ such that every $X \in U_1(g,0,1)$ has a real pencil of degree the first even integer $\geqslant [\frac{1}{2}(g+3)]$. We start with a genus $g$ real hyperelliptic curve $Y$ with $Y(\mathbb{R}) = \emptyset$ and such that the hyperelliptic pencil $R \in \mathrm{Pic}^2(Y)$ is defined over $\mathbb{R}$ ([7, 6.1 and 6.2]). Call $\sigma\colon Y \to Y$ the anti-holomorphic involution giving the real structure of $Y$. Let $y$ be the first integer such that $4y+3 \geqslant g+2$. Since $\sigma$ is not the hyperelliptic involution, for any $y$ general points $P_1,\ldots,P_y$ of $Y(\mathbb{C})$ we have $h^0(Y, R(P_1+\ldots+P_y+\sigma(P_1)+\ldots+\sigma(P_y))) = 2$, $h^0(Y, R^{\otimes 2}(2P_1+\ldots+2P_y+2\sigma(P_1)+\ldots+2\sigma(P_y))) = 3$ and $h^1(Y, R^{\otimes 2}(2P_1+\ldots+2P_y+2\sigma(P_1)+\ldots+2\sigma(P_y))) = 0$. Hence the real pencil $R(P_1+\ldots+P_y+\sigma(P_1)+\ldots+\sigma(P_y))$ propagates to an open neighborhood of $Y$ ([4, Appendix]). Since for every $X \in U(g,0,1)$ near $Y$ we have a smooth pencil near $R(P_1+\ldots+P_y+\sigma(P_1)+\ldots+\sigma(P_y)))$, we also have a pencil invariant for the anti-holomorphic involution. Now we continue as in the proof of Theorem 2 because if $X \in U(g,0,1)$, $L$ is real, $\sigma$ induces the real structure of $X$ and $h^1(X, L^{\otimes 2}) \leqslant 2z$, then for $z$ general points $P_1,\ldots,P_z$ of $X(\mathbb{C})$ we have $h^1(X, L^{\otimes 2}(P_1+\ldots+P_z+\sigma(P_1)+\ldots\sigma(P_z))) = 0$ and the pencil $L(P_1+\ldots+P_z+\sigma(P_1)+\ldots\sigma(P_z))$ is $\sigma$-invariant. We have a new difficulty with respect to the proof of Theorem 2. Since $c > 0$, it is not true that, calling $\sigma$ the anti-holomorphic involution inducing the real structure, every $\sigma$-invariant line bundle is real, but in our situation we have more: we have a morphism $f\colon X \to \mathbb{P}^1$ which is invariant with respect to the usual real structure of $\mathbb{P}^1$. Every such morphism $f$ is induced by a real spanned line bundle because the graph of $f$ is a real subcurve of $X \times \mathbb{P}^1$. □

*References*


[1] *E. Arbarello, M. Cornalba, P. Griffiths and J. Harris*: Geometry of Algebraic Curves, I. Springer-Verlag, 1985.

[2] *S. Chaudary*: The Brill-Noether theorem for real algebraic curves. Ph.D. thesis. Duke University, 1995.

[3] *M. Coppens*: One-dimensional linear systems of type II on smooth curves. Ph.D. thesis. Utrecht, 1983.

[4] *M. Coppens and G. Martens*: Linear series on a general $k$-gonal curve. Abh. Math. Sem. Univ. Hamburg *69* (1999), 347–371.

[5] *S. Diaz*: A bound on the dimension of complete subvarieties of $\mathcal{M}_g$. Duke Math. J. *51* (1984), 405–408.
[6] *C. J. Earle*: On moduli of closed Riemann surfaces with symmetries. Advances in the Theory of Riemann Surfaces. Annals of Math. Studies 66. Princeton, N. J., 1971, pp. 119–130.
[7] *B. Gross and J. Harris*: Real algebraic curves. Ann. scient. Éc. Norm. Sup., $4^e$ série *14* (1981), 157–182.
[8] *J. Harris and D. Mumford*: On the Kodaira dimension of the moduli space of curves. Invent. Math. *67* (1982), 23–86.
[9] *M. Homma*: Separable gonality of a Gorenstein curve. Mat. Contemp. To appear.
[10] *M. Seppälä*: Teichmüller spaces of Klein surfaces. Ann. Acad. Sci. Fenn. Ser. A I. Math. Dissertationes *15* (1978), 1–37.
[11] *M. Seppälä*: Real algebraic curves in the moduli spaces of complex curves. Compositio Math. *74* (1990), 259–283.
[12] *C. S. Seshadri*: Fibrés vectoriels sur les courbes algébriques. Astérisque 96. Soc. Math. France, 1982.

*Author's address*: Dept. of Mathematics, University of Trento, 38050 Povo (TN), Italy, e-mail: `ballico@science.unitn.it`.